Practice Problems 1: Moral Hazard

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1. Contracting under Coercion

A firm wishes to employ an agent. The agent has limited liability so wages are nonnegative, but the firm can punish the agent in a way that is unproductive (e.g. by humiliating them). The firm also can lower the worker's outside option by purchasing "guns" to harass the agent. More precisely, the timing is:

1. The firm chooses the level of guns, g, at cost $k(g)$.
2. The firm offers a contract $\langle w(y), p(y)\rangle$ to the agent, where $y \in \{L, H\} = \{0, 1\}$ is the output, $w \geq 0$ is the wage, and $p \geq 0$ is the nonpecuniary punishment. If the agent rejects the contract, they receive $\overline{u} - g$.
3. If the agent accepts the contract, he chooses effort $a \in [0, 1]$ at cost $c(a)$, giving rise to output $\Pr(y = H) = a$.

Payoffs for the firm and worker are

$$\Pi = zy - w - k(g)$$
$$U = w - p - c(a)$$

where z is the price of output. For simplicity, suppose $k(\cdot)$ and $c(\cdot)$ are increasing, convex and satisfy the appropriate Inada conditions (so we don't have to worry about boundary problems).

(a) Write down the firm's problem subject to the (IR) constraint and the (IC) constraint.

(b) Simplify the problem using the first-order approach. Is this approach valid here?

(c) Argue that the optimal contract has $p_H = 0$ and $w_L = 0$, and that the (IR) constraint binds.

(d) Use the (IR) and (IC) constraints to write profits as follows:

$$\Pi = za - ac(a) + a(1 - a)c'(a) - a\overline{u} + ag - k(g)$$

Henceforth, let us assume Π is concave in a.

For $x \in \Re^n$, $t \in \Re^n$, a function $f(x,t)$ is supermodular in (x,t) if all the cross-partial derivatives are positive. Topkis proved that if $f(x,t)$ is supermodular then the optimal solution

$$x^*(t) = \text{argmax}_x f(x,t)$$

is increasing in the parameter t.

(e) How does the optimal choice of a and g, $(a^*(z,\overline{u}), g^*(z,\overline{u}))$, vary in the price of output and the outside option? Provide an intuition.

2. Moral Hazard and Option Contracts

A principal (P) and an agent (A) play the following game.

1. P announces an option contract (T, B).
2. A accepts or rejects the contract. Rejection yields utility $\overline{U}$
3. A chooses effort e^A. This action is observable but not verifiable. Effort costs the agent e^A and yields revenue $R(e^A)$, where $R(\cdot)$ is increasing and concave.
4. P chooses whether to keep the project or sell it to the agent. If he keeps the project, he pays the agent T and payoffs are

   $$U_P = R(e_A) - T \qquad U_A = T - e_A$$

   If P sells the project to the agent, he receives B and payoffs are

   $$U_P = B \qquad U_A = R(e_A) - B - e_A$$

Let e_A^* maximise $R(e_A) - e_A$. A contract is first–best if it implements e_A^* and yields the agent utility $U_A = \overline{U}$.

Let $B = R(e_A^*) - T$ and $T - e_A^* = \overline{U}$. Show this contract implements the first–best. Provide an intuition

3. Debt Contracts

An entrepreneur has access to a project requiring one unit of capital. If taken, the project succeeds with probability p and produces output $R(p)$, or fails with probability $1-p$ and produces 0. The entrepreneur can costlessly choose $p \in [0,1]$. This choice is unobservable to investors.

The entrepreneur is risk neutral and has initial wealth $w \in [0,1]$. The entrepreneur must raise the additional capital by issuing debt to perfectly competitive risk neutral investors.[1] This debt is secured only by the assets of the project. Both the investors and the entrepreneur have available a safe investment paying an interest rate 0 if they do not invest.

(a) For $w \in [0,1]$, determine the equation that defines the equilibrium relationship between w and p. (Assume an interior solution for p).

(b) Let $R(p) = 5 - 4p$. If $w = 1$, what value of p would the entrepreneur choose? If instead, $w \in (\frac{7}{32}, 1)$, show there are 2 possible equilibrium choices for p. Which of these solutions is more reasonable? What happens if $w < \frac{7}{32}$?

(c) Let $R(p) = 5 - 4p$. Plot the entrepreneur's expected final wealth as a function of initial wealth $w \in [0,1]$. Discuss the effect of agency costs on the return to wealth.

