Lecture Notes - Dynamic Moral Hazard

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1 Dynamic Moral Hazard

- Effects
  - Consumption smoothing
  - Statistical inference
  - More strategies
  - Renegotiation
- Non-separable technologies
  - One action $a \in \{0, 1\}$, many outputs q_t
    * Example (Mirrlees): $dq_t = adt + \sigma dZ_t$; then $q_1 \sim N(a, \sigma^2)$
    * Basically, like single-period model
    * But awesome inference: $\frac{f(q|L)}{f(q|H)} \to \infty$ as $q \to -\infty$
    * Approximate first-best
      · Pay flat wage for $q \in (q^*, \infty)$ where $q^* << 0$, and punish agent hard when $q < q^*$
      · States $(-\infty, q^*)$ much more likely for $a = 0$

[Figure: $f(q|L), f(q|H)$]

-   - Many actions $a_1, a_2, \cdots, a_T$, one output q_T
    * Agent does not learn anything
    * Can choose $a_1, a_2, \cdots, a_T$ simultaneously
- Main model: Separable technologies
  - Time $t \in \{1, 2, \cdots\}$

- Action $a_t \in A \subseteq \mathbb{R}$
- Output q_t (observable) with separable independent pdf $f(q_t|a_t)$
- Preferences $u(c_t) - g(a_t)$ time-separable, stationary
- Reference utility $\overline{u}$ per period
- Principal gets $q_t - w_t$

1.1 Asymptotic Efficiency

- Fudenberg, Holmstrom, Milgrom, JET, 1990

1.1.1 Setup

- ∞ periods, common discount factor δ
- Output $q_t \in [\underline{q}, \overline{q}]$
- Actions $a_t \in A$
- First best action a^* and quantity $q^* = \mathbb{E}[q|a^*] > \underline{q}$
- Time-separable and stationary

1.1.2 Result

Proposition 1 *If everybody is patient, first-best is almost achievable:* $\forall \varepsilon, \exists \overline{\delta}, \forall \delta \geq \overline{\delta}$ *there is a contract generating agent utility greater than* $u(q^*) - g(a^*) - \varepsilon$ *(and yielding at least* 0 *to the principal).*

- Statement assumes that agent proposes contract and has to satisfy principal's IR constraint
- If principal proposes, can also get first-best

Idea:

- Make agent residual claimant
- He can build up savings (through principal) and then smooth his consumption

Proof.

- Agent's wealth w_t

- If wealth is high, $w_t \geq \left(q^* - \underline{q}\right)/\delta$, consume

$$q_t = q^* + \underbrace{(1-\delta)\, w_t - \widetilde{\varepsilon}}_{\geq 0}$$

  - Earnings q^*
  - Interest $(1-\delta)\, w_t$
  - save a little $\widetilde{\varepsilon} \in \left(0; (1-\delta)\left(q^* - \underline{q}\right)/\delta\right)$

- If wealth is low, $w_t \leq \left(q^* - \underline{q}\right)/\delta$, consume

$$q_t = \underline{q} + (1-\delta)\, w_t$$

  - Minimal earning $\underline{q}$
  - Interest $(1-\delta)\, w_t$

- This is pretty arbitrary. The point is that wealth grows

$$\mathbb{E}\left[w_{t+1}\right] - w_t \geq \min\left\{\left(q^* - \underline{q}\right)/\delta, \widetilde{\varepsilon}\right\} > 0$$

- Thus, wealth is a submartingale with bounded increments and eventually exceeds any threshold forever with probability one

$$\begin{aligned} \lim_{t\to\infty} p_t &= 1 \text{ where} \\ p_t &= \Pr\left(w_\tau \geq x \text{ for all } \tau \geq t\right) \\ x &= \left(q^* - \underline{q}\right)/\delta \end{aligned}$$

- Omitting non-negative terms gives lower bound on agent's utility

$$\begin{aligned} (1-\delta)\sum_{\tau=0}^{\infty} \delta^\tau \left(u\left(q_\tau\right) - g\left(a^*\right)\right) &\geq \delta^t p_t (u\left(q^*\right) - g\left(a^*\right)) \\ &\geq u\left(q^*\right) - g\left(a^*\right) - \varepsilon \end{aligned}$$

  when we choose δ and p_t close enough to 1

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1.2 Marginal Cost of Utility is Martingale

