Lecture Notes - Mechanism Design

Simon Board and Moritz Meyer-ter-Vehn

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1 Introduction Adverse Selection

- Agents have private information, their type θ_i
- Examples
  - Selling stuff: consumer knows his preference; seller knows quality of product
  - Regulating natural monopolies: firms know their production cost
  - Taxing and redistributing income: worker knows productivity or disutility from labor
  - Credit markets: entrepreneur knows risk of project
  - Insurance: Insuree knows idiosyncratic risk
- Asymmetric information can cause inefficiencies
  - Akerlof: market collapse
  - Monopoly pricing: Deadweight loss
- Mechanism design approach
  - Principal (usually uninformed) proposes mechanism = gameform & outcome function
  - Agents accept/reject mechanism
  - Agents play the game and outcomes are determined
- Alternative approach: Signalling
  - Informed party proposes contract
  - In equilibrium contract proposal signals type
- Plan of attack

- Single-agent
- Multi-agent
- Dynamics

2 Single Agent - Non-Linear Pricing

- Quasi-linear model
- Consumer (agent)
  - Type: taste θ
  - Utility $u = \theta q - t$: taste*quality - price
- Firm's profit: $t - c(q)$ (c increasing, convex)
- Revelation mechanism
  - Firm asks agent for type θ
  - Consumer reports θ (IC constraint)
  - Mechanism specifies $q(\theta), t(\theta)$
- Taxation mechanism
  - Firm offers menu of contracts $T(q)$
  - Consumer with type θ picks most favorable contract

Lemma 1 (Revelation principle & Taxation principle) *These two approaches are equivalent*

Proof. Taxation principle:

- Revelation mechanism $q(\cdot), t(\cdot)$ incentive compatible
- Let

$$T(q) = \begin{cases} t(\theta) & \text{if } q = q(\theta) \\ \infty & \text{if not} \end{cases}$$

- If $q(\cdot), t(\cdot)$ incentive compatible, then type θ buys $q(\theta)$

Revelation principle:

- Ask agent for θ and play taxation mechanism for him

- I.e.: $q(\theta) = \max_q \{\theta q - T(q)\}$ and $t(\theta) = T(q(\theta))$

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- Can incorporate (IR) constraint by setting $T(0) = 0$ (or $-\overline{u}$)
- Utility of type θ who reports $\widehat{\theta}$

$$\begin{aligned} u\left(\theta;\widehat{\theta}\right) &= \theta q\left(\widehat{\theta}\right) - t\left(\widehat{\theta}\right) \\ u(\theta) &= u(\theta;\theta) \end{aligned}$$

- Principal's problem: Choose $q(\cdot), t(\cdot)$ to maximize

$$\mathbb{E}_\theta\left[t(\theta) - c(q(\theta))\right]$$

such that

$$u(\theta;\theta) \geq 0 \quad \text{(IR)}$$

$$u(\theta;\theta) \geq u\left(\theta;\widehat{\theta}\right) \quad \text{(IC)}$$

- First best
  - Substitute (IR) into objective
  $$\max_{q(\theta)} \mathbb{E}_\theta\left[q(\theta)\,\theta - c(q(\theta))\right]$$
  - Consumers get 0-utility
  - Pointwise Maximization: $\theta = c'(q(\theta))$
  - Violates (IC):

$$\begin{aligned} u(\theta_H;\theta_L) &= \theta_H q_L^* - t_L^* \\ &= (\theta_H - \theta_L)\, q_L^* + \theta_L q_L^* - t_L^* \\ &> 0 = u(\theta_H;\theta_H) \end{aligned}$$

  - Draw picture

2.1 Two types $\theta_H > \theta_L$

- $1 - \pi = \Pr(\theta_H)$

- Principal maximizes

$$\max_{q_H,t_H,q_L,t_L} \{(1-\pi)(t_H - c(q_H)) + \pi(t_L - c(q_L))\}$$

such that

$$\begin{aligned}
\theta_H q_H - t_H &\geq 0 && \text{(IR H)}\\
\theta_L q_L - t_L &\geq 0 && \text{(IR L)}\\
\theta_H q_H - t_H &\geq \theta_H q_L - t_L && \text{(IC H)}\\
\theta_L q_L - t_L &\geq \theta_L q_H - t_H && \text{(IC L)}
\end{aligned}$$

- Will now see that (IR L) and (IC H) are binding and that the other constraint can be replaced with a monotonicity constraint

  - Assume $q_L > 0$ (Otherwise just serve θ_H efficiently and extract all rents)
  - (IR H) is slack:

    $$\begin{aligned}
    \theta_H q_H - t_H &\geq \theta_H q_L - t_L \text{ (by IC H)}\\
    &> \theta_L q_L - t_L\\
    &\geq 0 \text{ (by IR L)}
    \end{aligned}$$

    because of (IR L), (IC H), and $\theta_H q_L - t_L > \theta_L q_L - t_L$
  - (IR L) binds: If not can increase t_L, t_H by ε
  - Monotonicity: Adding (IC H) and (IC L) yields

    $$\begin{aligned}
    \theta_H q_H - \theta_L q_H &\geq \theta_H q_L - \theta_L q_L\\
    (\theta_H - \theta_L) q_H &\geq (\theta_H - \theta_L) q_L\\
    q_H &\geq q_L && \text{(Mon)}
    \end{aligned}$$

  - (IC H) binds: If not, increase t_H by ε because (IR H) is slack
  - (IC L) is redundant:

    $$\begin{aligned}
    t_H - t_L &= \theta_H (q_H - q_L) \text{ (by IC H)}\\
    &\geq \theta_L (q_H - q_L) \text{ (by Mon)}
    \end{aligned}$$

    because (IC H) binds and $q_H > q_L$

- Program becomes

$$\max_{q_H, t_H, q_L, t_L} \left\{ (1-\pi)(t_H - c(q_H)) + \pi (t_L - c(q_L)) \right\}$$

such that

$$\begin{aligned} u(\theta_L) &= \theta_L q_L - t_L = 0 && \text{(IR L)} \\ u(\theta_H) &= \theta_H q_H - t_H = \theta_H q_L - t_L = (\theta_H - \theta_L) q_L && \text{(IC H)} \\ q_H &\geq q_L && \text{(Mon)} \end{aligned}$$

- Relax program by ignoring (Mon)

$$\begin{aligned} & (1-\pi)(\theta_H q_H - (\theta_H - \theta_L) q_L - c(q_H)) + \pi (\theta_L q_L - c(q_L)) \\ = \; & (1-\pi) \left(\underbrace{\theta_H q_H - c(q_H)}_{\text{Welfare}} \right) + \pi \left(\underbrace{\theta_L q_L - c(q_L)}_{\text{Welfare}} - \underbrace{\frac{1-\pi}{\pi}}_{\text{likelihood ratio}} \underbrace{(\theta_H - \theta_L) q_L}_{\text{Rents}} \right) \end{aligned}$$

- Solution

$$\begin{aligned} c'(q_H) &= \theta_H \\ c'(q_L) &= \theta_L - \frac{1-\pi}{\pi} (\theta_H - \theta_L) \end{aligned}$$

- Check Monotonicity

$$\begin{aligned} c'(q_H) &= \theta_H \\ &> \theta_L - \frac{1-\pi}{\pi} (\theta_H - \theta_L) = c'(q_L) \\ q_H &> q_L \end{aligned}$$

- Properties
  - Low type is inefficiently underserved
    - Lowers rents of high type (picture)
    - Squeeze passengers in economy class
  - Lowest type gets no rents (that would be a waste)
  - Efficiency at the top
    - Nobody can afford to mimic high type
    - Serve him optimally (and tax him)

  - High type indifferent between contracts
    - ensured by making economy class uncomfortable
  - Quality increases in type (picture)

2.2 Continuum of types $\left[\underline{\theta};\overline{\theta}\right]$

- $\theta \in \left[\underline{\theta};\overline{\theta}\right]$ with pdf f
- Principal's problem:

$$\max_{q(\theta),t(\theta)} \mathbb{E}\left[t(\theta) - c(q(\theta))\right]$$

  such that

$$u(\theta;\theta) \geq 0 \quad \text{(IR)}$$
$$u(\theta;\theta) \geq u\left(\theta;\widehat{\theta}\right) \quad \text{(IC)}$$

Theorem 2 *$q(\cdot), t(\cdot)$ is incentive compatible iff*

$$u(\theta) = u(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(s)\,ds \quad \textit{(Payoff Equivalence)}$$
$$q(\theta) \text{ increasing} \quad \textit{(Monotonicity)}$$

Proof.

