Practice Problems 2: Asymmetric Information

October 12, 2015

1 Single-Agent Problems

1. Nonlinear Pricing with Two Types

Suppose a seller of wine faces two types of customers, θ_1 and θ_2, where $\theta_2 > \theta_1$. The proportion of type θ_1 agents is $\pi \in [0, 1]$. Let q be the quality of the wine and t the price.

Let type θ_1 buy contract (q_1, t_1) and type θ_2 buy (q_2, t_2). The cost of production is zero, $c(q) = 0$, and the seller maximises profit $\pi t_1 + (1 - \pi)t_2$

(a) Suppose agent θ_i has utility

$$u(\theta_i) = \theta_i q - \frac{1}{2}q^2 - t$$

Derive the first–best and profit–maximising qualities.

(b) Suppose agent θ_i has utility

$$u(\theta_i) = \theta_i(q - \frac{1}{2}q^2) - t$$

Derive the first–best and profit–maximising qualities.

2. Nonlinear Pricing with Three Types

Consider the nonlinear pricing model with three types, $\theta_3 > \theta_2 > \theta_1$. The utility of agent θ_i is

$$u(\theta_i) = \theta_i q - t$$

Denote the bundle assigned to agent θ_i by (q_i, t_i). We now have six (IC) constraint and three (IR) constraints. For example, (IC_1^2) says that θ_1 must not want to copy θ_2, i.e.

$$\theta_1 q_1 - t_1 \geq \theta_1 q_2 - t_2 \qquad (\text{IC}_1^2)$$

The firm's profit is

$$\sum_{i=1}^{3} \pi_i [t_i - c(q_i)]$$

where π_i is the proportion of type θ_i agents and $c(q)$ is increasing and convex.

(a) Show that (IR_2) and (IR_3) can be ignored.
(b) Show that $q_3 \geq q_2 \geq q_1$.
(c) Using (IC_2^1) and (IC_3^2) show that we can ignore (IC_3^1). Using (IC_2^3) and (IC_1^2) show that we can ignore (IC_1^3).
(d) Show that (IR_1) will bind.
(e) Show that (IC_2^1) will bind.
(f) Show that (IC_3^2) will bind.
(g) Assume that $q_3 \geq q_2 \geq q_1$. Show that (IC_1^2) and (IC_2^3) can be ignored.

3. Sequential Screening with Different Priors

At time I, a principal signs a contract $\langle q_1, t_1, q_2, t_2 \rangle$ with an agent for trade conducted at time II. At the time of contracting, the principal and agent are both uninformed of the agent's period II utility.

At time II, the state $s \in \{1, 2\}$ is revealed. The agent's utility in state s is $u_s(q) - t$. The cost to the principal of providing quantity q is $c(q)$ in both states. A contract $\langle q_1, t_1, q_2, t_2 \rangle$ then specifies the quantity $q \in \Re_+$ and transfer $t \in \Re$ in both states of the world. Assume that $u_1'(q) > u_2'(q)$ $(\forall q)$. For technical simplicity, also assume that utility functions are increasing and concave, while the cost function is increasing and convex.

The agent and principal have different priors over the state. The principal is experienced and knows that state 1 will occur with probability p. The agent is mistaken, and believes that state 1 will occur with probability θ. Assume that $\theta > p$, so the agent is more confident than the principal.

(a) Suppose that the state s is publicly observable. The principal thus maximises her profit

$$\Pi = p[t_1 - c(q_1)] + (1 - p)[t_2 - c(q_2)]$$

subject to the individual rationality constraint of the agent,

$$\theta[u_1(q_1) - t_1] + (1-\theta)[u_2(q_2) - t_2] \geq 0$$

Describe the principal's profit–maximising contract.

For the rest of this question, suppose the state s is only observed by the agent.

(b) Show that your optimal contract from (a) is not incentive compatible after the state has been revealed.

(c) Suppose the principal maximises her profit subject to individual rationality and incentive compatibility. Derive the optimal contract. [Hint: you can ignore one of the (IC) constraints and later show that it does not bind at the optimal solution].

4. Theory of A Market Maker

Suppose a risk–neutral agent wishes to trade one unit of a share with a risk–neutral intermediary. That is, the agent can buy one share, sell one share, or choose not to trade. All parties start with a common prior on the value of the share, $\theta \sim g(\theta)$. The game is as follows.

