Contract Theory: Final

9:00am–12:00pm, 6th December, 2010

1. Reputation and Exit

Time is continuous and infinite. A firm has quality $q \in \{L, H\}$. The market believes the firm is high quality with probability $x = \Pr(q = H)$. If the firm has reputation x its instantaneous profits are $x - k$, where k is the fixed cost of operating. The firm also has the option to shut down, obtaining 0 thereafter. Its future discounted profits are

$$\int_{t=0}^{\tau} e^{-rt}(x_t - k)\, dt$$

where r is the interest rate and τ the (random) time when it shuts down.

We suppose the market learns through good news events (e.g. the firm is featured on the front page of the Wall St Journal). If the firm has low quality, a signal never arrives. If the firm has high quality, a signal arrives with probability λdt.

First, suppose the firm *does not* know its own type. Both the firm and the market start with a prior x_0.

(a) Both the market and the firm learn about the firm's quality over time. Verify that reputation evolves as follows. If a signal arrives x jumps to one, otherwise

$$dx = -\lambda x(1 - x)dt$$

(b) Denote the firm's value function by $V(x)$. Taking $V(\cdot)$ as given, at what reputation x^* does the firm decide to exit? Will the firm be making a loss at this exit point?

(c) Calculate $V(1)$.

(d) Characterise $V(x)$ in the form of a differential equation.

Next, suppose the firm *does* know its own type.

(e) Denote the firms' value functions by $V_L(x)$ and $V_H(x)$. Characterise the equilibrium exit decisions for the low and high quality firms. Will the low quality firm be making a loss at her exit point, x_L^*?

(f) Characterise $V_L(x)$ and $V_H(x)$ in the form of differential equations.

2. Motivating Information Acquisition

A potential house buyer (principal) hires a real estate broker (agent) to collect information about a house. The house has quality $q \in \{L, H\}$. A high quality house delivers utility 1 to the principal, a low quality house delivers utility -1 (this is net of the price paid). The prior is $\Pr(q = H) = \gamma$. Both the agent's and principal's utility are quasi-linear.

The agent invests effort e at cost $c(e)$ into observing a signal $s \in \{G, B\}$. The signal is informative with probability

$$\Pr(s = G|q = H) = \Pr(s = B|q = L) = \frac{1}{2} + e =: \eta(e)$$

The signal provides 'hard' information, so the agent cannot lie about the value of the signal. The cost function $c(e)$ is increasing and convex, and obeys $c'''(e) > 0$. To obtain internal optima assume that $c'(0) = 0$, $c''(0) = 0$ and $\lim_{e \to 1/2} c(e) = \infty$.

After observing the signal, the principal can choose to buy the house or not. If her decision to buy is independent of the signal, there is no reason to have the agent exert effort. Hence we assume the principal buys if $s = G$ and does not buy if $s = B$.

(a) Suppose the principal can observe the agent's effort choice. Show the welfare maximising effort satisfies the first order condition $c'(e) = 1$. [Here, welfare is the sum of the agents and principal's utility].

Now, consider the second best contract, where e is not observed by the principal. A contract consists of a wage $w_G \geq 0$ when the good signal is observed, and a wage $w_B \geq 0$ when the bad signal is observed [Note the limited liability constraint; there is no other (IR) constraint].

(b) Write down the agent's utility. Show the agent's optimal level of effort satisfies the first order condition $(w_G - w_B)(2\gamma - 1) = c'(e)$.

(c) Using the FOC in (b), what can we say about how the optimal wages change in the prior, γ? Is it always possible to motivate positive effort? Provide an intuition.

(d) Suppose $\gamma > 1/2$. Replacing the agent's (IC) constraint with their first-order condition, show the principal's optimal effort satisfies the first order condition

$$1 = c''(e)\left[e + \frac{1}{2(2\gamma - 1)}\right] + c'(e)$$

so that effort is increasing in γ.

3. Durable Goods with Varying Demand

A monopolist sells a good over two periods, $t \in \{1, 2\}$, with zero marginal cost. The discount rate is δ for both the firm and the agents. The firm chooses prices $\{p_1, p_2\}$.

Each period a demand curve enters the market $f_t(\theta)$ with support $[0, 1]$ and cumulative distribution $F_t(\theta)$. Once they enter, an agent has value θ for one unit of the good. If an agent buys in period $s \in \{1, 2\}$, his utility is

$$u = (\theta - p_s)\delta^s$$

Agents who enter in period 1 can thus buy in either period 1 or 2 (or never). An agent who enters in period 2 can only buy in period 2 (or never).

(a) An agent with value θ who is born in period t has equilibrium utility

$$u(\theta, t) = \max_{\tau \geq t} (\theta - p_\tau)\delta^\tau$$

Show that aggregate equilibrium utility for cohort t is given by

$$E[u(\theta, t)] = \int_0^1 \delta^{\tau(\theta,t)}[1 - F_t(\theta)]d\theta$$

where $\tau(\theta, t)$ is the time agent (θ, t) buys.

(b) Show that the firm's aggregate profits are given by

$$\Pi = \int_0^1 \delta^{\tau(\theta,1)} m_1(\theta)d\theta + \int_0^1 \delta^{\tau(\theta,2)} m_2(\theta)d\theta$$

where

$$m_t(\theta) := \theta f_t(\theta) - [1 - F_t(\theta)]$$

Assume that $m_t(\theta)$ is increasing in θ for each cohort.

(c) Given the firm charges prices $\{p_1, p_2\}$, argue that we can characterise the allocations by cutoffs $\{\theta_1^*, \theta_2^*\}$, where θ_t^* is the lowest type that buys at time t. That is, an agent buys in time s if their value is above θ_s^*, independent of the cohort to which they belong.

(d) Suppose demand is identical in the two periods, $m_1(\theta) = m_2(\theta)$. What are the optimal cutoffs and prices?

(e) Suppose demand is rising over time, $m_1(\theta) \geq m_2(\theta)$. What are the optimal cutoffs and prices? [Note: the demand is rising here because the static monopoly price is rising].

(f) Suppose demand is falling over time, $m_1(\theta) \leq m_2(\theta)$. What are the optimal cutoffs and prices?