- Rogerson, Econometrica, 1985

1.2.1 Principal's Problem

- Two periods $t \in \{1, 2\}$, no discounting
- Let $a = (a_1, a_2(q_1))$ be agent's action plan
- Principal chooses $a, w_1(q_1), w_2(q_1, q_2)$ to maximize

$$
\begin{aligned}
\mathbb{E}\left[(q_1 - w_1(q_1) + q_2 - w_2(q_1, q_2))\,|a\right] &\text{ subject to } && : \\
\mathbb{E}\left[u(w_1(q_1)) - g(a_1) + u(w_2(q_1, q_2)) - g(a_2)\,|a\right] &\geq && \mathbb{E}\left[...|\widetilde{a}\right] \qquad \text{(IC)} \\
\mathbb{E}\left[u(w_1(q_1)) - g(a_1) + u(w_2(q_1, q_2)) - g(a_2)\,|a\right] &\geq && 2\overline{u} \qquad \text{(IR)}
\end{aligned}
$$

- Note: Agent can't save or borrow

1.2.2 Result

Proposition 2 *The optimal long-term contract* $a^*, w_1^*(q_1), w_2^*(q_1, q_2)$ *satisfies*

$$
\frac{1}{u'(w_1^*(q_1))} = \mathbb{E}\left[\frac{1}{u'(w_2^*(q_1, q_2))}|q_1, a^*\right] \qquad (*)
$$

for all q_1.

Idea:

- LHS is marginal cost of providing utility today
- RHS is expected marginal cost of providing utility tomorrow
- Agent is indifferent between receiving utility today or tomorrow
- If LHS<RHS principal could profit by front-loading utility

Proof.

- Let $w_1^*(q_1), w_2^*(q_1, q_2)$ be optimal contract
- Fix q_1
- Shift ε utility to period 1

$$
\begin{aligned}
u(\widehat{w}_1(q_1)) &= u(w_1^*(q_1)) + \varepsilon \\
u(\widehat{w}_2(q_1, q_2)) &= u(w_2^*(q_1, q_2)) - \varepsilon
\end{aligned}
$$

- Does not affect agent's IC or IR constraint
- By first-order Taylor approximation

$$\begin{aligned}\widehat{w}_1(q_1) &= w_1^*(q_1) + \frac{\varepsilon}{u'(w_1^*(q_1))} \\ \widehat{w}_2(q_1,q_2) &= w_2^*(q_1,q_2) - \frac{\varepsilon}{u'(w_2^*(q_1,q_2))}, \forall q_2\end{aligned}$$

- Thus, effect on Revenue

$$\widehat{R} - R^* = -\varepsilon\left(\frac{1}{u'(w_1^*(q_1))} - \mathbb{E}\left[\frac{1}{u'(w_2^*(q_1,q_2))}|q_1,a^*\right]\right)$$

- As ε can be chosen positive or negative, optimality requires that the term in parantheses vanishes

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1.2.3 Discussion

- Properties of optimal long-term contract a^*, w^*
  - Inertia: RHS increasing in $w_1^*(q_1)$, consumtion smoothing
  - Complex: $w_2^* \neq w_2(q_2)$
- Agent would like to save - not borrow
  - Apply Jensen's inequality to (*)
  - $f(x) = 1/x$ is a convex function, thus $1/(\mathbb{E}[x]) \leq \mathbb{E}[1/x]$

$$u'(w_1^*(q_1)) = 1/\mathbb{E}\left[\frac{1}{u'(w_2^*(q_1,q_2))}\right] \leq \mathbb{E}\left[u'(w_2^*(q_1,q_2))\right]$$

  - Intuition:
    - If "=", then principal can get some IC_2 for free by front-loading wages
    - That is, offer $w_1(q_1) + dw, w_2(q_1,q_2) - dw$
    - Keep them hungry

1.3 Short-term Contracts

- Fudenberg, Holmstrom, Milgrom, JET, 1990

1.3.1 Setup

- 2 periods, no discounting
- Time separable technology and preferences
- Agent can save, but principal can monitor this
  - Funny assumption, but necessary for tractability and result
  - Maybe reasonable in developing countries when saving is through landlord
- Outside utility $\overline{u} = u(\overline{q})$

