Homework 2: Dynamic Moral Hazard

Due: Wed 20th November

Question 0 (Normal learning model)

Suppose that $z_t = \theta + \epsilon_t$, where $\theta \sim N(m_0, 1/h_0)$ and $\epsilon_t \sim N(0, 1/h_\epsilon)$ are IID. Show that

$$\theta | z_1 \sim N\left(\frac{h_\epsilon z_1}{h_0 + h_\epsilon} + \frac{h_0 m_0}{h_0 + h_\epsilon}, \frac{1}{h_0 + h_\epsilon}\right)$$

Question 1 (Teams and Collusion)

Consider Holmstrom's model of moral hazard in teams. N agents work in a team with joint output $x(a_i, \ldots, a_N)$, where a_i is the effort of agent i and $g(a_i)$ is is the increasing, convex cost function. Utilities are given by $u_i = t_i - g(a_i)$, where t_i is a transfer.

(a) Show that by introducing a principal (agent $N+1$) who does not participate in the production process, we can sustain an efficient effort profile as a Nash equilibrium using a differentiable balanced-budget output-sharing rule, i.e. $\sum_i t_i(x) = x$ $(\forall x)$. The principal's utility is utility $u_{N+1} = t_{N+1}$. Construct the contract so the principal makes zero utility in equilibrium.

(b) Suppose the principal can collude with one agent (call her agent k). That is, the colluders secretly write a side contract based on x to increase their joint payoff (other agents are unaware of the side contract). Show the scheme in (a) is susceptible to collusion.

(c) Suppose we restricted ourselves to differentiable output–sharing schemes that are invulnerable to collusion. Show that it is impossible to sustain the efficient effort profile.

Question 2 (Hidden Savings)

There are two periods. In period 1 the agent (privately) chooses to consume c. In period 2 they choose effort $a \in \{L, H\}$ at cost $g(a)$, where $g(H) > g(L)$. Output is binomial, $q \in \{0, 1\}$, where the probability that $q = 1$ given action $a \in \{L, H\}$ is p_a and $p_H > p_L$. The principal commits to the wage schedule at the start of the game. Wages are paid in period 2: denote the wage paid in state $q \in \{0, 1\}$ by (w_1, w_0).

Suppose the agent's utility is given by

$$u(c_a) + p_a u(w_1 - c_a) + (1 - p_a)u(w_0 - c_a) - g(a)$$

where $u(\cdot)$ is increasing and strictly concave, and c_a is the consumption of the agent in period 1 if they plan to take action a in period 2.

Suppose the principal wishes to implement high effort. The two–period (IC) constraint says that

$$\begin{aligned} u(c_H) + p_H u(w_1 - c_H) + (1 - p_H)u(w_0 - c_H) - g(H) \\ \geq u(c_L) + p_L u(w_1 - c_L) + (1 - p_L)u(w_0 - c_L) - g(L) \end{aligned} \tag{1}$$

(a) Show that $w_1 > w_0$ and $c_H > c_L$.

(b) Use (1) to show that the second–period (IC) constraint (after c_H has been chosen) is slack.

(c) Why does this matter?

Question 3 (Short–term and long–term contracts)

Suppose there are three periods, $t \in \{1, 2, 3\}$. Each period a principal and an agent must share a good; let $x_t \in \mathbb{R}$ be the share obtained by the agent. The principal gets $\sum_t \pi_t(x_t)$ and the agent gets $\sum_t u_t(x_t)$, where $\pi_t(x_t)$ is decreasing in x_t and $u_t(x_t)$ is increasing in x_t. The agent's outside option is a share of the assets $(\underline{x}_1, \underline{x}_2, \underline{x}_3)$.

(a) Suppose the principal can write a long term contract. Write down the program of maximising profit subject to individual rationality.

(b) Now suppose the principal offered a spot contract each period. Using backwards induction derive the optimal sequence of spot contracts. Explain why this may differ from the long–term contract.

(c) Suppose the principal offers two–period contracts. In the first period they offer $({}_1x_1,\ {}_1x_2)$. If it is rejected the agent gets $\underline{x}_1$. At the start of the second period a new contract $({}_2x_2,\ {}_2x_3)$ may be proposed by the principal. If this is rejected the agent gets ${}_1x_2$ if they accepted the first contract or $\underline{x}_2$ otherwise. In the third period a spot contract is offered to the agent. If this is rejected, the agent gets ${}_2x_3$ if they accepted the second contract, or $\underline{x}_3$ otherwise. Show that if $\lim_{x\to-\infty} u_t(x) = -\infty$ and $\lim_{x\to\infty} u_t(x) = \infty$ then this can implement the optimal long term contract.

(d) Provide an example (outside options, utility functions, profit function) where the two–period contracts cannot implement the long–term contract.

Question 4 (Dynamic Contracts with Hidden Wage Offers)

A risk neutral firm employs a risk averse worker. There are infinite periods, with discount rate $\delta \in (0, 1)$.