- Only if
  - Take $\theta' > \theta$. Then

$$\theta'\left(q\left(\theta'\right) - q(\theta)\right) \geq t\left(\theta'\right) - t(\theta) \geq \theta\left(q\left(\theta'\right) - q(\theta)\right)$$

    and hence $q\left(\theta'\right) - q(\theta) \geq 0$
  - Payoff Equivalence follows by Envelop Theorem

$$\frac{du}{d\theta}(\theta) = \frac{\partial u}{\partial \theta}(\theta;\theta) + \frac{\partial u}{\partial \widehat{\theta}}(\theta;\theta) = q(\theta)$$

    (real proof in Milgrom, Segal Ectra. 2002)

- If

$$\begin{aligned} u\left(\theta'\right) &= u(\theta) + \int_{\theta}^{\theta'} q(s)\,ds \text{ (by Payoff Equivalence)} \\ &\geq u(\theta) + \int_{\theta}^{\theta'} q(\theta)\,ds \text{ (by Monotonicity)} \\ &= u(\theta) + \left(\theta' - \theta\right) q(\theta) \\ &= u\left(\theta';\theta\right) \end{aligned}$$

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- Principal's problem:

$$\max_{q(\theta),t(\theta)} \mathbb{E}\left[t(\theta) - c(q(\theta))\right]$$

such that

$$\begin{aligned} u(\theta) &= u(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(s)\,ds && \text{(IC FOC)} \\ u(\underline{\theta}) &= 0 && \text{(IR low)} \\ & q(\theta) \text{ increasing} && \text{(Monotonicity)} \end{aligned}$$

- Relax Montonicity

$$\begin{aligned} \mathbb{E}\left[t(\theta) - c(q(\theta))\right] &= \mathbb{E}\left[\theta q(\theta) - u(\theta) - c(q(\theta))\right] \\ &= \mathbb{E}\left[\theta q(\theta) - \int_{\underline{\theta}}^{\theta} q(s)\,ds - c(q(\theta))\right] \\ &= \int_{\underline{\theta}}^{\overline{\theta}} \left(\theta q(\theta) - \int_{\underline{\theta}}^{\theta} q(s)\,ds - c(q(\theta))\right) f(\theta)\,d\theta \\ &= -\int_{\underline{\theta}}^{\overline{\theta}} \int_{s}^{\overline{\theta}} q(s) f(\theta)\,d\theta ds + \int_{\underline{\theta}}^{\overline{\theta}} \left(\theta q(\theta) - c(q(\theta))\right) f(\theta)\,d\theta \\ &= -\int_{\underline{\theta}}^{\overline{\theta}} q(s)\left(1 - F(s)\right) ds + \int_{\underline{\theta}}^{\overline{\theta}} \left(\theta q(\theta) - c(q(\theta))\right) f(\theta)\,d\theta \\ &= \int_{\underline{\theta}}^{\overline{\theta}} \left(\theta q(\theta) - \frac{1 - F(\theta)}{f(\theta)} q(\theta) - c(q(\theta))\right) f(\theta)\,d\theta \\ &= \int_{\underline{\theta}}^{\overline{\theta}} \left(\left(\theta - \frac{1 - F(\theta)}{f(\theta)}\right) q(\theta) - c(q(\theta))\right) f(\theta)\,d\theta \end{aligned}$$

- Marginal Revenue: $MR(\theta) = \theta - \frac{1-F(\theta)}{f(\theta)}$

- Selling $q(\theta)$ generates
  - Surplus $\theta q(\theta)$
  - Costs $c(q(\theta))$
  - Consumer rents $\frac{1-F(\theta)}{f(\theta)} q(\theta)$
- First order condition

$$MR(\theta) = c'(q(\theta))$$

- If $MR(\theta)$ increasing (e.g. uniform, exponential) then q increasing

2.2.1 Quadratic Example

- Set-up: $\theta \sim U[0;1]; c(q) = q^2/2$
- Marginal Revenue: $MR(\theta) = 2\theta - 1$
- Marginal Cost: $c'(q) = q$
- Optimal contract $q(\theta) = (2\theta - 1)_+$

2.2.2 Linear Cost

- $c(q) = cq$ with $q \in [0;1]$
- No haggling theorem:

$$\begin{aligned} q(\theta) &= 1 \text{ if } MR(\theta) > c \\ q(\theta) &= 0 \text{ if } MR(\theta) < c \end{aligned}$$

- Same as classic monopoly problem (draw picture)
  - Profit: $p(x)x - cx$
  - Marginal revenue: $m(x) = p(x) + p'(x)x$
  - But $x = 1 - F(\theta)$ and $dx/d\theta = -f(\theta)$
  - Hence $m(x) = \theta - \frac{1-F(\theta)}{f(\theta)} = MR(\theta)$

2.2.3 Further Remarks

- Generalizes to N agents with independent information: Optimal Auctions
- Quantity
  - With one customer, q could be quantity
  - With many consumers,
    * non-linear price schemes subject to resale constraints
    * non-linear costs links problem across consumers (Segal AER 03)
- Non-increasing marginal revenue $MR(\theta)$
  - Need to iron it
  - Replace $\int^{\theta} MR(s)\, ds$ with smallest convex envelop (draw picture)

3 Multi-Agent

- Examples
  - Auctions
  - Bilateral Trade
  - Production and Distribution in Society

3.1 Setup

- N agents i
- Private information θ^i; $\theta = \left(\theta^i\right)_{i \in 1 \ldots N}$
- Outcomes $y \in Y$; often allocation plus transfers $y = \left(k, \left(t^i\right)_{i \in 1 \ldots N}\right)$
- Utility $u^i = u^i(y; \theta)$
  - $u^i = u^i\left(y, \theta^i\right)$: private valuations
  - $u^i = v_k^i(\theta) - t^i$: quasi-linear utility
- Mechanism designer's objective: "Implement" a choice rule $\psi : \Theta \to Y$ to maximize, e.g.
  - Efficiency: $\psi(\theta)$ not Pareto-dominated given θ

- Quasi-linear $\psi = (q, t)$
  - Efficiency: $q(\theta)$ maximizes $\sum_i v_k^i(\theta)$
  - Revenue: Maximize $\mathbb{E}\left[\sum_i t^i(\theta)\right]$

- Mechanism m
  - Game form with strategy sets S_i
  - Outcome function mapping terminal nodes of game form to Y

- Mechanism m (partially) "x"-implements of choice rule ψ if there exists a strategy profile $s_* = \left(s_*^1, ..., s_*^n\right)$ such that
  - Terminal node of $s_*^1\left(\theta^1\right), ..., s_*^n\left(\theta^n\right)$ is mapped to $\psi(\theta)$
  - s_* is an "x"-equilibrium

- Full implementation: If this is true for every equilibrium s_*

3.2 Revelation Principle

- Set of all mechanisms has little structure
- Particular class of mechanisms: Revelation mechanisms: $S^i = \Theta^i$, i.e. strategy is to state a type $\widehat{\theta}^i$

Definition 3 *Choice rule $\psi : \Theta \to Y$ is incentive compatible wrt. equilibrium concept "x", if stating the truth $\widehat{\theta}^i = \theta^i$ is an "x"-equilibrium.*

Theorem 4 (Revelation Priniciple) *A choice rule ψ is (partially) implementable by any mechanism, if and only if it is incentive compatible.*

Proof. If:

- ψ incentive compatible -> ψ is implemented by the revelation mechanism

Only if:

- Let ψ be implemented by mechanism m with strategies $s_*^i\left(\theta^i\right)$
- Strategy $s_*^i\left(\theta^i\right)$ is better than any other strategy $s^{i\prime}$, in particular better than the equilibrium strategy of any other type $s_*^i\left(\widehat{\theta}^i\right)$
- Revelation mechanism: Agents report $\widehat{\theta}$, mechanism implements $\psi\left(\widehat{\theta}\right)$

- Need to verify: i optimally reports $\widehat{\theta}^i = \theta^i$
  - If agent reports $\widehat{\theta}^i \neq \theta^i$, achieves same result as when playing $s_*^i\left(\widehat{\theta}^i\right)$ in mechanism m
  - If agent reports $\widehat{\theta}^i = \theta^i$, achieves same result as when playing $s_*^i\left(\theta^i\right)$ in mechanism m
  - This is by assumption better

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- Very robust insight
  - Holds for all standard implementation concepts
  - If agents control actions a^i on top of common decision ψ (Myerson 1982, Ectra.) can replace any mechanism with centralized mechanism where
    * Agents report types $\widehat{\theta}^i$
    * Mechanism designer recommends actions $\widehat{a}^i$
    * In equilibrium agents are truthful $\widehat{\theta}^i = \theta^i$ and obedient $a^i = \widehat{a}^i$
  - If agents can act sequentially and acquire further information (Myerson 1986 Ectra) can replace any mechanism with centralized mechanism where
    * Agents report everything they have learned so far
    * Mechanism designer recommends actions $\widehat{a}^i$ (but nothing more)
    * In equilibrium agents are truthful and obedient
- Limitations
  - Full implementation: Revelation mechanism may have additional equilibria
  - Multiple Principals: Mechanism of one principal may reference mechanism of other principal...
  - Informed Principal: Proposing a mechanism serves as signal about own information
- Revelation mechanism not example of real-life mechanisms but limit on what is achievable; not robust to
  - Communication costs
  - Bounded rationality

3.3 Equilibrium concepts

- Dominant-strategy (strategy-proof) implementation: For all $i, \theta^i, \widehat{\theta}^i, \theta^{-i}, \widehat{\theta}^{-i}$

$$u^i\left(\psi\left(\theta^i, \widehat{\theta}^{-i}\right); \theta\right) \geq u^i\left(\psi\left(\widehat{\theta}^i, \widehat{\theta}^{-i}\right); \theta\right)$$

"Strategy is optimal no matter what"

- Ex-post implementation: For all $i, \theta^i, \widehat{\theta}^i, \theta^{-i}$

$$u^i\left(\psi\left(\theta^i, \theta^{-i}\right); \theta\right) \geq u^i\left(\psi\left(\widehat{\theta}^i, \theta^{-i}\right); \theta\right)$$

"Strategy is optimal against any opponent types who play equilibrium"

- Bayesian Nash implementation:
  - There is a common prior π over θ
  - Agents beliefs $\pi^i\left(\cdot|\theta^i\right)$ over Θ^{-i} are given by Bayesian updating
  - For all $i, \theta^i, \widehat{\theta}^i$

$$\mathbb{E}_{\pi^i\left(\cdot|\theta^i\right)}\left[u^i\left(\psi\left(\theta^i, \theta^{-i}\right); \theta\right)\right] \geq \mathbb{E}_{\pi^i\left(\cdot|\theta^i\right)}\left[u^i\left(\psi\left(\widehat{\theta}^i, \theta^{-i}\right); \theta\right)\right]$$