1.3.2 Principal's Problem

- Principal chooses $a_1, w_1 = w_1(q_1), s = s(q_1), a_2 = a_2(q_1), w_2 = w_2(q_1, q_2, s)$ to maximize

$$\begin{aligned} \mathbb{E}\left[(q_1 - w_1 + q_2 - w_2)\,|a\right] \text{ subject to } &: \\ \mathbb{E}\left[u(w_1 - s) - g(a_1) + u(w_2 + s) - g(a_2)\,|a\right] &\geq \mathbb{E}\left[\cdots|\widetilde{a}, \widetilde{s}\right] && \text{(IC)} \\ \mathbb{E}\left[u(w_1 - s) - g(a_1) + u(w_2 + s) - g(a_2)\,|a\right] &\geq 2\overline{u} && \text{(IR)} \end{aligned}$$

- Can ignore savings for the moment
  - Savings IC constraint not-binding because observable
  - Can choose $s(q_1) = 0$ in optimal contract because principal can save for the agent by adjusting w

1.3.3 Renegotiation and Spot Contracts

- After period 1 of contract a^*, w^*, the principal could offer the agent to change contract
- Optimally, he offers contract $\hat{a}_2, \hat{w}_2(q_2)$ to maximize

$$\begin{aligned} \mathbb{E}\left[q_2 - w_2(q_2)\,|a_2\right] \text{ subject to } &: && \text{(Seq-Eff)} \\ \mathbb{E}\left[u(\hat{w}_2(q_2)) - g(\hat{a}_2)\,|\hat{a}_2\right] &\geq \mathbb{E}\left[\cdots|\widetilde{a}_2\right] && \text{(IC')} \\ \mathbb{E}\left[u(\hat{w}_2(q_2)) - g(\hat{a}_2)\,|\hat{a}_2\right] &\geq \mathbb{E}\left[u(w_2^*(q_1, q_2)) - g(a_2^*)\,|a_2^*\right] && \text{(IR')} \end{aligned}$$

  where the last line captures the idea that the agent can insist on the original long-term contract

- Of course, $\hat{a}_2, \hat{w}_2(q_2)$ implicitly depend on q_1 through (IR')

- Call contract sequentially efficient, or renegotiation-proof if there is no such mutually beneficial deviation after any realization of q_1, and thus $\widehat{a}_2 = a_2^*$ and $\widehat{w}_2(q_2) = w_2^*(q_1, q_2)$.

Proposition 3 *The optimal long-term contract is renegotiation-proof.*

- If there was a profitable deviation after q_1, there is a weakly more profitable deviation where IR' is binding
  - not true in general games!
  - true here, because Pareto-frontier downward-sloping
- The original contract could then be improved by substituting the deviation into the original contract
  - Does not change agent's period 1's IC or IR because expected continuation utility after q_1 unchanged (but expected marginal utility after q_1 may have changed)
  - Agent's period 2 IC and IR satisfied

1.3.4 Spot Contracts

- The long-term contract a^*, w^* can be *implemented via spot contracts* if there is a saving strategy $s(q_1)$ for the agent such that the second period spot contract $\overline{a}_2, \overline{w}_2(q_2)$ maximizes

$$\mathbb{E}\left[q_2 - w_2(q_2) \,|a_2\right] \text{ subject to } : \tag{Spot}$$

$$\mathbb{E}\left[u\left(\overline{w}_2(q_2) + s(q_1)\right) - g(\overline{a}_2) \,|\overline{a}_2\right] \geq \mathbb{E}\left[\cdots|\widetilde{a}_2\right] \tag{IC-spot}$$

$$\mathbb{E}\left[u\left(\overline{w}_2(q_2) + s(q_1)\right) - g(\overline{a}_2) \,|\overline{a}_2\right] \geq u\left(\overline{q} + s(q_1)\right) \tag{IR-spot}$$

  yields the same actions $\overline{a}_2 = a_2^*$ and wages $\overline{w}_2(q_2) + s(q_1) = w_2^*(q_1, q_2)$ as the original contract.