4. Credible Wage Paths

There are two periods, with no discounting. The firm proposes a contract (w_0, w_s) which the agent accepts if the sum of period 1 and period 2 utilities exceeds $\overline{u}$ in expectation. Their utility function is given by the increasing, strictly concave function $u(\cdot)$.

In the first period the worker gets paid w_0 (if they accept the contract). They then produce q for the firm.

In the second period, the state of the world $s \in S$ is the realised with probability f_s. The firm offers w_s, while there is an outside offer, $\overline{w}_s$. The worker accepts the larger. If they work for the firm, the worker produces $q > \max_s \overline{w}_s$.

[1] A debt contract states that the first D dollars from the project goes to the investors.

(a) The firms problem is to maximise two–period profits subject to the first–period and second–period (IR) constraints. Write down this problem.

(b) Characterise the optimal wage path. If s is the state of the economy, how are wage affected by slumps and booms?

(c) Suppose the agent can commit to his period 2 behaviour in period 1. Describe the optimal contract.

5. Motivating Information Acquisition

A potential house buyer (principal) hires a real estate broker (agent) to collect information about a house. The house has quality $q \in \{L, H\}$. A high quality house delivers utility 1 to the principal, a low quality house delivers utility -1 (this is net of the price paid). The prior is $\Pr(q = H) = \gamma$. Both the agent's and principal's utility are quasi-linear.

The agent invests effort e at cost $c(e)$ into observing a signal $s \in \{G, B\}$. The signal is informative with probability

$$\Pr(s = G|q = H) = \Pr(s = B|q = L) = \frac{1}{2} + e =: \eta(e)$$

The signal provides 'hard' information, so the agent cannot lie about the value of the signal. The cost function $c(e)$ is increasing and convex, and obeys $c'''(e) > 0$. To obtain internal optima assume that $c'(0) = 0$, $c''(0) = 0$ and $\lim_{e \to 1/2} c(e) = \infty$.

After observing the signal, the principal can choose to buy the house or not. If her decision to buy is independent of the signal, there is no reason to have the agent exert effort. Hence we assume the principal buys if $s = G$ and does not buy if $s = B$.

(a) Suppose the principal can observe the agent's effort choice. Show the welfare maximising effort satisfies the first order condition $c'(e) = 1$. [Here, welfare is the sum of the agents and principal's utility].

Now, consider the second best contract, where e is not observed by the principal. A contract consists of a wage $w_G \geq 0$ when the good signal is observed, and a wage $w_B \geq 0$ when the bad signal is observed [Note the limited liability constraint; there is no other (IR) constraint].

(b) Write down the agent's utility. Show the agent's optimal level of effort satisfies the first order condition $(w_G - w_B)(2\gamma - 1) = c'(e)$.

(c) Using the FOC in (b), what can we say about how the optimal wages change in the prior, γ? Is it always possible to motivate positive effort? Provide an intuition.

(d) Suppose $\gamma > 1/2$. Replacing the agent's (IC) constraint with their first-order condition, show the principal's optimal effort satisfies the first order condition

$$1 = c''(e)\left[e + \frac{1}{2(2\gamma - 1)}\right] + c'(e)$$

so that effort is increasing in γ.

6. Debt Contracts and Information Acquisition

An Entrepreneur seeks financing I from a competitive market of Investors. E has a project that pays off random return q that is observed by both players ex-post, but not before the contract $\langle r(q) \rangle$ is signed.

Unlike previous models, the project will take place whether or not the investor makes the investment. Rather, the funding is used to pay E, who gains utility αI. We assume $\alpha \geq 1$, so that investment is efficient.

If the investor makes the investment, she is repayed according to $r(q) \in [0, q]$. Hence her payoff is $U_I = r(q) - I$, while E's utility is $U_E = q - r(q) + \alpha I$. In the first-best contract, any feasible $r(q)$ such that $E[r(q)] = I$ maximises E's utility.

(a) Suppose that, just before they sign the contract, the investor can pay to observe the success of the project, q. Suppose we wish to choose $r(q)$ to minimize the incentive for the investor to acquire information (i.e. minimize the increase in their payoffs obtained by acquiring). Argue that the optimal contract has $r(q) \geq \min\{q, I\}$ and therefore that a debt contract is optimal (although not necessarily uniquely optimal).

(b) Now suppose that, just before they sign the contract, the Entrepreneur can pay to observe the success of the project, q. Suppose we wish to choose $r(q)$ to minimize the incentive for the entrepreneur to acquire information. Again, argue that a debt contract is optimal.