  "Strategy is optimal in expectation, given beliefs $\pi^i\left(\cdot|\theta^i\right)$"

- Interim (robust) implementation (Bergemann, Morris 2005, Ectra)
  - Agents' types $\tau^i = \left(\theta^i, \pi^i\right)$ have two components
    - Payoff relevant information θ^i
    - Beliefs π^i over τ^{-i}
  - For all $i, \tau^i = \left(\theta^i, \pi^i\right), s^{i\prime}$

$$\mathbb{E}_{\pi^i}\left[u^i\left(\psi\left(\theta^i, \theta^{-i}\right); \theta\right)\right] \geq \mathbb{E}_{\pi^i}\left[u^i\left(\psi\left(\widehat{\theta}^i, \theta^{-i}\right); \theta\right)\right]$$

  "Strategy is optimal in expectation, given beliefs π^i"
  - If all beliefs are possible, i.e. $\Pi^i = \Delta\left(\Theta^{-i}\right)$, interim implementation coincides with ex-post implementation

- Complete information implementation
  - Every agent knows θ, i.e. $\theta^i = \theta$

- For all $i, \theta, \widehat{\theta}$

$$u^i\left(\psi\left(\theta;\theta,...,\theta\right);\theta\right) \geq u^i\left(\psi\left(\widehat{\theta};\theta,...,\theta\right);\theta\right)$$

  "Truth telling is Nash equilibrium"

3.4 The Dictator Theorem

- Gibbard 1973 Ectra, Satterthwaite 1975 JET
- $N \geq 2$ agents
- $Y \geq 3$ outcomes
- Types $\theta^i = u^i$ strict preference order over Y
- Choice rule ψ
  - Exhaustive if $\psi\left(\Theta\right) = Y$
  - Dictatorial with dictator i if for all types θ the rule picks i's favorite outcome, i.e. $\psi\left(\theta\right) = a$ such that $u^i\left(a\right) > u^i\left(b\right)$ for all $b \in Y$

Theorem 5 *Every exhaustive, strategy-proof choice rule is dictatorial*

- Not true for $Y = 2$: Voting with any majority rule is strategy proof.
- Bayesian implementation unclear: Prior over ordinal preferences messy

Lemma 6 (Monotonicity) *Every strategy-proof choice rule is monotone: If $\psi\left(u\right) = a$ and a is at least as preferable under v than it is under u, i.e. for all i and b*

$$u^i\left(a\right) \geq u^i\left(b\right) \Rightarrow v^i\left(a\right) \geq v^i\left(b\right)$$

then $\psi\left(v\right) = a$.

Proof.

- Suppose first that $v = \left(v^1, u^{-1}\right)$ and $\psi\left(v\right) = a'$.
- By strategy-proofness of agent 1

$$\begin{aligned} u^1\left(a\right) &\geq u^1\left(a'\right) \\ v^1\left(a\right) &\geq v^1\left(a'\right) \\ v^1\left(a'\right) &\geq v^1\left(a\right) \end{aligned}$$

  and therefore $a = a'$

- This argument can be repeated for every agent

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Lemma 7 (Pareto) *If $v^i(a) > v^i(b)$ for all i, then $\psi(v) \neq b$.*

Proof.

- Suppose to the contrary that $\psi(v) = b$
- Let w be such that

$$w^i(a) > w^i(b) > w^i(c) \text{ and } w^i(c) = v^i(c) \text{ for all } c \neq a, b$$

- As $\psi(v) = b$ and b is as least as preferable under w than it is under v we have $\psi(w) = b$
- As ψ is exhaustive, there exists u with $\psi(u) = a$
- As a is at least as preferable under w as under u we have $\psi(w) = a$

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Lemma 8 (2 agents) *The Theorem holds for $N = 2$.*

Proof.

- Consider the preference profile

$$u = \left(u^1, u^2\right) = \begin{pmatrix} a & b \\ b & a \\ c & c \end{pmatrix}$$

- By Lemma Pareto we have $\psi(u) \neq c$
- Assume $\psi(u) = a$
- Then for

$$v = \begin{pmatrix} a & b \\ b & c \\ c & a \end{pmatrix}$$

  we have $\psi(v) \neq c$ by Lemma Pareto, and $\psi(v) \neq b$ by strategy proofness of agent 2 with preference u^2

- Thus, $\psi(v) = a$
- By Lemma Monotonicity, we have $\psi(w) = a$ whenever agent 1 ranks a highest

- Thus, agent 1 is a dictator for a
- Let $Y^1 \subseteq Y$ be the set of outcomes for which 1 is the dictator, and similarly Y^2
- Let Y^0 be the set of outcomes for which no agent is a dictator
- $\#Y^0 \leq 1$ by the above
- Thus as $\#Y = 3$, if $Y^1 \neq \emptyset$ then $Y^2 = \emptyset$
- If $a \in Y^1$ and $b \in Y^0$, consider again u
- We know that either $b \in Y^2$ or $a \in Y^1$, but $Y^2 = \emptyset$

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Proof of Theorem.

- By induction, assume the Theorem holds for $p < N$ agents
- Let φ be voting rule for two agents defined by

$$\varphi\left(u^1, v\right) = \psi\left(u^1, v, \cdots, v\right)$$

  - φ is exhaustive because ψ is, and by Lemma Pareto
  - φ is strategy proof
    - Consider deviation to w by agent 2
    - Let $a_k := \psi\left(u^1, \underbrace{w, \cdots, w}_{k \text{ times}}, v, \cdots, v\right)$
    - $v\left(a_{k-1}\right) \geq v\left(a_k\right)$, because ψ is strategy-proof
    - Thus, $v\left(\varphi\left(u^1, v\right)\right) = v\left(a_0\right) \geq v\left(a_{N-1}\right) = v\left(\varphi\left(u^1, w\right)\right)$
  - Thus φ is dictatorial
  - If agent 1 is the dictator, we are done

- If agent 2 is the dictator for φ, fix u^1_* and consider

$$\xi\left(u^2, \cdots, u^N\right) = \psi\left(u^1_*, u^2, \cdots, u^N\right)$$

  - ξ is strategy-proof
  - ξ is exhaustive (because 2 is dictator for φ)
  - By induction, ξ is dictatorial with dictator 2, say

- Consider, finally,

$$\varrho\left(u^1,u^2\right)=\psi\left(u^1,u^2,u^3_*,\cdots,u^N_*\right)$$

for arbitrarily fixed $u^3_*,\cdots,u^N_*$

  - ρ is exhaustive (because 2 is dictator for ξ)
  - ρ is strategy proof (because ψ is)
  - Thus ρ is dictatorial
  - But 1 cannot be the dictator because type u^1_* does not get his most prefered outcome as 2 is the dictator of ξ
  - Thus, 2 is the dictator of ρ for any $u^3_*,\cdots,u^N_*$

- Thus, 2 is the dictator ofo ψ

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3.4.1 Restricted preference domains

- Single-peaked preferences:
  - Exists uniform order on Y and for all u^i there is y such that u^i increasing below y and decreasing above y
  - Median voter rule, e.g., is strategy proof

- Quasi-linear, private value utility:
  - Surplus-maximizing choice rule $q(\theta)=\arg\max_k\left\{\sum_i v^i_k\left(\theta^i\right)\right\}$
  - can be made strategy-proof by VCG transfers $t^i(\theta)=\sum_{j\neq i}v^j_{q\left(\theta^{-i}\right)}\left(\theta^j\right)$, where $q\left(\theta^{-i}\right)$ maximizes $\sum_{j\neq i}v^j_k\left(\theta^j\right)$

4 Quasi-linear, private values

- $u^i = v^i\left(k;\theta^i\right) - t^i$
- Direct mechanism $(q,t):\Theta \to K \times \mathbb{R}^N$

4.1 VCG: Efficiency

- Mechanism ex-post efficient: $q^*(\theta)$ maximizes $\sum_i v^i\left(k;\theta^i\right)$ for all θ

Theorem 9 (Vickrey, Clarke, Groves) *For quasi-linear utilities, the efficient allocation rule q is dominant-strategy implementable.*

Proof.

- Let $t^i\left(\widehat{\theta}\right) = -\sum_{j\neq i} v^j\left(q^*\left(\widehat{\theta}\right);\widehat{\theta}^j\right)$: Value of other agents
- Thus, for any $\theta^{-i}, \widehat{\theta}^{-i}$, the utility from reporting $\widehat{\theta}^i$ is total social surplus

$$\begin{aligned} u^i\left(\theta^i;\widehat{\theta}^i\right) &= v^i\left(q^*\left(\widehat{\theta}\right);\theta^i\right) - t^i\left(\widehat{\theta}\right) \\ &= v^i\left(q^*\left(\widehat{\theta}\right);\theta^i\right) + \sum_{j\neq i} v^j\left(q^*\left(\widehat{\theta}\right);\widehat{\theta}^j\right) \\ &= \sum v^j\left(q^*\left(\widehat{\theta}^i,\widehat{\theta}^{-i}\right);\theta^i,\widehat{\theta}^j\right) \end{aligned}$$

- To maximize this over $\widehat{\theta}^i$, type θ^i should report truthfully.

