Homework 3: Asymmetric Information

1. Public Goods Provision

A firm is considering building a public good (e.g. a swimming pool). There are n agents in the economy, each with IID private value $\theta_i \in [0,1]$. Agents' valuations have density $f(\theta)$ and distribution $F(\theta)$. Assume that

$$MR(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)}$$

is increasing in θ. The cost of the swimming pool is cn, where $c > 0$.

First suppose the government passes a law that says the firm cannot exclude people from entering the swimming pool. A mechanism thus consists of a build decision $P(\theta_1, \dots, \theta_n) \in [0,1]$ and a payment by each agent $t_i(\theta_1, \dots, \theta_n) \in \Re$. The mechanism must be individually rational and incentive compatible. [Note: When showing familiar results your derivation can be heuristic.]

(a) Consider an agent with type θ_i, whose utility is given by

$$\theta_i P - t_i$$

Derive her utility in a Bayesian incentive compatible mechanism.

(b) Given an build decision $P(\cdot)$, derive the firm's profits.

(c) What is the firm's optimal build decision?

(d) Show that $E[MR(\theta)] = 0$.

(e) Show that as $n \to \infty$, so the probability of provision goes to zero. [You might wish to use the Chebyshev inequality, which says that $\Pr(|Z - E[Z]| \geq \alpha) \leq \frac{\mathrm{Var}(Z)}{\alpha^2}$ for a random variable Z.]

Next, suppose the firm can exclude agents. A mechanism now consists of a build decision $P(\theta_1, \dots, \theta_n) \in [0,1]$, a participation decision for each agent $x_i(\theta_1, \dots, \theta_n) \in [0,1]$ and a payment $t_i(\theta_1, \ldots, \theta_n) \in \Re$. Agent i's utility is now given by

$$\theta_i x_i P - t_i$$

The cost is still given by cn, where n is the number of agents in the population.

(f) Solve for the firm's optimal build decision $P(\cdot)$ and participation rule $x_i(\cdot)$.

(g) Suppose $n \to \infty$. Show there exists a cutoff c^* such that the firm provides the pool with probability one if $c < c^*$, and with probability zero if $c > c^*$.

2. Costly State Verification

There is a risk–neutral entrepreneur E who has a project with privately observed return y with density $f(y)$ on $[0, Y]$. The project requires investment $I < E[y]$ from an outside creditor C.

A contract is defined by a pair $(s(y), B(y))$ consisting of payment and verification decision. If an agent reports y they pay $s(y) \leq y$ and are verified if $B(y) = 1$ and not verified if $B(y) = 0$. If the creditor verifies E they pay exogenously given cost c and get to observe E's type.

The game is as follows:

- E chooses $(s(y), B(y))$ to raise I from a competitive financial market.
- Output y is realised.
- E claims the project yields $\hat{y}$. If $B(\hat{y}) = 0$ then E pays $s(\hat{y})$ and is not verified. If $B(\hat{y}) = 1$ then C pays c and observes E's true type. If they are telling the truth they pay $s(y)$; if not, then C can take everything.
- Payoffs. E gets $y - s(y)$, while C gets $s(y) - cB(y) - I$.

(a) Show that a contract is incentive compatible if and only if there exists a D such that $s(y) = D$ when $B(y) = 0$ and $s(y) \leq D$ when $B(y) = 1$.

Consider E's problem:

$$\begin{aligned}
\max_{s(y),B(y)} \quad & E[y - s(y)] \\
\text{s.t.} \quad & s(y) \leq y \qquad (MAX) \\
& E[s(y) - cB(y) - I] \geq 0 \qquad (IR) \\
& s(y) \leq D \quad \forall y \in B^V \qquad (IC1) \\
& s(y) = D \quad \forall y \notin B^V \qquad (IC2)
\end{aligned}$$

where B^V is the verification region (where $B(y) = 1$).

(b) Show that constraint (IR) must bind at the optimum. [Hint: Proof by contradiction.]

Now E's problem becomes

$$\begin{aligned}
\min_{s(y),B(y)} \quad & E[cB(y)] \\
\text{s.t.} \quad & (MAX),\ (IC1),\ (IC2) \\
& E[s(y) - cB(y) - I] = 0 \qquad (IR)
\end{aligned}$$

(c) Show that any optimal contract $(s(y), B(y))$ has a verification range of the form $B^V = [0, D]$ for some D. [Hint: Proof by contradiction.]

