Contract Theory: Final

9:00am–12:00pm, 11th December, 2012

1. Set Asides and Subsidies

Two agents, H and L, compete in an auction. Agent H has value $v_H \sim f_H(\cdot)$, while agent L has value $v_L \sim f_L(\cdot)$. The support of both distributions is bounded away from zero, while the seller has value $v_0 = 0$.

A mechanism $\langle p_i(v), t_i(v) \rangle$ gives the probability of agent $i \in \{L, H\}$ winning the good, and their payment. The government wishes to maximize welfare, subject to giving the good to agent L with at least α probability, i.e.,

$$E_v[p_L(v_L, v_H)] \geq \alpha$$

To motivate this, one can view L as a disadvantaged bidder (e.g. a small firm) that the government wishes to keep in business. [Hint: You can solve the problem by ignoring the (IC) constraints and then verifying that there are payments that implement this allocation.]

(a) What is the government's optimal mechanism?

Under a set-aside policy the government sets β such that: with probability β only L competes; and with probability $1 - \beta$ they run a welfare maximizing auction. Under a subsidy policy, the government subsidizes L's bid by a fixed amount (e.g. in a second-price auction).

(b) Does the government prefer a set-aside policy or a subsidy policy? Provide intuition.

2. Contracting under Coercion

A firm wishes to employ an agent. The agent has limited liability so wages are nonnegative, but the firm can punish the agent in a way that is unproductive (e.g. by humiliating them). The firm also can lower the worker's outside option by purchasing "guns" to harass the agent. More precisely, the timing is:

1. The firm chooses the level of guns, g, at cost $k(g)$.

2. The firm offers a contract $\langle w(y), p(y) \rangle$ to the agent, where $y \in \{L, H\} = \{0, 1\}$ is the output, $w \geq 0$ is the wage, and $p \geq 0$ is the nonpecuniary punishment. If the agent rejects the contract, they receive $\overline{u} - g$.

3. If the agent accepts the contract, he chooses effort $a \in [0, 1]$ at cost $c(a)$, giving rise to output $\Pr(y = H) = a$.

Payoffs for the firm and worker are

$$\Pi = zy - w - k(g)$$
$$U = w - p - c(a)$$

where z is the price of output. For simplicity, suppose $k(\cdot)$ and $c(\cdot)$ are increasing, convex and satisfy the appropriate Inada conditions (so we don't have to worry about boundary problems).

(a) Write down the firm's problem subject to the (IR) constraint and the (IC) constraint.

(b) Simplify the problem using the first-order approach. Is this approach valid here?

(c) Argue that the optimal contract has $p_H = 0$ and $w_L = 0$, and that the (IR) constraint binds.

(d) Use the (IR) and (IC) constraints to write profits as follows:

$$\Pi = za - ac(a) + a(1 - a)c'(a) - a\overline{u} + ag - k(g)$$

Henceforth, let us assume Π is concave in a.

For $x \in \Re^n$, $t \in \Re^n$, a function $f(x, t)$ is supermodular in (x, t) if all the cross-partial derivatives are positive. Topkis proved that if $f(x, t)$ is supermodular then the optimal solution

$$x^*(t) = \text{argmax}_x f(x, t)$$

is increasing in the parameter t.

(e) How does the optimal choice of a and g, $(a^*(z, \overline{u}), g^*(z, \overline{u}))$, vary in the price of output and the outside option? Provide an intuition.

3. Robust Trading Mechanisms.

Myerson and Satterthwaite (1982) characterize the set of Bayesian (IC) and Interim (IR) mechanisms with balanced budgets. What allocations could we achieve if we insisted that the mechanism be robust to agents holding a range of beliefs about their opponent?

There is a buyer with value $v \in [0,1]$ and seller with cost $c \in [0,1]$. A mechanism consists of a allocation $q(v,c) \in \{0,1\}$ and a transfer $t(v,c) \geq 0$. The assumption that $q \in \{0,1\}$ will make part (d) easier. Payoffs are

$$U_B(v,c) = q(v,c)v - t(v,c)$$
$$U_S(v,c) = t(v,c) - q(v,c)c$$

Rather than specify beliefs about their opponent's payoffs, we insist on ex-post (IC):

$$q(v,c)v - t(v,c) \geq q(\hat{v},c)v - t(\hat{v},c) \tag{ICB}$$
$$t(v,c) - q(v,c)c \geq t(v,\hat{c}) - q(v,\hat{c})c \tag{ICS}$$

and ex-post (IR)

$$q(v,c)v - t(v,c) \geq 0 \tag{IRB}$$
$$t(v,c) - q(v,c)c \geq 0 \tag{IRS}$$

(a) Show that the (IR) constraints imply $q(v,c) = t(v,c) = 0$ if $v < c$.

To avoid trivialities, we hereafter assume that $v \geq c$.

(b) Show that $q(v,c)$ is increasing in v and decreasing in c.

(c) Using the envelope theorem, show that (ICB) and (ICS) and part (a) imply

$$q(v,c)v - t(v,c) = \int_c^v q(x,c)dx$$
$$t(v,c) - q(v,c)c = \int_c^v q(v,x)dx$$

And therefore that

$$q(v,c)(v-c) = \int_c^v [q(x,c) + q(v,x)]dx \tag{ICBS}$$

(d) Show that (ICBS) implies that any implementable mechanism take the form of a posted price mechanism. That is, there exists a price p such that

$$q(v,c) = 1 \quad \text{iff } v \geq p \geq c$$

To prove this, it's easiest to use a graphical approach. Since allocations are bang-bang, one can plot the trade region in (v,c) space. Under a price mechanism, it should look like a rectangular block. With any other trade region, there should be a point that contradicts (ICBS).