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4.1.1 Externality mechanism (Clarke 71)

$$\begin{aligned} q^{*,-i}\left(\widehat{\theta}\right) \quad &: = \arg\max_k \sum_{j\neq i} v^j\left(k,\theta^j\right) \\ t^i\left(\widehat{\theta}\right) \quad &= \underbrace{-\sum_{j\neq i} v^j\left(q^*\left(\widehat{\theta}\right);\widehat{\theta}^j\right)}_{\textbf{Value of } j\neq i \textbf{ with } i} + \underbrace{\sum_{j\neq i} v^j\left(q^{*,-i}\left(\widehat{\theta}\right);\widehat{\theta}^j\right)}_{\textbf{Value of } j\neq i \textbf{ without } i} \end{aligned}$$

- Agent i pays the externality of his report on others
- This is still strategy-proof, because $\sum_{j\neq i} v^j\left(q^{*,-i}\left(\widehat{\theta}\right);\widehat{\theta}^j\right)$ does not depend on $\widehat{\theta}^i$

4.1.2 Extensions

- Insight "Efficienct Allocations are Strategy-proof when externalities are internalized" is very robust
- No topological assumptions on type space (connectedness, dimensionality, etc.)
- Extends to Dynamic Mechanisms: (Bergemann, Valimaki Ectra 2002)
  1. Agent acquires information θ^i at cost c^i
  2. Agent reports information $\widehat{\theta}^i$ to principal
  3. Principal decides on outcome and transfers
     $\Rightarrow$ VCG transfers gives full informational externality to agent

4.1.3 Limitations

- Private values:
  - If $v^i(k) = v^i\left(k, \theta^i, \theta^{-i}\right)$, then agent j's perceived value v^j depends directly on i's report $\widehat{\theta}^i$
  - This distorts i's incentive to report truthfully
- Budget-balance constraint: $\sum_i t^i(\theta) = 0$ for all θ

Theorem 10 (Green, Laffont) *If Θ^i is sufficiently rich, no efficient, strategy-proof mechanism is budget-balanced.*

4.2 Payoff Equivalence

- θ^i independently distributed with pdf $f^i\left(\theta^i\right)$
- Interim expected utility

$$U^i\left(\theta^i, \widehat{\theta}^i\right) = \int_{\Theta^{-i}} \left(v^i\left(q\left(\widehat{\theta}^i; \theta^{-i}\right), \theta^i\right) - t^i\left(\widehat{\theta}^i; \theta^{-i}\right)\right) f^{-i}\left(\theta^{-i}\right) d\theta^{-i}$$

- In BNE: $U\left(\theta^i\right) = U\left(\theta^i, \theta^i\right) \geq U^i\left(\theta^i, \widehat{\theta}^i\right)$

Theorem 11 (Milgrom, Segal (2002)) *If Θ^i connected (and other regularity assumptions are satisfied), and (q, t) is BNE, then*

$$U^i\left(\theta^{i\prime}\right) = U\left(\theta^i\right) + \int_{\theta^i}^{\theta^{i\prime}} \partial_1 U(\tau, \tau)\, d\tau.$$

- ∂_1 is partial derivative wrt. true type

$$\partial_1 U(\tau, \tau) = \int_{\Theta^{-i}} \partial_{\theta^i} v^i\left(q(\theta), \theta^i\right) f^{-i}\left(\theta^{-i}\right) d\theta^{-i}$$

- Without independence would get additional term $u_i(\theta)\,\partial_{\theta^i} f^{-i}\left(\theta^i, \theta^{-i}\right)$
- Integral is path-integral from θ^i to $\theta^{i\prime}$

4.2.1 Object allocation with one-dimensional types

- $\theta^i \in \left[\underline{\theta}_i, \overline{\theta}_i\right]$ with cdf F^i, often symmetric
- $v^i\left(q(\theta), \theta^i\right) = q(\theta)\,\theta^i$
- $q(\theta) = \left(q^i(\theta)\right)_{i\in N, i=0} \in \Delta(N+1)$ stochastic allocation among N agents and principal 0
- Risk-neutrality: $u^i = q^i\theta^i - t^i$
- Expected allocation: $Q\left(\theta^i\right) = \int_{\Theta^{-i}} q\left(\theta^i, \theta^{-i}\right) dF^{-i}\left(\theta^{-i}\right)$

Proposition 12 *Mechanism (q,t) is Bayesian incentive compatible iff*

$$U^i\left(\theta^{i\prime}\right) = U\left(\theta^i\right) + \int_{\theta^i}^{\theta^{i\prime}} Q(\tau)\,d\tau \qquad \text{(IC-FOC)}$$

$$Q(\tau) \text{ non-decreasing} \qquad \text{(MON)}$$

Proof. Just as in single-agent case. ■

4.2.2 Calculating bidding functions

- First-price auction: $u^i = \left(\theta^i - \beta\left(\theta^i\right)\right) q^i(\theta)$
- Ex-ante symmetric bidders $\theta^i \sim f\left(\theta^i\right)$
- In efficient equilibrium:
  - $U^i(\underline{\theta}) = 0$
  - $Q^i\left(\theta^i\right) = F(\tau)^{N-1}$
  - $U^i\left(\theta^i\right) = \int_{\underline{\theta}}^{\theta^i} F(\tau)^{N-1} d\tau$
- On the other hand $U^i\left(\theta^i\right) = \left(\theta^i - \beta\left(\theta^i\right)\right) Q^i\left(\theta^i\right)$

- Thus

$$\beta\left(\theta^i\right) = \theta^i - \frac{\int_{\underline{\theta}}^{\theta^i} F\left(\tau\right)^{N-1}}{F\left(\theta^i\right)^{N-1}}$$

- Uniform case with $\left[\underline{\theta};\overline{\theta}\right] = [0;1]$: $\beta\left(\theta^i\right) = \theta^i - \frac{\left(\theta^i\right)^{N-1}/N}{\left(\theta^i\right)^{N-1}} = \frac{N-1}{N}\theta^i$

4.3 Optimal Auctions

- Seller chooses q, t to maximize

$$\begin{aligned} R &= \mathbb{E}\left[\sum_i t^i\left(\theta\right)\right] \\ &= \mathbb{E}\left[\sum_i q^i\left(\theta\right) v^i\left(\theta^i\right) - U^i\left(\theta^i\right)\right] \end{aligned}$$

  subject to

  - Resource constraint: $\sum_i q^i\left(\theta\right) \leq 1$
  - IC-FOC
  - MON
  - IR: $U^i\left(\underline{\theta}_i\right) \geq 0$

- Bidders' rents

$$\begin{aligned} \mathbb{E}\left[U^i\left(\theta^i\right)\right] &= \mathbb{E}\left[\underbrace{U^i\left(\underline{\theta}_i\right)}_{=0} + \int_{\underline{\theta}}^{\theta^i} Q^i\left(\tau\right) d\tau\right] \\ &= \int_{\underline{\theta}_i}^{\overline{\theta}^i} Q^i\left(\theta^i\right)\left(1 - F^i\left(\theta^i\right)\right) d\theta^i \end{aligned}$$

  by the same argument as in the single-agent case

- Back to the principal

$$\begin{aligned} R &= \sum_i \int_{\Theta^i}\left(Q^i\left(\theta^i\right)\theta^i - Q^i\left(\theta^i\right)\frac{1 - F^i\left(\theta^i\right)}{f^i\left(\theta^i\right)}\right) dF^i\left(\theta^i\right) \\ &= \int_{\Theta}\sum_i q^i\left(\theta\right)\left(\theta^i - \frac{1 - F^i\left(\theta^i\right)}{f^i\left(\theta^i\right)}\right) dF\left(\theta\right) \\ &= \int_{\Theta}\sum_i q^i\left(\theta\right) MR^i\left(\theta^i\right) dF\left(\theta\right) \end{aligned}$$

- Optimal mechanism:
  - Allocate object to bidder i with highest marginal revenue $MR^i\left(\theta^i\right)$ if ≥ 0
  - Otherwise, keep it

4.3.1 Optimal "real" mechanisms

- Define r^i by $MR^i\left(r^i\right) = 0$
- Optimal mechanism does not sell to types $\theta^i < r^i$
- In symmetric setting, any standard auction with reserve price r and symmetric monotone equilibrium is optimal
  - Second price
  - First price
  - All pay
- Example:
  - $\left[\underline{\theta}, \overline{\theta}\right] = [0;1]$, uniform distribution
  - $MR\left(\theta^i\right) = \theta^i - \frac{1-F\left(\theta^i\right)}{f\left(\theta^i\right)} = 2\theta^i - 1$
  - $r = 1/2$
- Reserve price serves same function as monopoly price:
  - Sacrifice surplus from low types to extract rents from high types
  - Interesting: Optimal reserve price does not depend on N
- Entry fees e:
  - Alternative feature to extract rents from low types
  - Optimal mechanism sets e to make type $\theta^i = r$ indifferent about entry
- Asymmetries:
  - Bidder j has ex-ante higher types than bidder i, say $\frac{f^j(\theta)}{1-F^j(\theta)} < \frac{f^i(\theta)}{1-F^i(\theta)}$
  - Then $MR^i(\theta) = \theta - \frac{1-F^i(\theta)}{f^i(\theta)} > \theta - \frac{1-F^j(\theta)}{f^j(\theta)} = MR^j(\theta)$
  - Thus, if $\theta^i = \theta^j - \varepsilon$, still allocate to i
  - Favor weak bidders, and extract more from high types of strong bidders