Proposition 4 *A renegotiation-proof contract can be implemented by spot contracts.*

- For 2, let $\hat{a}_2, \hat{w}_2(q_2)$ be optimal continuation contract after q_1
- Set $s = s(q_1)$ so that $u(\overline{q} + s) = \mathbb{E}\left[u\left(\hat{w}_2(q_2)\right) - g(\hat{a}_2)|\hat{a}_2\right]$
- Then, $\overline{a}_2 = \widehat{a}_2, \overline{w}_2(q_2) = \widehat{w}_2(q_2) - s$ solves (Spot)

1.3.5 Discussion

- Rationale for Short-Term Contracting
- Separates incentive-provision from consumption smoothing
- Yields recursive structure of optimal long-term contract - Memory of contract can be captured by one state variable: savings
- Generalizes to
  - T periods
  - Preferences where a_1 does not affect trade-off between a_2 and c_2

1.4 Optimal Linear Contracts

- Holmstrom, Milgrom, Econometrica 1987

1.4.1 Setup

- 2 periods, no discounting
- Time separable technology and preferences
- Funny utility function

$$u(w_1, w_2, a_1, a_2) = -\exp\left(-\left(w_1 + w_2 - g(a_1) - g(a_2)\right)\right)$$

  - Consumption at the end (-> no role for savings)
  - Monetary costs of effort
  - CARA - no wealth effects

- Outside wage w per period
- Optimal static contract a^s, w^s

1.4.2 Result

Proposition 5 *1. The optimal long-term contract repeats the optimal static contract:*

$$w_1^*(q_1) = w^s(q_1) \text{ and } w_2^*(q_1, q_2) = w^s(q_2)$$

2. If q is binary, or Brownian, the optimal contract is linear in output: $w^*(q_1, q_2) = \alpha + \beta(q_1 + q_2)$

Idea: CARA makes everything separable

Proof.

- Principal chooses a^*, w^* to maximize

$$\begin{aligned} \mathbb{E}\left[q_1 - w_1(q_1) + q_2 - w_2(q_1, q_2) \mid a\right] \text{ subject to } &: \\ \mathbb{E}\left[-\exp\left(-\left(w_1^*(q_1) + w_2^*(q_1, q_2) - g(a_1^*) - g(a_2^*)\right)\right) \mid a^*\right] &\geq \mathbb{E}\left[-\exp(...) \mid \widetilde{a}\right] \quad \text{(IC)} \\ \mathbb{E}\left[-\exp\left(-\left(w_1^*(q_1) + w_2^*(q_1, q_2) - g(a_1^*) - g(a_2^*)\right)\right) \mid a^*\right] &\geq u(2w) \quad \text{(IR)} \end{aligned}$$

- Can choose $w_2^*(q_1, q_2)$ so that $\mathbb{E}\left[-\exp\left(-\left(w_2^*(q_1, q_2) - g(a_2^*)\right)\right) \mid a_2^*\right] = u(w)$ for all q_1 (make IR binding in period 2)

- Add $\Delta(q_1)$ to all $w_2^*(q_1,q_2)$
- Subtract $\Delta(q_1)$ from $w_1^*(q_1)$
- Does not affect $w_1^*(q_1)+w_2^*(q_1,q_2)$ for any realization (q_1,q_2)
- Principal and agent only care about this sum

- Sequential efficiency implies that in the second period after realization of q_1, principal chooses $\widehat{a}_2, \widehat{w}_2$ to maximize

$$\begin{aligned} \mathbb{E}\left[q_2 - w_2(q_2)\,|a_2\right] \text{ subject to } &: \\ -\exp\left(-\left(w_1^*(q_1) - g\left(a_1^*\right)\right)\right)\mathbb{E}\left[\exp\left(-\left(\widehat{w}_2(q_2) - g\left(\widehat{a}_2\right)\right)\right)|\widehat{a}_2\right] &\geq -\exp(...)\,\mathbb{E}\left[\exp(...)|\widetilde{a}_2\right] \quad \text{(IC 2)} \\ -\exp(...)\,\mathbb{E}\left[\exp\left(-\left(\widehat{w}_2(q_2) - g\left(\widehat{a}_2\right)\right)\right)|\widehat{a}_2\right] &\geq -\exp(...)\,u(w) \quad \text{(IR 2)} \end{aligned}$$