(c) Suppose the cost of information acquisition are c_E and c_I for the entrepreneur and investor. What is the highest level of investment I that is sustainable if we do not wish either party to acquire information?

[Note: If you get stuck, you might find it easier to work with $\alpha = 1$.]

7. Moral Hazard with Persistent Effort

An agent chooses effort $e \in \{e_L, e_H\}$ at time 0 at cost $c(e) \in \{0, c\}$. At time $t \in \{1, 2\}$, output $y_t \in \{y_1, \ldots, y_N\}$ is realized according to the IID distribution $\Pr(y_t = y_n|e) = f(y_n|e)$.

A contract is a pair of wages $\langle w_1(y_1), w_2(y_1, y_2)\rangle$. The agent's utility is then

$$u(w_1(y_1)) + u(w_2(y_1, y_2)) - c(e)$$

where $u(\cdot)$ is increasing and concave, while the firm's profits are

$$y_1 + y_2 - w_1(y_1) - w_2(y_1, y_2)$$

where we ignore discounting. The agent has outside option $2u_0$. Also, assume the principal wishes to implement effort e_H.

(a) What is the first best contract, assuming effort is observable? [Note: A formal derivation is not necessary].

(b) Suppose the firm cannot observe the agent's effort. Set up the firm's problem.

(c) Characterise the optimal first-period and second-period wages.

(d) Suppose output is binomial, $y_t \in \{y_L, y_H\}$. Let $f(y_H|e_L) = \pi_L$ and $f(y_H|e_H) = \pi_H$. How do wages vary over time? In particular, can you provide a full ranking of wages across the different states and time periods?

8. Short–term and long–term contracts

Suppose there are three periods, $t \in \{1, 2, 3\}$. Each period a principal and an agent must share a good; let $x_t \in \mathbb{R}$ be the share obtained by the agent. The principal gets $\sum_t \pi_t(x_t)$ and the

agent gets $\sum_t u_t(x_t)$, where $\pi_t(x_t)$ is decreasing in x_t and $u_t(x_t)$ is increasing in x_t. The agent's outside option is a share of the assets $(\underline{x}_1, \underline{x}_2, \underline{x}_3)$.

(a) Suppose the principal can write a long term contract. Write down the program of maximising profit subject to individual rationality.

(b) Now suppose the principal offered a spot contract each period. Using backwards induction derive the optimal sequence of spot contracts. Explain why this may differ from the long–term contract.

(c) Suppose the principal offers two–period contracts. In the first period they offer $({}_1x_1,\ {}_1x_2)$. If it is rejected the agent gets $\underline{x}_1$. At the start of the second period a new contract $({}_2x_2,\ {}_2x_3)$ may be proposed by the principal. If this is rejected the agent gets ${}_1x_2$ if they accepted the first contract or $\underline{x}_2$ otherwise. In the third period a spot contract is offered to the agent. If this is rejected, the agent gets ${}_2x_3$ if they accepted the second contract, or $\underline{x}_3$ otherwise. Show that if $\lim_{x\to-\infty} u_t(x) = -\infty$ and $\lim_{x\to\infty} u_t(x) = \infty$ then this can implement the optimal long term contract.

(d) Provide an example (outside options, utility functions, profit function) where the two–period contracts cannot implement the long–term contract.

9. Teamwork and Tournaments

Two agents work in a team. They simultaneously choose effort e_i at cost $c(e_i)$, yielding output $y(e_1+e_2)$. Utility is linear, so an agent's utility equals the share of the output they receive minus the cost of effort. Assume c is differentiable, strictly convex with $c'(0) = 0$ and $\lim_{e\to\infty} c'(e) = \infty$. Assume y is differentiable and concave.

(a) Write down the FOC for the first-best effort, e^*

Now, suppose there is a measure of which agent produces the most output. Given (e_1, e_2), agent 1 wins with probability $p(e_1, e_2)$ and agent 2 wins with $1 - p(e_1, e_2)$. Suppose we give the winner share s of the output, while the loser gains share $1 - s$. Assume p is increasing in e_1, differentiable and symmetric in (e_1, e_2), so that $p(e_1, e_2) = 1 - p(e_2, e_1)$

(b) Write down the FOC for the agent's problem. Derive an expression for s in order to implement the first-best.

(c) Suppose $y = \alpha(e_1 + e_2)$, $c(e) = e^2/2$ and $p = \frac{e_1^r}{e_1^r + e_2^r}$, where $r \geq 1$ reflects the responsiveness of the signal to effort. Which shares $(s, 1-s)$ implement first-best?