4.4 Inefficiency of Bilateral Trade

- Buyer with value $v \sim F\left[\underline{v}, \overline{v}\right]$
- Seller with cost $c \sim G\left[\underline{c}, \overline{c}\right]$
- Mechanism: $\left(q, t^B, t^S\right)$
  - Probability of trade $q\left(v, c\right) \in [0; 1]$
  - Payment by buyer $t^B\left(v, c\right)$
  - Payment to seller $t^S\left(v, c\right)$
- Efficiency requires

$$q\left(v, c\right) = 1 \text{ iff } v > c \tag{Eff}$$

- Individual rationality constraints:

$$U^B\left(v\right) = \int \left(q\left(v, c\right) v - t^B\left(v, c\right)\right) dG\left(c\right) \geq 0 \text{ for all } v \tag{IR B}$$

$$U^S\left(c\right) = \int \left(t^S\left(v, c\right) - q\left(v, c\right) c\right) dF\left(v\right) \geq 0 \text{ for all } c \tag{IR S}$$

- Budget-balance constraint:

$$t^B\left(v, c\right) \geq t^S\left(v, c\right) \text{ for all } \left(v, c\right) \tag{BB Ex-Post}$$

$$\mathbb{E}\left[t^B\left(v, c\right)\right] \geq \mathbb{E}\left[t^S\left(v, c\right)\right] \tag{BB Ex-Ante}$$

- For mechanism $m = (q, t)$, define expected surplus and revenue

$$S\left(m\right) = \int q\left(v, c\right)\left(v - c\right) dF\left(v\right) dG\left(c\right)$$

$$R\left(m\right) = \int \left(t^B\left(v, c\right) - t^S\left(v - c\right)\right) dF\left(v\right) dG\left(c\right)$$

Theorem 13 (Myerson-Satterthwaite 1983) *Assume continuous, overlapping type spaces, i.e.* $\underline{v} \leq \underline{c} < \overline{v} \leq \overline{c}$. *The efficient allocation rule* $q\left(v, c\right) = 1$ *iff* $v > c$ *is not implementable under Bayesian-Nash IC, IR, and BB Ex-ante.*

Proof.

- Consider mechanism VCG: If $v > c$

$$\begin{aligned} q(v,c) &= 1 \\ t^B(v,c) &= c \text{ (buyer pays his externality)} \\ t^S(v,c) &= v \text{ (seller receives his externality)} \end{aligned}$$

  otherwise, $q = t^B = t^S = 0$

- By VCG this is strategy-proof
- The lowest-type buyer $\underline{v}$ never trades: $U^B_{VCG}(\underline{v}) = 0$
- The highest-cost seller $\bar{c}$ never trades: $U^S_{VCG}(\bar{c}) = 0$
- Mechanism has deficit of $R(VCG) = \mathbb{E}\left[t^B(v,c) - t^S(v,c)\right] = \int_{v>c} (c-v)\, dF(v)dG(c) < 0$
- Both agents capture full surplus, but there's only one surplus to go around
- And every efficient, IC, IR mechanism $m = (q, t')$ has at least this deficit
  - By Efficiency

$$S(VCG) = S(m) = \int_{v>c} (v-c)\, dF(v)\, dG(c)$$

  - By Rev. Equiv.

$$\begin{aligned} U^B_m(v) - U^B_m(\underline{v}) &= U^B_{VCG}(v) - U^B_{VCG}(\underline{v}) = \int_{\underline{v}}^{v} Q^B(\tau)\, d\tau \\ U^S_m(c) - U^s_m(\bar{c}) &= U^S_{VCG}(c) - U^S_{VCG}(\bar{c}) = \int_{c}^{\bar{c}} Q^S(\tau)\, d\tau \end{aligned}$$

  - Thus,

$$\begin{aligned} R(m) &= S(m) - \left(\mathbb{E}\left[U^B_m(v)\right] + \mathbb{E}\left[U^S_m(c)\right]\right) \\ &= S(VCG) - \left(\mathbb{E}\left[U^B_{VCG}(v)\right] + \mathbb{E}\left[U^S_{VCG}(c)\right]\right) \\ &\quad + U^B_{VCG}(\underline{v}) + U^S_{VCG}(\bar{c}) - U^B_m(\underline{v}) - U^s_m(\bar{c}) \\ &\leq R(VCG) \\ &< 0 \end{aligned}$$

■

- The Theorem continues to hold if $[\underline{v}, \overline{v}] \neq [\underline{c}, \overline{c}]$ as long as efficient trade is not certain, i.e. $\underline{v} < \overline{c}$
  - Need to change VCG payments to

$$\begin{aligned} t^B(v,c) &= \max\{c; \underline{v}\} \\ t^S(v,c) &= \min\{\overline{c}; v\} \end{aligned}$$

    to ensure $U^B_{VCG}(\underline{v}) = U^S_{VCG}(\overline{c}) = 0$
  - This mechanism still has a deficit: if $v > c$

$$t^B(v,c) = \max\{c; \underline{v}\} < \min\{\overline{c}; v\} = t^S(v,c) \text{ if } c > \underline{v}$$

4.5 Correlated Values - Full Surplus Extraction

- Cremer, McLean (1985), (1988)
- Often types are correlated: common quality component x (unknown to seller) correlates the conditionally independent signals of the agents
  - Signal $\theta^i = x + \varepsilon^i$ where ε^i independent
  - Private values $v^i = \theta^i$
  - Common values x
- Revenue equivalence breaks down

$$\begin{aligned} \frac{d}{d\theta^i}U(\theta^i) &= \frac{d}{d\theta^i}\int_{\Theta^{-i}}\left(q^i\left(\widehat{\theta}^i;\theta^{-i}\right)\theta^i - t^i\left(\widehat{\theta}^i;\theta^{-i}\right)\right)f^{-i}\left(\theta^{-i}|\theta^i\right)d\theta^{-i} \\ &= Q^i(\theta^i) + \int_{\Theta^{-i}} u^i\left(\widehat{\theta}^i;\theta^{-i}\right)\frac{d}{d\theta^i}f^{-i}\left(\theta^{-i}|\theta^i\right)d\theta^{-i} \\ &= ??? \end{aligned}$$

- Thus (IC) plus binding (IR) constraints for "lowest type" does not pin down transfers and utility
- Let $\Theta^i = \left\{\underline{\theta}, \cdots, \overline{\theta}\right\}$ for $i = 1, \cdots, N$ (McAfee, Reny ('92) generalizes this to continuous types)
- Assume that type distribution is "generic" in that the M beliefs $\pi^i\left(\cdot|\theta^i\right) = \frac{f(\theta^i,\cdot)}{f(\Theta^i,\cdot)} \in \Delta\left(\Theta^{-i}\right)$ are linearly independent

$$\underbrace{\begin{pmatrix} \pi^i\left(\underline{\theta},\cdots,\underline{\theta}|\underline{\theta}\right) & \cdots & \pi^i\left(\overline{\theta},\cdots,\overline{\theta}|\underline{\theta}\right) \\ \cdots & \cdots & \cdots \\ \pi^i\left(\underline{\theta},\cdots,\underline{\theta}|\overline{\theta}\right) & \cdots & \pi^i\left(\overline{\theta},\cdots,\overline{\theta}|\overline{\theta}\right) \end{pmatrix}}_{M \text{ rows, } (N-1)M \text{ columns}}$$

has full rank M

- This is "generically" satisfied (for parameter set with full Lebesgue measure) as long as $\#\Theta^{-i} \geq \#\Theta^i$
- It is violated for statistically independent types $\theta^i \Rightarrow$ beliefs $\pi^i\left(\cdot|\theta^i\right) = \pi^i(\cdot)$ are indepent of θ^i

Theorem 14 (Cremer, McLean) *For generic information structures, and any IC, IR mechanism (q,t) there exists an alternative (IC) mechanism $(q, t+\ell)$, where IR binds for all types.*

In particular, there exists an ex-post efficient mechanism that extracts all surplus.

Proof.

- Calculate interim utilities $U^i\left(\theta^i\right)$ in (q,t)
- Require agent i to accept a lottery $\ell = \ell\left(\theta^{-i}\right)$ that solves

$$\begin{pmatrix} \mathbb{E}\left[\ell\left(\theta^{-i}\right)|\theta^i = \underline{\theta}\right] \\ \cdots \\ \mathbb{E}\left[\ell\left(\theta^{-i}\right)|\theta^i = \overline{\theta}\right] \end{pmatrix} = \begin{pmatrix} \pi^i\left(\underline{\theta}, \cdots, \underline{\theta}|\underline{\theta}\right) & \cdots & \pi^i\left(\overline{\theta}, \cdots, \overline{\theta}|\underline{\theta}\right) \\ \cdots & \cdots & \cdots \\ \pi^i\left(\underline{\theta}, \cdots, \underline{\theta}|\overline{\theta}\right) & \cdots & \pi^i\left(\overline{\theta}, \cdots, \overline{\theta}|\overline{\theta}\right) \end{pmatrix} \begin{pmatrix} \ell\left(\underline{\theta}, \cdots, \underline{\theta}\right) \\ \cdots \\ \ell\left(\underline{\theta}, \cdots, \underline{\theta}\right) \end{pmatrix} = \begin{pmatrix} U^i\left(\underline{\theta}\right) \\ \cdots \\ U^i\left(\overline{\theta}\right) \end{pmatrix} \quad (*)$$

  if he wants to participate in (q,t)
- This lottery exists by assumption
- Now, $U^i_{(q,t+\ell)}\left(\theta^i; q, t+\ell\right) = U^i_{(q,t)}\left(\theta^i\right) - \mathbb{E}\left[\ell\left(\theta^{-i}\right)|\theta^i\right] = 0$
- If rows $\pi^i\left(\cdot|\theta^i\right)$ are almost (but not quite) linearly dependent, the lottery ℓ satisfying (*) needs large payments $\ell\left(\theta^{-i}\right) \gg 0$

