Homework 2: Dynamic Moral Hazard

0. Normal learning model

Suppose that $z_t = \theta + \epsilon_t$, where $\theta \sim N(m_0, 1/h_0)$ and $\epsilon_t \sim N(0, 1/h_\epsilon)$ are IID. Show that

$$\theta | z_1 \sim N\left(\frac{h_\epsilon z_1}{h_0 + h_\epsilon} + \frac{h_0 m_0}{h_0 + h_\epsilon}, \frac{1}{h_0 + h_\epsilon}\right)$$

1. Teams and Collusion

Consider Holmstrom's model of moral hazard in teams. N agents work in a team with joint output $x(a_i, \ldots, a_N)$, where a_i is the effort of agent i and $g(a_i)$ is is the increasing, convex cost function. Utilities are given by $u_i = t_i - g(a_i)$, where t_i is a transfer.

(a) Show that by introducing a principal (agent $N+1$) who does not participate in the production process, we can sustain an efficient effort profile as a Nash equilibrium using a differentiable balanced-budget output-sharing rule, i.e. $\sum_i t_i(x) = x$ $(\forall x)$. The principal's utility is utility $u_{N+1} = t_{N+1}$. Construct the contract so the principal makes zero utility in equilibrium.

(b) Suppose the principal can collude with one agent (call her agent k). That is, the colluders secretly write a side contract based on x to increase their joint payoff (other agents are unaware of the side contract). Show the scheme in (a) is susceptible to collusion.

(c) Suppose we restricted ourselves to differentiable output–sharing schemes that are invulnerable to collusion. Show that it is impossible to sustain the efficient effort profile.

2. Hidden Savings

There are two periods. In period 1 the agent (privately) chooses to consume c. In period 2 they choose effort $a \in \{L, H\}$ at cost $g(a)$, where $g(H) > g(L)$. Output is binomial, $q \in \{0, 1\}$, where the probability that $q = 1$ given action $a \in \{L, H\}$ is p_a and $p_H > p_L$. The principal commits to the wage schedule at the start of the game. Wages are paid in period 2: denote the wage paid in state $q \in \{0, 1\}$ by (w_1, w_0).

Suppose the agent's utility is given by

$$u(c_a) + p_a u(w_1 - c_a) + (1 - p_a)u(w_0 - c_a) - g(a)$$

where $u(\cdot)$ is increasing and strictly concave, and c_a is the consumption of the agent in period 1 if they plan to take action a in period 2.

Suppose the principal wishes to implement high effort. The two–period (IC) constraint says that

$$\begin{aligned} u(c_H) + p_H u(w_1 - c_H) + (1 - p_H)u(w_0 - c_H) - g(H) \\ \geq u(c_L) + p_L u(w_1 - c_L) + (1 - p_L)u(w_0 - c_L) - g(L) \end{aligned} \tag{1}$$

(a) Show that $w_1 > w_0$ and $c_H > c_L$.

(b) Use (1) to show that the second–period (IC) constraint (after c_H has been chosen) is slack.

(c) Why does this matter?

3. Dynamic Contracts and Disaster Prevention

A firm employs a single agent. Time is continuous $t \in [0, \infty)$. Both players are risk neutral. The agent has discount rate ρ, while the firm has discount rate r. Assume $\rho > r$.

The agent exerts private effort to avert a disaster (e.g. an oil spill, or a product explosion). The agent chooses effort $a_t \in \{0, 1\}$ corresponding to low/high effort. Under high effort, a disaster arrives according to Poisson rate λ; under low effort a disaster arrives at Poisson rate $\lambda + \delta$. If the agent chooses low effort, he also gains flow benefit b. Let N_t be the number of disasters that have occurred by time t. The agent's flow wage is $w_t \geq 0$. The firm earns μ flow profits but loses K every time there is a disaster. If the firm fires the agent and liquidates the project then both parties get zero thereafter.

The agent's wealth at time t is thus

$$V_t = E_t \left[\int_t^\tau e^{-\rho(s-t)} [w_s + (1 - a_s)b] ds \right]$$

where τ is the (random) liquidation time. The firm's time t profits are

$$\Pi_t = E_t \left[\int_t^{\tau} e^{-r(s-t)} [\mu ds - w_s ds - K dN_s] \right]$$

The firm chooses to a contract $\{w_t, \tau\}$ to maximize her time-0 profits. Assume that δ is sufficiently large that she wishes to implement high effort, $a_t = 1$, at all times.

(a) Observe that a contract can be described recursively using V as a state variable. Write the agent's HJB equation and use it to derive a condition for high effort to be (IC). When evaluating how the agent's wealth changes over time, you will find it useful to separate between the increase if there is no disaster, denoted $\dot{V}$, and the decrease if there is a disaster, denoted H.

(b) Let the firm's profit function be denoted $\Pi(V)$. Write the firm's HJB assuming $a_t = 1$ (and ignoring (IC)).

(c) Assume $\Pi(V)$ is concave. Using the agent's HJB to substitute for $\dot{V}$ in the firm's HJB, argue that there is a V^* such that the the firm only pays the agent when his wealth exceeds this threshold.