(d) Show that any optimal contract $(s(y), B(y))$ sets $s(y) = y$ when $B(y) = 1$. [Hint: Proof by contradiction.]

(e) A contract is thus characterised by D. Which D maximises E's utility? Can you give a financial interpretation to this contract?

3. Ironing

Consider the continuous–type price discrimination problem from class, where the principal chooses $q(\theta)$ to maximise

$$E[q(\theta)MR(\theta) - c(q(\theta))]$$

subject to $q(\theta)$ increasing in θ.

For $v \in [0,1]$, let
$$H(v) = \int_0^v MR(F^{-1}(x))dx$$
be the expected marginal revenue up to $\theta = F^{-1}(v)$. Let $\overline{H}(v)$ be the highest convex function under $H(v)$. Then define $\overline{MR}(\theta)$ by
$$\overline{H}(v) = \int_0^v \overline{MR}(F^{-1}(x))dx$$
Finally, let $\Delta(\theta) = H(F(\theta)) - \overline{H}(F(\theta))$.[1]

(a) Argue that $\Delta(\theta) > 0$ implies $\overline{MR}(\theta)$ is flat. Also argue that $\Delta(\underline{\theta}) = \Delta(\overline{\theta}) = 0$.

(b) Since $q(\theta)$ is an increasing function, show that
$$E[q(\theta)MR(\theta) - c(q(\theta))] = E[q(\theta)\overline{MR}(\theta) - c(q(\theta))] - \int_{\underline{\theta}}^{\overline{\theta}} \Delta(\theta)dq(\theta)$$

(c) Derive the profit–maximising allocation $q(\theta)$.

4. Financing of Investments

An entrepreneur (E) has assets $\theta \sim G[\underline{\theta}, \overline{\theta}]$ that it privately knows. It also has an opportunity to make a further investment that yields random returns v that are independent of θ. In order to make this investment, E needs an investor (I) to inject funds. The two parties can only contract on the total output of the firm, $y = \theta + v$. Suppose $y \sim f(\cdot|\theta)$ obeys MLRP.

We suppose I offers E a menu of contracts. Using the revelation principle, we can denote the contracts $\{t(y, \tilde{\theta})\}$, where $\tilde{\theta}$ is the agent's report. We consider contracts $t \in [0, y]$ that are weakly increasing in y. If E does not invest, he makes $U^{NI} = \theta$; if E does invest he makes $U^I = E[y - t(y, \tilde{\theta})|\theta]$ and I makes $\pi = E[t(y, \tilde{\theta})]$.

(a) Show that any mechanism $\{t(y, \tilde{\theta})\}$ induces investment from an interval of the form $[\underline{\theta}, \hat{\theta}]$.

Suppose I wishes to implement a cutoff $\hat{\theta}$.

[1] Note, it is important that we take the convex hull in quantile space. If we use θ–space, then $\Delta(\theta) > 0$ implies $\overline{MR}(\theta)f(\theta)$ is flat, which is not particularly useful.

(b) Argue that I can limit herself to contracts $\{t(y)\}$ that are independent of E's report

(c) A debt contract is defined by $t^D(y) = \min\{y, D\}$. Show that a debt contract is the cheapest way to implement $\hat{\theta}$. That is, pick any arbitrary contract $\{t(y)\}$ such that the same types invest, and show that moving to a debt contract raises expected payments to I.

[Useful fact: A function $\phi(y)$ is single-crossing in y if $\phi(y) \leq 0$ for $y < y^*$ and $\phi(y) \geq 0$ for $y > y^*$. If $y \sim f(\cdot|\theta)$ obeys MLRP and $\phi(y)$ is single-crossing in y, then $E[\phi(y)|\theta]$ is single-crossing in θ.]

5. Dynamic Mechanism Design

A firm sells to a customer over $T = 2$ periods. There is no discounting.

The consumer's per-period utility is

$$u = \theta q - p$$

where $q \in \Re$ is the quantity of the good, and p is the price. The agent's type $\theta \in \{\theta_L, \theta_H\}$ is privately known. In period 1, $\Pr(\theta = \theta_H) = \mu$. In period 2, the agent's type may change. With probability $\alpha > 1/2$, her type remains the same; with probability $1 - \alpha$ her type switches (so a high type becomes a low type, or a low type becomes a high type).