■

- If (q,t) is Bayes-Nash IC, then so is $(q, t+\ell)$
- If (q,t) is strategy-proof, then so is $(q, t+\ell)$
- Alternative Bayesian mechanism
  - Assume types perfectly correlated, i.e. $\Pr\left(\theta^{-i} = (x, \cdots, x)\,|\theta^i = x\right) = 1$ for any x
  - Set $t^i\left(\widehat{\theta}\right) = \infty$ unless $\widehat{\theta}^1 = \cdots = \widehat{\theta}^N$
  - This makes any allocation rule q Bayesian incentive compatible (but not ex-post IC or strategy-proof)
- Example:

- $N = 2$ and $\Theta = \{0, 1\}$ with $f\left(\theta^i, \theta^j\right) = \begin{pmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{pmatrix}$, so $\pi^i\left(\theta^{-i}, \theta^i\right) = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$
- (q, t) second-price auction without reserve
- $U(0) = 0$
- $U(1) = \pi^i\left(\theta^{-i} = 0 | \theta^i = 1\right)(1 - 0) = 1/3$
- So $\ell\left(\theta^{-i}\right)$ must satisfy

$$\begin{pmatrix} \mathbb{E}\left[\ell\left(\theta^{-i}\right)|\theta^i = 0\right] \\ \mathbb{E}\left[\ell\left(\theta^{-i}\right)|\theta^i = 1\right] \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix} \begin{pmatrix} \ell(0) \\ \ell(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}$$

- Take $\ell(1) = 2/3$ and $\ell(0) = -1/3$, i.e. make bidder i bet on $\theta^{-i} = 0$ at odds $1:2$
  * $\theta^i = 0$ thinks $\theta^{-i} = 0$ likely and is indifferent about bet
  * $\theta^i = 1$ thinks $\theta^{-i} = 0$ unlikely and does not like the bet, but is indifferent when it is coupled to the auction

4.5.1 Limitations

- Surplus-extraction mechanism satisfies
  - Dominant-strategy, ex-post IC
  - But only Interim IR!
- Ex-post IR constraint binds because of lottery ℓ
- If prior π is almost independent, then $\ell \gg 0$, and $u^i\left(\widehat{\theta}\right) \ll 0$ for some $\widehat{\theta}$
- Limitations of full-surplus extraction
  - Risk-aversion
  - Collusion
  - Uncertainty about beliefs (Neeman, JET, 2004, Heifetz, Neeman, Ectra, 2006)
- Agents have beliefs and maximize subjective expected utility, BUT designer may not know beliefs
- (Finite) type space $T^i = \Theta^i \times \Pi^i$ where $\Pi^i \subseteq \Delta\left(T^{-i}\right)$, with type $t^i = \left(\theta^i, \pi^i\right)$
  - Payoff relevant type θ^i
  - Belief type $\pi^i(\cdot)$

- In classical Bayesian setting with correlated prior $\pi \in \Delta(\Theta)$, payoff types θ^i and belief types π^i are one-to-one
- Cremer McLean: Belief type π^i is not really private, and gives payoff type θ^i away $\Rightarrow$ agent's rents can be extracted
- Consider larger type space T^i where beliefs do not determine preferences (BDP)
- I.e. exists two types $t^i = (\theta^i, \pi^i), t^{i\prime} = (\theta^{i\prime}, \pi^i)$ with
  - Different payoff characterisitics $\theta^i < \theta^{i\prime}$
  - Identical beliefs π^i
- Interim expected utility

$$U^i\left(t^i; \widehat{t^i}\right) = \int_{T^{-i}} \left(q^i\left(\widehat{t^i}, t^{-i}\right)\theta^i - p^i\left(\widehat{t^i}, t^{-i}\right)\right) d\pi^i\left(t^{-i}\right)$$

- Now, consider mechanism $(q, p) : T \to N \times \mathbb{R}^N$ that satisfies

$$\begin{aligned} U\left(t^i; t^i\right) &\geq U^i\left(t^i; \widehat{t^i}\right) && \text{(Interim IC)} \\ U\left(t^i; t^i\right) &\geq 0 && \text{(Interim IR)} \end{aligned}$$

  for all $t^i, \widehat{t^i} \in T^i$

- If $t^i = (\theta^i, \pi^i)$ gets served, i.e. $Q(t^i) > 0$, then $t^{i\prime} = (\theta^{i\prime}, \pi^i)$ will get positive rents (Neeman 2004)

$$\begin{aligned} U\left(t^i; t^i\right) &\geq 0 && \text{(Interim IC)} \\ U\left(t^{i\prime}; t^{i\prime}\right) &\geq U\left(t^{i\prime}; t^i\right) && \text{(Interim IR)} \\ &= \int_{T^{-i}} \left(q^i\left(t^i, t^{-i}\right)\theta^{i\prime} - p^i\left(t^i, t^{-i}\right)\right) d\pi^i\left(t^{-i}\right) \\ &= Q\left(t^i\right)\left(\theta^{i\prime} - \theta^i\right) + U\left(t^i; t^i\right) \\ &> 0 \end{aligned}$$

- Heifetz, Neeman (2006): When beliefs $\pi^i(\cdot|\theta^i)$ are derived from common prior π on T, then for "generic" priors $\pi \in \Delta(T)$ BDP is not satisfied, and full surplus extraction is not possible

5 Dynamic Screening

- 2-period price discrimination
- Buyer

$$u = \sum_{t=1,2} q_t \theta_t - p_t$$

- Seller

$$\sum_{t=1,2} \left(p_t - c\left(q_t\right)\right)$$

- Mechanism:

$$q_1\left(\theta_1\right), p_1\left(\theta_1\right)$$
$$q_2\left(\theta_1, \theta_2\right), p_2\left(t_1, t_2\right)$$

5.1 Commitment

- Seller can commit to mechanism
  - Ratchet Effect: No unilateral deviation (violating IR) by principal after learning θ_1 in period 1
  - Renegotiation-proof: No bilateral deviation (satisfying IR) by principal and agent in period 2

5.1.1 Constant Type

- $\theta_1 = \theta_2 = \theta$

Proposition 15 *Optimal 2-period contract repeats optimal short-term contract.*

Proof.

- Consider θ with $q_1\left(\theta\right) \neq q_2\left(\theta\right)$
- Mechanism can be replicated by repetition of static randomized mechanism
- But optimal short-term mechanism is deterministic: $q\left(\theta\right) = 1$ iff $MR\left(\theta\right) > 0$

■

- Consumer gets informational rent twice
- After learning θ, seller has interest to change contract, even if he has to compensate buyer

5.1.2 Independent Type

- θ_1, θ_2 independent
- Buyer initially only knows θ_1

Proposition 16 *Optimal contract* $q_1^*(\theta_1), q_2^*(\theta_1, \theta_2)$*, satisfies*

$$
\begin{aligned}
MR(\theta_1) &= c'(q_1^*(\theta_1)) && \textit{(Period 1)} \\
\theta_2 &= c'(q_2^*(\theta_2)) && \textit{(Period 2)}
\end{aligned}
$$

Proof.

- Buyer will receive informational rents of at least $\mathbb{E}[u(\theta)] = \int q_1(\theta_1)(1 - F(\theta_1))\, d\theta_1$ from knowing θ_1 before contract is signed
- Seller faces usual trade-off for $q_1^*(\theta_1)$ when maximizing $R = \mathbb{E}[S(\theta) - u(\theta) - c(q(\theta))]$
- But seller faces no trade-off for $q_2^*(\theta_2)$
  - Allocate efficiently: $\theta_2 = c'(q_2^*(\theta_2))$
  - Charge expected 2nd period value $p_2 = \mathbb{E}[\theta_2 q_2^*(\theta_2)]$ in period 1
- Buyer does not get information rent for θ_2 because contract is signed before θ_2 is learned

■

- Renegotiation constraints not binding: Seller sells efficiently and reaps all rents
- Consumer gets informational rent only for θ_1

5.1.3 Correlated Types

- Courty, Li, 2000, REStud
- Types

$$
\begin{aligned}
\theta_1 &\sim F(\theta_1) \\
\theta_2 &\sim G(\theta_2|\theta_1)
\end{aligned}
$$

- Period 2

- IC 2

$$
\begin{aligned}
u_2\left(\widehat{\theta}_1, \theta_2 | \theta_1, \theta_2\right) &= q_2\left(\widehat{\theta}_1, \theta_2 | \theta_1, \theta_2\right)\theta_2 - p_2\left(\widehat{\theta}_1, \theta_2 | \theta_1, \theta_2\right) \geq \\
&\geq q_2\left(\widehat{\theta}_1, \widehat{\theta}_2 | \theta_1, \theta_2\right)\theta_2 - p_2\left(\widehat{\theta}_1, \widehat{\theta}_2 | \theta_1, \theta_2\right) = u_2\left(\widehat{\theta}_1, \widehat{\theta}_2 | \theta_1, \theta_2\right)
\end{aligned}
$$

- Envelope Theorem

$$
u_2\left(\widehat{\theta}_1, \theta_2 | \theta_1, \theta_2\right) = \int_{\underline{\theta}}^{\theta_2} q_2\left(\widehat{\theta}_1, s\right) ds
$$

- Smart accounting

$$
\mathbb{E}\left[u_2\left(\widehat{\theta}_1, \theta_2 | \theta_1, \theta_2\right) \mid \left(\widehat{\theta}_1, \theta_1\right)\right] = \int_{\underline{\theta}}^{\overline{\theta}} q_2\left(\widehat{\theta}_1, \theta_2\right)\left(1 - G\left(\theta_2 | \theta_1\right)\right) d\theta_2
$$