(d) Using (IC), what is the optimal punishment H?

(e) When is the project shut down?

4. Dynamic Contracts with Hidden Wage Offers

A risk neutral firm employs a risk averse worker. There are infinite periods, with discount rate $\delta \in (0, 1)$.

In period t, the firm's payoff is

$$\pi = q - w_t$$

where q is some fixed output, and w_t is the wage. The worker obtains

$$u(w_t).$$

Each period the worker obtains a wage offer $\overline{w}_t$ with a strictly positive density $f(\cdot)$, distribution

$F(\cdot)$ and support $[0,1]$. These wage offers are IID and are *not observed* by the firm. Denote $\overline{V} = E[u(\overline{w})]/(1-\delta)$. Assume $q > 1$.

The firm offers the worker a contract $\{w_t\}$ that consists of a series of wages. These do not depend on the outside offers.[1]

Each period proceeds as follows. First, the worker sees the outside wage offer $\overline{w_t}$. Second, the worker chooses whether to quit or stay. If he quits, he never works for the firms again and obtains $u(\overline{w_t}) + \delta\overline{V}$. If he stays, he's paid according to the contract and the game proceeds to the next period.

Notation: Let V equal the agent's promised utility at the start of the period, which includes the value of the outside job, if the worker quits. Let V_+ be promised utility at the end of the period.

(a) The worker quits if his outside wage offer exceeds a threshold, w^*. How is w^* determined?

(b) Write down the firm's profit $\Pi(V)$ as a function of the wage w and V_+.

(c) Write down the promise keeping constraint. [The PK constraint says that the principal delivers the utility it promises, V].

(d) The firm maximises profit subject to (i) the promise keeping constraint, (ii) w^* being determined by the equation in (a). Assume V is sufficiently large so that $\Pi(V)$ is decreasing. Also assume that $\Pi(V)$ is concave. Show that the optimal choices of w and V_+ are related by the equation

$$-\Pi'(V_+) = \frac{1}{u'(w)}$$

(e) Suppose we are in a steady state, so $V_+ = V$ and wages are constant. Show that the probability of quitting is zero, i.e., $w^* \geq 1$. You can either do this via the FOC from part (d) and the envelope theorem, or from a direct argument.

5. Relational Contracting

Suppose a firm employs two workers. It signs a stationary relational contract (w^i, b^i, e^i) with each worker i. The firm gets profit $y(e^i) - W^i$ from each worker, while the agents get $W^i - c^i(e^i)$,

[1]One might allow the agent to make reports to the firm. We do not allow this here.

where $W^i = w^i + b^i$. Outside utility/profits equal 0.

First, consider a bilateral contract, where deviation by the firm or agent in relationship i leads to Nash reversion in this relationship only.

(a) Characterise the self–enforcing contracts by no deviation constraints on both agents and the principal.

(b) Sum across the constraints to derive conditions on surplus needed to sustain a relationship. [Note: This surplus condition is also sufficient for a contract to be self–enforcing.]

Second, consider a joint contract where deviation by the firm or any worker leads all workers to revert to noncooperation.

(c) Characterise the self–enforcing contracts by no deviation constraints on both agents and the principal.

(d) Sum across the constraints to derive a condition of surplus needed to sustain a relationship. [Note: This surplus condition is also sufficient for a contract to be self–enforcing.]

(e) Show that the total surplus is higher under the joint contract than under bilateral contracts. Intuitively, when is the joint contract strictly better? In this case, why is it better?

6. Reputation and Exit

Time is continuous and infinite. A firm has quality $q \in \{L, H\}$. The market believes the firm is high quality with probability $x = \Pr(q = H)$. If the firm has reputation x its instantaneous profits are $x - k$, where k is the fixed cost of operating. The firm also has the option to shut down, obtaining 0 thereafter. Its future discounted profits are

$$\int_{t=0}^{\tau} e^{-rt}(x_t - k)\, dt$$

where r is the interest rate and τ the (random) time when it shuts down.

We suppose the market learns through good news events (e.g. the firm is featured on the front page of the Wall St Journal). If the firm has low quality, a signal never arrives. If the firm has high quality, a signal arrives with probability λdt.

First, suppose the firm *does not* know its own type. Both the firm and the market start with a prior x_0.

(a) Both the market and the firm learn about the firm's quality over time. Verify that reputation evolves as follows. If a signal arrives x jumps to one, otherwise

$$dx = -\lambda x(1-x)dt$$

(b) Denote the firm's value function by $V(x)$. Taking $V(\cdot)$ as given, at what reputation x^* does the firm decide to exit? Will the firm be making a loss at this exit point?

(c) Calculate $V(1)$.

(d) Characterise $V(x)$ in the form of a differential equation.

Next, suppose the firm *does* know its own type.

(e) Denote the firms' value functions by $V_L(x)$ and $V_H(x)$. Characterise the equilibrium exit decisions for the low and high quality firms. Will the low quality firm be making a loss at her exit point, x_L^*?

(f) Characterise $V_L(x)$ and $V_H(x)$ in the form of differential equations.