- Period one factors out (this is because there are no wealth effects)

- Optimal second period contract $\widehat{a}_2, \widehat{w}_2$ is the optimal short-term contract, $\widehat{a}_2 = a^s, \widehat{w}_2(q_2) = w^s(q_2)$, independent of q_1

- Taken $\widehat{a}_2, \widehat{w}_2$ as given, the principal chooses $\widehat{a}_1, \widehat{w}_1$ to maximize

$$\begin{aligned} \text{maximize } \mathbb{E}\left[q_1 - w_1(q_1)\,|a_1\right] \text{ subject to } &: \\ -\mathbb{E}\left[\exp\left(-\left(\widehat{w}_1(q_1) - g\left(\widehat{a}_1\right)\right)\right)|\widehat{a}_1\right]\mathbb{E}\left[\exp\left(-\left(\widehat{w}_2(q_2) - g\left(\widehat{a}_2\right)\right)\right)|\widehat{a}_2\right] &\geq -\mathbb{E}\left[...|\widetilde{a}_1\right]\mathbb{E}\left[...|\widetilde{a}_2\right] \quad \text{(IC 1)} \\ -\mathbb{E}\left[\exp\left(-\left(\widehat{w}_1(q_1) - g\left(\widehat{a}_1\right)\right)\right)|\widehat{a}_1\right]\mathbb{E}\left[\exp\left(-\left(\widehat{w}_2(q_2) - g\left(\widehat{a}_2\right)\right)\right)|\widehat{a}_2\right] &\geq u(2w) \quad \text{(IR 1)} \end{aligned}$$

- This is again the static problem, proving (1)

- (2) follows because every function of q binary is linear, and a Brownian motion is approximated by a binary process

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1.4.3 Discussion

- Not very general, but extends to any number of periods
- Stationarity not so suprising:
  - technology independent
  - no consumption-smoothing
  - no wealth-effects

- no benefits from long-term contracting

- Agent benefits from ability to adjust actions according to realized output
  - Consider generalization with $t \in [0; T]$ and $dq_t = adt + \sigma dW_t$, so that $q_T \sim N\left(a, \sigma^2 T\right)$
  - If agent cannot adjust his action, principal can implement first-best via tail-test and appropriate surplus
  - Tail-test does not work if agent can adjust effort
    * Can slack at first...
    * ... and only start working if q_t drifts down to far
  - More generally with any concave, say, reward function $w\left(q_T\right)$, agent will
    * work in steep region, after bad realization
    * shirk in flat region, after good realization
  - Providing stationary incentives to always induce the static optimal a^* is better

[Figure: Optimal contracts $w(q_T)$]

1.5 Continuous Time (Sannikov 2008)

1.5.1 Setup

- Continuous time $t \in [0; \infty)$, discount rate r
- Think about time as tiny discrete increments dt; remember $rdt \approx 1 - e^{-rdt}$
- Time separable technology

$$dX_t = a_t dt + dZ_t$$

- Brownian Motion Z_t (also called Wiener process) characterized by
  - Sample paths Z_t continuous almost surely
  - Increments independent and stationary with distribution $Z_{t+\Delta} - Z_t \sim \mathcal{N}(0, \Delta)$

[Figure: Z_t]

- Wealth of agent

$$W = r \int_{t=0}^{\infty} e^{-rt} \left(u\left(c_t\right) - g\left(a_t\right)\right) dt$$

(the "r" annuitizes the value of the agent and renders it comparable to u and g)

- Cost function g with $g(0) = 0$, $g' > 0$, $g'' > 0$
- Consumption utility with $u(0) = 0, u' > 0, u'' < 0, \lim_{x\to\infty} u'(x) = 0$
- Consumption = wage; no hidden savings

- Revenue of firm

$$\begin{aligned}\Pi &= r\mathbb{E}\left[\int e^{-rt} dX_t\right] - r\int e^{-rt} c_t dt \\ &= r\int e^{-rt}(a_t - c_t)\, dt\end{aligned}$$