- Period 1
  - Expected utility of type θ_1 when stating $\widehat{\theta}_1$

$$
u\left(\theta_1; \widehat{\theta}_1\right) = \theta_1 q_1\left(\widehat{\theta}_1\right) - p_1\left(\widehat{\theta}_1\right) + \int_{\underline{\theta}}^{\overline{\theta}} q_2\left(\widehat{\theta}_1, \theta_2\right)\left(1 - G\left(\theta_2 | \theta_1\right)\right) d\theta_2
$$

  - Envelope theorem

$$
u'\left(\theta_1\right) = q_1\left(\theta_1\right) - \int_{\underline{\theta}}^{\overline{\theta}} q_2\left(\widehat{\theta}_1, \theta_2\right) \frac{\partial}{\partial \theta_1} G\left(\theta_2 | \theta_1\right) d\theta_2
$$

  - Smart accounting

$$
\begin{aligned}
\mathbb{E}\left[u\left(\theta_1\right)\right] &= \mathbb{E}\left[u'\left(\theta_1\right)\frac{1 - F\left(\theta_1\right)}{f\left(\theta_1\right)}\right] \\
&= \mathbb{E}_{\theta_1}\left[\mathbb{E}_{\theta_2|\theta_1}\left[q_1\left(\theta_1\right) - q_2\left(\widehat{\theta}_1, \theta_2\right)\frac{\partial_{\theta_1} G\left(\theta_2 | \theta_1\right)}{g\left(\theta_2 | \theta_1\right)}\right]\frac{1 - F\left(\theta_1\right)}{f\left(\theta_1\right)}\right]
\end{aligned}
$$

- Optimal quantities

$$
\begin{aligned}
\theta_1 - \frac{1 - F\left(\theta_1\right)}{f\left(\theta_1\right)} &= c'\left(q_1\left(\theta_1\right)\right) \\
\theta_2 + \underbrace{\frac{\partial_{\theta_1} G\left(\theta_2 | \theta_1\right)}{g\left(\theta_2 | \theta_1\right)}}_{\text{Informativeness}} \frac{1 - F\left(\theta_1\right)}{f\left(\theta_1\right)} &= c'\left(q_2\left(\theta_1, \theta_2\right)\right)
\end{aligned}
$$

- Both over- and under-provision of quality in optimal contract
- Usually $\partial_{\theta_1} G\left(\theta_2 | \theta_1\right) < 0$, so under-provision

- No distortion for highest type θ_1
- No distortion for types θ_2 in ranges where $\partial_{\theta_1} G(\theta_2|\theta_1) = 0$
- Example:
  - Additive FOSD structure
$$\theta_2 = \frac{1}{2}\theta_1 + \frac{1}{2}\varepsilon$$
    where $\varepsilon \sim H[-\infty, \infty]$
  - Then
$$\begin{aligned} G(\theta_2|\theta_1) &= \Pr\left(\frac{1}{2}\theta_1 + \frac{1}{2}\varepsilon \leq \theta_2\right) \\ &= \Pr(\varepsilon \leq 2\theta_2 - \theta_1) \\ &= H(2\theta_2 - \theta_1) \\ \partial_{\theta_1} G(\theta_2|\theta_1) &= -h(2\theta_2 - \theta_1) \end{aligned}$$

5.2 No Commitment

- Laffont, Tirole (1990)
- Short-term contracts
- Fixed types (no problems if types are independent)
- Continuum of types $[\underline{\theta}, \overline{\theta}]$

Proposition 17 *In the optimal contract there is no non-trivial interval of types* (θ_0, θ_1) *that separates in the first round.*

- Once θ is revealed, rents in period 2 are zero
- Lying $\widehat{\theta} = \theta - \varepsilon$, has first order gain in period 2 and only second order loss in period 1

Proof.

- For $\theta, \widehat{\theta} = \theta - \varepsilon \in (\theta_0, \theta_1)$
$$\begin{aligned} u_2\left(\widehat{\theta}, \theta\right) &= \left(\theta - \widehat{\theta}\right) q_2\left(\widehat{\theta}\right) \\ u_2(\theta, \theta) &= 0 \end{aligned}$$

- Incentive constraints

$$\begin{aligned} \theta q_1(\theta) - p_1(\theta) &\geq \theta q_1\left(\widehat{\theta}\right) - p_1\left(\widehat{\theta}\right) + \left(\theta - \widehat{\theta}\right) q_2\left(\widehat{\theta}\right) && \text{(IC)} \\ \widehat{\theta} q_1\left(\widehat{\theta}\right) - p_1\left(\widehat{\theta}\right) &\geq \widehat{\theta} q_1(\theta) - p_1(\theta) && \text{(IC')} \end{aligned}$$

- Then

$$\begin{aligned} 0 &\geq \theta q_1'(\theta) - p_1'(\theta) \text{ by (IC')} \\ &\geq \left(\theta - \widehat{\theta}\right) q_2\left(\widehat{\theta}\right) \text{ by (IC)} \end{aligned}$$

- This cannot be if $q_2\left(\widehat{\theta}\right) > 0$

■

- Thus, optimal contract must feature pooling intervals
- Analogy Crawford, Sobel (1982)

5.3 Durable Goods

5.3.1 Commitment

Fixed Demand

- Distribution $F(\theta)$ of buyers, available in period $1, 2, \cdots, \infty$
- Consumer utility, when reporting $\widehat{\theta}$

$$u_i = \delta^{t\left(\widehat{\theta}\right)} q\left(\widehat{\theta}\right)\theta - p\left(\widehat{\theta}\right)$$

  and receiving object in $t\left(\widehat{\theta}\right)$

- Firm maximizes

$$\max_{t(\theta)} \mathbb{E}\left[MR(\theta)\,\delta^{t(\theta)}\right] = \int m(\theta)\,\delta^{t(\theta)} d\theta$$

  where

$$m(\theta) = MR(\theta) f(\theta) = \theta f(\theta) - (1 - F(\theta))$$

- Solution (Stokey 1979)

$$\tau(\theta) = \begin{cases} 1 & \text{if } m(\theta) \geq 0 \\ \infty & \text{if } m(\theta) \leq 0 \end{cases}$$

- No-haggling
- Result remains if new cohort of $F(\theta)$ buyers enters every round

Varying Demand

- $F_t(\theta)$ in $t = 1, 2$
- Firm maximizes

$$\max_{\tau(t,\theta)\geq t} \sum_{t=1,2} \int m_t(\theta)\, \delta^{\tau(t,\theta)} d\theta$$

- Solution (Board 2008)
- Period 1: Sell to all types $\left[\theta_1^*, \overline{\theta}_1\right]$ where

$$m_1(\theta_1^*) = 0$$

- Period 2:
  - Face $F_2(\theta)$ and $F_1(\theta|\theta \leq \theta_1^*)$
  - Marginal revenue

$$M_2(\theta) = m_2(\theta) + \min\{0; m_1(\theta)\}$$

  - When $\theta < \theta_1^*$, i.e. when selling to types left-over from period 1 with $m_1(\theta) < 0$, damage revenue in period 1 (never happens if F_t independent of t)
  - Optimally, sell to $[\theta_2^*, \infty]$ where

$$M_2(\theta_2^*) = 0$$

5.3.2 No Commitment - Coase

- Coase conjecture: Durable-goods monopolist always faces competition: Himself in the future
- When buyers are patient, seller cannot extract any profits
- Solution: Renting

2 periods

- Bulow 1982
- $\theta \sim U[0; 1]$
- Serve $[\theta^*; 1]$ in period 1

- In period 2, $p_2 = \theta^*/2$
- Consumer indifference

$$\begin{aligned}
\theta^*(p_1) - p_1 &= \delta\left(\theta^*(p_1) - p_2\right) \\
p_1 &= (1 - \delta/2)\,\theta^*(p_1) \\
\theta^*(p_1) &= kp_1 \text{ where } k = (1 - \delta/2)^{-1}
\end{aligned}$$

- Period 1: Choose p_1 (or equivalently choose $\theta^*(p_1)$) to maximize

$$\begin{aligned}
\pi &= (1 - \theta^*(p_1))\,p_1 + \delta\left(\theta^*(p_1)/2\right)^2 \\
&= (1 - kp_1)\,p_1 + \delta k^2 p_1^2/4
\end{aligned}$$

- First order condition

$$\begin{aligned}
0 &= 1 - 2kp_1^* + \delta k^2 p_1^*/2 \\
p_1^* &= \left(2k - \delta k^2/2\right)^{-1} \\
&= k^{-1}\frac{1}{2 - \delta k/2} \\
&= \frac{1 - \delta/2}{2 - \frac{\delta}{2-\delta}} \\
&= \frac{(2-\delta)(2-\delta)}{8 - 6\delta}
\end{aligned}$$

- Example: $\delta = 0$
  - $p_1^* = 4/8 = 1/2$
  - $\theta_1^* = 1/2$ - like in static model
  - $p_2^* = 1/4$
  - Profit $\pi = \frac{1}{2}\frac{1}{2} + 0\frac{1}{4}\frac{1}{4} = 1/4$, like in static model
- Example: $\delta = 1$
  - $p_1^* = 1/2$
  - $\theta_1^*(1/2) = (1 - 1/2)^{-1}\,1/2 = 1$ - everybody waits
  - $p_2^* = 1/2$
  - Profit $\pi = 0 + 1\frac{1}{2}\frac{1}{2} = 1/4$ - like in static model
  - Got lucky that there game ends after period 2