The firm chooses a mechanism to maximise the sum of its profits. The per-period profit is given by

$$\pi = p - \frac{1}{2}q^2$$

A mechanism consists of period 1 allocations $\langle q_L, q_H \rangle$, period 2 allocations $\langle q_{LL}, q_{LH}, q_{HL}, q_{HH} \rangle$, and corresponding prices, where q_{LH} is the quantity allocated to an agent who declares L in period 1 and H in period 2.

(a) Consider period $t = 2$. Fix the first period type, θ. Assume in period 2 that the low-type's (IR) constraint binds, the high type's (IC) constraint binds and we can ignore the other constraints. Characterize the second period rents obtained by the agents, $U_{\theta L}$ and $U_{\theta H}$, as a function of $\{q_{LL}, q_{LH}, q_{HL}, q_{HH}\}$

(b) Consider period $t = 1$. Assume the low-type's (IR) constraint binds, the high type's (IC) constraint binds and we can ignore the other constraints. Derive the lifetime rents obtained by the agents, U_L and U_H, as a function of $\{q_L, q_H, q_{LL}, q_{LH}, q_{HL}, q_{HH}\}$.

(c) Derive the firm's total expected profits.

(d) Assume the firm does not want to exclude, i.e. that $\Delta := \theta_H - \theta_L$ is sufficiently small. Derive the profit-maximizing allocations $\{q_L, q_H, q_{LL}, q_{LH}, q_{HL}, q_{HH}\}$. In particular, show that q_{HL} is first-best. Can you provide an intuition for this result?

(Bonus) Suppose T is arbitrary. Can you derive the form of the optimal mechanism?

6. Robust Trading Mechanisms.

Myerson and Satterthwaite (1982) characterize the set of Bayesian (IC) and Interim (IR) mechanisms with balanced budgets. What allocations could we achieve if we insisted that the mechanism be robust to agents holding a range of beliefs about their opponent?

There is a buyer with value $v \in [0,1]$ and seller with cost $c \in [0,1]$. A mechanism consists of a allocation $q(v,c) \in \{0,1\}$ and a transfer $t(v,c) \geq 0$. The assumption that $q \in \{0,1\}$ will make part (d) easier. Payoffs are

$$U_B(v,c) = q(v,c)v - t(v,c)$$
$$U_S(v,c) = t(v,c) - q(v,c)c$$

Rather than specify beliefs about their opponent's payoffs, we insist on ex-post (IC):

$$q(v,c)v - t(v,c) \geq q(\hat{v},c)v - t(\hat{v},c) \qquad \text{(ICB)}$$
$$t(v,c) - q(v,c)c \geq t(v,\hat{c}) - q(v,\hat{c})c \qquad \text{(ICS)}$$

and ex-post (IR)

$$q(v,c)v - t(v,c) \geq 0 \qquad \text{(IRB)}$$
$$t(v,c) - q(v,c)c \geq 0 \qquad \text{(IRS)}$$

(a) Show that the (IR) constraints imply $q(v,c) = t(v,c) = 0$ if $v < c$.

To avoid trivialities, we hereafter assume that $v \geq c$.

(b) Show that $q(v,c)$ is increasing in v and decreasing in c.

(c) Using the envelope theorem, show that (ICB) and (ICS) and part (a) imply

$$q(v,c)v - t(v,c) = \int_c^v q(x,c)dx$$
$$t(v,c) - q(v,c)c = \int_c^v q(v,x)dx$$

And therefore that

$$q(v,c)(v-c) = \int_c^v [q(x,c) + q(v,x)]dx \tag{ICBS}$$

(d) Show that (ICBS) implies that any implementable mechanism take the form of a posted price mechanism. That is, there exists a price p such that

$$q(v,c) = 1 \quad \text{iff } v \geq p \geq c$$

To prove this, it's easiest to use a graphical approach. Since allocations are bang-bang, one can plot the trade region in (v,c) space. Under a price mechanism, it should look like a rectangular block. With any other trade region, there should be a point that contradicts (ICBS).