1.5.2 Agent's problem

- Contract: a_t, c_t as function of $X_{s\leq t}$
- Recursive formulation: a_t, c_t, W_t as function of $X_{s\leq t}$
- 'Promise-keeping' constraint

$$\begin{aligned}W_t &= rdt\,(u(c_t) - g(a_t)) + (1 - rdt)\,\mathbb{E}[W_{t+dt}|a_t] \qquad \text{(PK)} \\ rW_t dt &= rdt\,(u(c_t) - g(a_t)) + \mathbb{E}[dW_t|a_t]\end{aligned}$$

- Assume that value increments dW_t are linear in output increments dX_t with wealth dependent sensitivity $rb(W_t)$

$$\begin{aligned}dW_t &= rb(W_t)\, dX_t = rb(W_t)(a_t dt + dZ_t) \\ \mathbb{E}[dW_t|a_t] &= rb(W_t)\,\mathbb{E}[dX_t] = rb(W_t)\,\mathbb{E}[a_t dt + dZ_t] = rb(W_t)\, a_t dt\end{aligned}$$

- First-order condition

$$g'(a(W)) = b(W) \qquad \text{(IC)}$$

- Evolution of wealth governed by

$$\begin{aligned}dW_t &= \mathbb{E}[dW_t] + (dW_t - \mathbb{E}[dW_t]) \\ &= r(W_t - (u(c_t) - g(a_t)))\, dt + rb(W_t)(dX_t - a_t dt) \\ &= r(W_t - (u(c_t) - g(a_t)))\, dt + rb(W_t)\, dZ_t\end{aligned}$$

  - Drifting
    * up when wealth and interest rW_t are high
    * down when consumption c_t is high

    - up when effort a_t is high
  - Wiggling
    - up, when production exceeds expectations $dX_t > a_t dt$
    - down, when production falls short of expectations $dX_t < a_t dt$

1.5.3 Firm's problem

- Choose a_t, c_t as function of $X_{s \leq t}$ to maximize Π subject to (IC) and (IR)
- Example: Retiring agent with wealth W_t
  - Instruct agent not to take any effort $a_t = 0$
  - Pay out wealth as annuity $u(c_t) = W_t$
  - Firm profit from this contract $\Pi_0(u(c)) = -c$

[Figure: Π_0 and Π]

- Evolution of profit

$$\begin{aligned} \Pi_t &= r(a_t - c_t)\,dt + (1 - r dt)\,\mathbb{E}[\Pi_{t+dt}] \\ r\Pi_t dt &= r(a_t - c_t)\,dt + \mathbb{E}[d\Pi_t] \end{aligned}$$

- Principal's expected profit Π_t is function of state variable, i.e. of agent's wealth $\Pi(W_t)$
- The expected value of the increment $\mathbb{E}[d\Pi_t]$ can be calculated with Ito's Lemma

Lemma 6 (Ito) *Consider the stochastic process W_t governed by*

$$dW_t = \gamma(W_t)\,dt + \sigma dZ_t$$

and a process $\Pi_t = \Pi(W_t)$ that is a function of this original process. Then the expected increment of Π_t is given by

$$\mathbb{E}[d\Pi(W)] = \left[\gamma(W)\,\Pi'(W) + \frac{1}{2}\sigma^2\Pi''(W)\right] dt$$

Proof. By Taylor expansion

$$\begin{aligned} \Pi(W_{t+dt}) - \Pi(W_t) &= \Pi(W_t + \gamma(W_t)\,dt + \sigma dZ_t) - \Pi(W_t) \\ &= (\gamma(W_t)\,dt + \sigma dZ_t)\,\Pi'(W_t) + \frac{1}{2}(\gamma(W_t)\,dt + \sigma dZ_t)^2\,\Pi''(W_t) + o(dt) \\ &= (\gamma(W_t)\,dt + \sigma dZ_t)\,\Pi'(W_t) + \frac{1}{2}\sigma^2 dZ_t^2\Pi''(W_t) + o(dt) \\ \mathbb{E}[d\Pi_t] &= \gamma(W_t)\,dt\Pi'(W_t) + \frac{1}{2}\sigma^2 dt\Pi''(W_t) + o(dt) \end{aligned}$$

- The reason, the Ito term $\frac{1}{2}\sigma^2\Pi''(W_t)$ comes in is that W_t is oscillating so strongly, with stdv. $\sqrt{dt}$ in every dt.