- Example: $\delta = 1/2$
  - $p_1^* = 9/20$
  - $\theta_1^* = (1 - 1/4)^{-1} 9/20 = 3/5$
  - $p_2^* = 3/10$
  - Profit: $\pi = 9/20 * 2/5 + 1/2 * 9/100 = 22.5/100 < 1/4$
- Indifference types $\theta_t^* (\delta)$ increasing in δ

Continuous Time

- Fuchs, Skrzypacz, AER 2009
- Period length $\Delta \approx 0$
- Interest rate r
- Strategies
  - Seller quotes price $p = p(\theta)$ as a function of history; Skimming property: History can be summarized by set of remaining types $[\underline{\theta}; \theta]$
  - Buyer chooses cutoff $\theta_+ = \theta_+ (p)$: Types $[\underline{\theta}, \theta_+]$ wait, $[\theta_+, \theta]$ buy
- Draw graph of $\theta(t), p(t)$
- Problem
  - In discrete time, this becomes very messy (Gul, Sonnenschein, Wilson 1986)
  - In continuous time, there is no equilibrium because the seller will try to cut the price ever faster (in the limit the price drops to zer instantly, which is not an equilibrium)
- Trick:
  - Arrival of clone with probability $\lambda\Delta \approx 1 - e^{\lambda\Delta}$ -> object is sold at auction for θ
  - This puts a lower bound on the seller's opportunity value of selling
  - If seller just waits and only sells at arrival, i.e. $\theta_+ (p) = \overline{\theta}, p(\theta) = \overline{\theta}$, payoff is

$$V = \int e^{-rt} \lambda e^{-\lambda t} \mathbb{E}[\theta] \, dt = \frac{\lambda}{r + \lambda} \mathbb{E}[\theta]$$

  - This is not an equilbirium, but we'll see that this is actually the profit he'll make in equilibrium

- Indifference condition of buyer $\theta_+ = \theta_+(p)$ as function of price today $p(\theta)$ and price tomorrow $p(\theta_+)$

$$\theta_+ - p(\theta) = e^{-r\Delta} e^{-\lambda\Delta} (\theta_+ - p(\theta_+)) \tag{Buyer}$$

- (Buyer) allows us to write $p = p(\theta, \theta_+)$: Price that makes θ_+ indifferent, given $p(\theta_+)$

- Seller's payoff in period t when $[\underline{\theta}, \theta]$ types remain and types (θ_+, θ) buy in next period

$$V(\theta) = \underbrace{\Delta\lambda\mathbb{E}\left[\widetilde{\theta}|\widetilde{\theta} \leq \theta\right]}_{\text{Clone arrives}} + e^{-\lambda\Delta}\left[\underbrace{\left(\frac{F(\theta) - F(\theta_+)}{F(\theta)}\right) p(\theta, \theta_+)}_{\text{Buyer buys}} + \underbrace{\frac{F(\theta_+)}{F(\theta)} e^{-r\Delta} V(\theta_+)}_{\text{Buyer waits}}\right]$$

- Seller maximizes RHS through choice of θ_+ (note that θ_+ determines $p(\theta, \theta_+)$)

- Subtracting $e^{-r\Delta}V(\theta)$ on both sides, we can rewrite

$$\begin{aligned}
r\Delta V(\theta) &= \max_{\theta_+} \left\{ \begin{array}{c} \Delta\lambda\mathbb{E}\left[\widetilde{\theta}|\widetilde{\theta} \leq \theta\right] + \underbrace{e^{-\lambda\Delta}}_{\approx 1} \left(\frac{F(\theta) - F(\theta_+)}{F(\theta)}\right) p(\theta, \theta_+) + \\ \left(e^{-\lambda\Delta} \frac{F(\theta_+)}{F(\theta)} - 1\right) \underbrace{e^{-r\Delta} V(\theta_+)}_{\approx V(\theta)} + \underbrace{e^{-r\Delta}}_{\approx 1} (V(\theta_+) - V(\theta)) \end{array} \right\} \\
&= \max_{\theta_+} \left\{ \begin{array}{c} \Delta\lambda\left(\mathbb{E}\left[\widetilde{\theta}|\widetilde{\theta} \leq \theta\right] - V(\theta)\right) + \left(\frac{F(\theta) - F(\theta_+)}{F(\theta)}\right) p(\theta, \theta_+) \\ + \underbrace{e^{-\lambda\Delta}}_{\approx 1} \left(\frac{F(\theta_+)}{F(\theta)} - 1\right) V(\theta) + (V(\theta_+) - V(\theta)) \end{array} \right\} \\
&= \underbrace{\Delta\lambda}_{\Pr(Arrival)} \underbrace{\left(\mathbb{E}\left[\widetilde{\theta}|\widetilde{\theta} \leq \theta\right] - V(\theta)\right)}_{\text{Gain at arrival}} + \max_{\theta_+} \left\{ \underbrace{\frac{f(\theta)}{F(\theta)} (\theta - \theta_+)}_{\Pr(Sale)} \underbrace{(p(\theta, \theta_+) - V(\theta))}_{\text{Gain at sale}} - \underbrace{(V(\theta) - V(\theta_+))}_{\text{Depreciation}} \right\}
\end{aligned}$$

- In the continuous-time limit seller chooses "sales rate" $d\theta/dt$ to maximize $rV(\theta)$

$$\begin{aligned}
rV(\theta) &= \lambda\left(\mathbb{E}\left[\widetilde{\theta}|\widetilde{\theta} \leq \theta\right] - V(\theta)\right) + \max_{d\theta/dt \in [-\infty, 0]} \left\{ -\frac{d\theta}{dt} \frac{f(\theta)}{F(\theta)} (p(\theta) - V(\theta)) + V'(\theta) \frac{d\theta}{dt} \right\} \\
&= \lambda\left(\mathbb{E}\left[\widetilde{\theta}|\widetilde{\theta} \leq \theta\right] - V(\theta)\right) + \left(\underbrace{\frac{f(\theta)}{F(\theta)} (p(\theta) - V(\theta)) - V'(\theta)}_{x}\right) \max_{d\theta/dt \in [-\infty, 0]} \left\{ -\frac{d\theta}{dt} \right\}
\end{aligned}$$

- Indifference condition (Buyer) becomes

$$
\begin{aligned}
-\left(p(\theta)-p(\theta_{+})\right)/\Delta &= (r+\lambda)(\theta-p(\theta)) \\
\underbrace{-p'(\theta)\frac{d\theta}{dt}}_{\text{Gains from waiting}} &= \underbrace{(r+\lambda)(\theta-p(\theta))}_{\text{Cost of waiting}} \qquad \text{(Buyer C)}
\end{aligned}
$$

- Now show that gain from selling fast $x=0$

$$
\frac{f(\theta)}{F(\theta)}\left(p(\theta)-V(\theta)\right)=V'(\theta) \qquad \text{(Seller)}
$$

- If $x>0$
  - Seller should screen infinitely fast: $\frac{d\theta}{dt}=-\infty$
  - This is not compatible with (Buyer C): The gains from waiting are infinite if the price is falling instantly
- If $x<0$
  - Seller should stop screening: $\frac{d\theta}{dt}=0$
  - By (Buyer C) all buyers with $\theta>p(\theta)$ will buy
  - No sales afterwards
  - Seller can then increase revenue by lowering the price after all
- Thus $x=\frac{f(\theta)}{F(\theta)}\left(p(\theta)-V(\theta)\right)-V'(\theta)=0$
- This implies
  - Seller is indifferent between all prices, could get her payoff by setting $p=\infty$ and waiting rather than lowering the price

$$
V(\theta)=\frac{\lambda}{r+\lambda}\mathbb{E}\left[\widetilde{\theta}|\widetilde{\theta}\leq\theta\right] \qquad \text{(Payoff Seller)}
$$

  - As for the derivative

$$
\begin{aligned}
V'(\theta) &= \frac{\lambda}{r+\lambda}\frac{d}{d\theta}\mathbb{E}\left[\widetilde{\theta}|\widetilde{\theta}\leq\theta\right] \\
&= \frac{\lambda}{r+\lambda}\frac{f(\theta)}{F(\theta)}\left(\theta-\mathbb{E}\left[\widetilde{\theta}|\widetilde{\theta}\leq\theta\right]\right)
\end{aligned}
$$

- Plugging this into $V'(\theta) = \frac{f(\theta)}{F(\theta)}(p(\theta) - V(\theta))$ yields

$$\begin{aligned}
\frac{\lambda}{r+\lambda}\left(\theta - \mathbb{E}\left[\widetilde{\theta}|\widetilde{\theta} \leq \theta\right]\right) &= p(\theta) - \frac{\lambda}{r+\lambda}\mathbb{E}\left[\widetilde{\theta}|\widetilde{\theta} \leq \theta\right] \\
p(\theta) &= \frac{\lambda}{r+\lambda}\theta
\end{aligned}$$

  price that makes seller indifferent between selling and waiting

- Screening rate $-\frac{d\theta}{dt}$ can be chosen to satisfy (Buyer C)

$$\begin{aligned}
-\frac{dp}{dt} &= -p'(\theta)\frac{d\theta}{dt} \\
&= (r+\lambda)(\theta - p(\theta)) \\
&= \frac{r+\lambda}{\lambda} r p(\theta) \qquad \text{(Screening Rate)}
\end{aligned}$$

- Coase Conjecture: As $\lambda \to 0$
  - (Payoff Seller) implies that $V(\theta) \to 0$
  - (Screening Rate) implies that $-\frac{dp}{dt} \to \infty$