In period t, the firm's payoff is

$$\pi = q - w_t$$

where q is some fixed output, and w_t is the wage. The worker obtains

$$u(w_t).$$

Each period the worker obtains a wage offer $\overline{w}_t$ with a strictly positive density $f(\cdot)$, distribution $F(\cdot)$ and support $[0, 1]$. These wage offers are IID and are *not observed* by the firm. Denote $\overline{V} = E[u(\overline{w})]/(1 - \delta)$. Assume $q > 1$.

The firm offers the worker a contract $\{w_t\}$ that consists of a series of wages. These do not depend on the outside offers.[1]

Each period proceeds as follows. First, the worker sees the outside wage offer $\overline{w}_t$. Second, the worker chooses whether to quit or stay. If he quits, he never works for the firms again and obtains $u(\overline{w_t}) + \delta\overline{V}$. If he stays, he's paid according to the contract and the game proceeds to the next period.

Notation: Let V equal the agent's promised utility at the start of the period, which includes the value of the outside job, if the worker quits. Let V_+ be promised utility at the end of the period.

(a) The worker quits if his outside wage offer exceeds a threshold, w^*. How is w^* determined?

[1] One might allow the agent to make reports to the firm. We do not allow this here.

(b) Write down the firm's profit $\Pi(V)$ as a function of the wage w and V_+.

(c) Write down the promise keeping constraint. [The PK constraint says that the principal delivers the utility it promises, V].

(d) The firm maximises profit subject to (i) the promise keeping constraint, (ii) w^* being determined by the equation in (a). Assume V is sufficiently large so that $\Pi(V)$ is decreasing. Also assume that $\Pi(V)$ is concave. Show that the optimal choices of w and V_+ are related by the equation

$$-\Pi'(V_+) = \frac{1}{u'(w)}$$

(e) Suppose we are in a steady state, so $V_+ = V$ and wages are constant. Show that the probability of quitting is zero, i.e., $w^* \geq 1$. You can either do this via the FOC from part (d) and the envelope theorem, or from a direct argument.

Question 5 (Relational Contracting)

Suppose a firm employs two workers. It signs a stationary relational contract (w^i, b^i, e^i) with each worker i. The firm gets profit $y(e^i) - W^i$ from each worker, while the agents get $W^i - c^i(e^i)$, where $W^i = w^i + b^i$. Outside utility/profits equal 0.

First, consider a bilateral contract, where deviation by the firm or agent in relationship i leads to Nash reversion in this relationship only.

(a) Characterise the self–enforcing contracts by no deviation constraints on both agents and the principal.

(b) Sum across the constraints to derive conditions on surplus needed to sustain a relationship. [Note: This surplus condition is also sufficient for a contract to be self–enforcing.]

Second, consider a joint contract where deviation by the firm or any worker leads all workers to revert to noncooperation.

(c) Characterise the self–enforcing contracts by no deviation constraints on both agents and the principal.

(d) Sum across the constraints to derive a condition of surplus needed to sustain a relationship. [Note: This surplus condition is also sufficient for a contract to be self–enforcing.]

(e) Show that the total surplus is higher under the joint contract than under bilateral contracts. Intuitively, when is the joint contract strictly better? In this case, why is it better?

Question 6 (Reputation and Exit)

Time is continuous and infinite. A firm has quality $q \in \{L, H\}$. The market believes the firm is high quality with probability $x = \Pr(q = H)$. If the firm has reputation x its instantaneous profits are $x - k$, where k is the fixed cost of operating. The firm also has the option to shut down, obtaining 0 thereafter. Its future discounted profits are

$$\int_{t=0}^{\tau} e^{-rt}(x_t - k)\, dt$$

where r is the interest rate and τ the (random) time when it shuts down.

We suppose the market learns through good news events (e.g. the firm is featured on the front page of the Wall St Journal). If the firm has low quality, a signal never arrives. If the firm has high quality, a signal arrives with probability λdt.

First, suppose the firm *does not* know its own type. Both the firm and the market start with a prior x_0.

(a) Both the market and the firm learn about the firm's quality over time. Verify that reputation evolves as follows. If a signal arrives x jumps to one, otherwise

$$dx = -\lambda x(1-x)dt$$

(b) Denote the firm's value function by $V(x)$. Taking $V(\cdot)$ as given, at what reputation x^* does the firm decide to exit? Will the firm be making a loss at this exit point?

(c) Calculate $V(1)$.

(d) Characterise $V(x)$ in the form of a differential equation.

Next, suppose the firm *does* know its own type.

(e) Denote the firms' value functions by $V_L(x)$ and $V_H(x)$. Characterise the equilibrium exit decisions for the low and high quality firms. Will the low quality firm be making a loss at her exit point, x_L^*?

(f) Characterise $V_L(x)$ and $V_H(x)$ in the form of differential equations.