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- In the case at hand we have

$$\begin{aligned}\gamma(W_t) &= r(W_t - (u(c_t) - g(a_t))) \\ \sigma &= rg'(a_t)\end{aligned}$$

and we get

$$r\Pi(W) = \max_{a,c} r(a-c) + r(W - (u(c) - g(a)))\Pi'(W) + \frac{1}{2}r^2 g'(a)^2 \Pi''(W) \qquad (*)$$

- So the principal chooses plans $a = a(W) > 0$ and $c = c(W)$ to maximize the RHS of (*)
- The agent has to retire at some point W_r
  - Marginal utility of consumption $u'(c) \to 0$
  - Marginal cost of effort $g'(a) \geq \varepsilon > 0$
- Boundary conditions
  - $\Pi(0) = 0$: If the agent's wealth is 0, he can achieve this by setting future effort $a_t = 0$, yielding 0 to the firm
  - $\Pi(W_r) = \Pi_0(W_r) = -u^{-1}(W_r)$: At some retirement wealth W_r, the agent retires
  - $\Pi'(W_r) = \Pi_0'(W_r)$: Smooth pasting: The profit function is smooth and equals Π_0 above W_r

Theorem 7 *There is a unique concave function $\Pi(W) \geq \Pi_0(W)$, maximizing (*) under the above boundary conditions. The action and consumption profiles $a(W), c(W)$ constitute an optimal contract.*

1.5.4 Properties of Solution

Properties of Π

- $\Pi(0) = 0$
- $\Pi'(0) > 0$: terminating the contract at $W = 0$ is inefficient, and $W > 0$ serves as insurance against termination

- $\Pi(W) < 0$ for large W, e.g. $W = W_r$, because agent has been promised a lot of continuation utility

Properties of $a^*(W)$

- Optimal effort $a^*(W)$ maximizes

$$ra + rg(a)\,\Pi'(W) + \frac{1}{2}r^2 g'(a)^2\,\Pi''(W)$$

  - Increased output ra
  - Compensating agent for effort through continuation wealth $rg(a)\,\Pi'(W)$ - positive for $W \approx 0$!
  - Compensating agent for income risk $\frac{1}{2}r^2 g'(a)^2\,\Pi''(W)$

- Monotonicity of $a^*(W)$ unclear

[Figure: $a^*(W)$]

-
  - $\Pi'(W)$ decreasing
  - $\Pi''(W)$ could be increasing or decreasing

- As $r \to 0$, $a^*(W)$ decreasing

Properties of $c^*(W)$

- Optimal consumption $c^*(W)$ maximizes

$$-rc - ru(c)\,\Pi'(W)$$

  and thus

$$\frac{1}{u'(c^*)} = -\Pi'(W) \text{ or } c^* = 0$$

  - $\frac{1}{u'(c^*)}$: cost of current consumption utility
  - $-\Pi'(W)$: cost of continuation utility - negative for small W

- Thus agent does not consume as long as W small

1.5.5 Extensions: Career Paths

- Performance-based compensation $c\left(W_t\right)$ serves as short-term incentive
- Now incorporate long-term incentives into model
- In baseline model principal's outside option was retirement $\Pi_0\left(u\left(c\right)\right) = -c$
- Can model, quitting, replacement or promotion by different outside options $\widetilde{\Pi}_0$
- This only changes the boundary conditions but not the differential equation determining Π
  - If agent can quit at any time with outside utility $\widetilde{W}$, then $\widetilde{\Pi}_0\left(u\left(c\right)\right) = \begin{cases} -c & \text{if } u\left(c\right) > \widetilde{W} \\ 0 & \text{if } u\left(c\right) = \widetilde{W} \end{cases}$
  - If agent can be replaced at profit D to firm, then $\widetilde{\Pi}_0\left(u\left(c\right)\right) = D - c$
  - If agent can be promoted at cost K, resulting into new value function Π_p, then $\widetilde{\Pi}_0\left(W\right) = \max\left\{\Pi_0\left(W\right); \Pi_p\left(W\right) - K\right\}$

[Figure: $\Pi(W)$ for three extensions]

- Ranking of $\Pi(W)$

$$\text{quitting} < \text{benchmark} < \text{replacement, promotion}$$