1. The intermediary sets bid price B and ask prices A. Assume the market for intermediaries is competitive, so they make zero profits on each trade.
2. With probability $1-\alpha \in (0,1)$ the agent is irrational, buying one share at price A and selling one share at price B.[1] With probability α the agent is rational. In this case, the agent receives a signal $s \in [\underline{s}, \overline{s}]$ with nondegenerate distribution $f(s|\theta)$, and chooses to buy at A or sell at B. Assume $f(s|\theta)$ obeys the MLRP.
3. The value of the share, θ, is revealed. The agent and intermediary receive their payoffs. The rational agent's payoffs are as follows: if he buys, he receives $\theta - A$; if he sells he receives $B - \theta$; and if he does not trade he receives 0. The intermediaries payoffs are the opposite.

(a) Fix prices (A, B). For which signals will the rational agent trade?

[1]This is rather unrealistic, but it makes the maths easier.

(b) Given the zero profit condition for the intermediary, how are equilibrium prices (A, B) determined?

(c) Show that, in equilibrium, $A \geq E[\theta] \geq B$. Show that some rational agents will not trade.

(d) Suppose α increases. Show how this affects (a) the equilibrium prices, and (b) the proportion of rational agents trading.

(e) What happens as $\alpha \to 1$?

5. Screening without Transfers

A principal employs an agent who privately observes the state of the world $\theta \in [\underline{\theta}, \overline{\theta}]$ which is distributed with density $f(\theta)$. The principal first makes a report to the principal who chooses an action $q \in \{1, 2\}$. Consider the following direct–revelation mechanism:

1. The principal commits to a mechanism $q(\hat{\theta}) \in \{1, 2\}$.
2. The agent observes the state θ.
3. The agent then sends a message to the principal $\hat{\theta}$.
4. The principal receives payoff $v(\theta, q)$ and the agent receive payoff $u(\theta, q)$.

(a) Suppose $u(\theta, q)$ is supermodular in that

$$u(\theta_H, q_H) + u(\theta_L, q_L) > u(\theta_H, q_L) + u(\theta_L, q_H)$$

for $\theta_H > \theta_L$ and $q_H > q_L$. Show incentive compatibility implies that $q(\theta)$ is increasing.

(b) Characterise the mechanism, $q(\cdot)$, that maximises the principal's expected payoff.

(c) Intuitively, what happens to the optimal mechanism as the principal's preferences converge to those of the agent's? That is, $v(\theta, q) \to u(\theta, q)$ in L^1.

6. Holdup and Private Information

Suppose a buyer invests b at cost $c(b)$, where $c(\cdot)$ is increasing and convex. Investment b induces a stochastic valuation v for one unit of a good. The valuation is observed by the buyer and is distributed according to $f(v|b)$.

The seller then makes a TIOLI offer to the buyer of a price p. The buyer accepts or rejects.

(a) First suppose the seller observes v. How much will the buyer invest?

For the rest of the question, suppose that the seller observes neither b nor v. Assume that buyer's and seller's optimisation problems are concave.

(b) Assume $f(v|b)$ satisfies the hazard rate order in that

$$\frac{f(v|b)}{1-F(v|b)} \quad \text{decreases in } b \tag{HR}$$

Derive the seller's optimal price. How does the optimal price vary with b?

(c) Derive the buyer's optimal investment choice. Notice that (HR) implies that $F(v|b)$ decreases in b. How does the optimal investment vary with the expected price, p?

(d) Argue that there will be a unique Nash equilibrium in (b, p) space.

(e) How does the level of investment differ from part (a)? Why?

7. Moral Hazard and Asymmetric Information

A firm employs an agent who is risk–neutral, but has limited liability (i.e. they cannot be paid a negative wage). There is no individual rationality constraint. The agent can choose action $a \in \{L, H\}$ at cost $\{0, c\}$. There are two possible outputs $\{q_L, q_H\}$. The high output occurs with probability p_L or p_H if the agent takes action L or H, respectively. The agent's payoff is

$$w - c(a)$$

where w is the wage and $c(a)$ the cost of the action. The principal's payoff is

$$q - w$$

where q is the output and w is the wage.

(a) Characterise the optimal wages and action.

Suppose there are two types of agents, $i \in \{1, 2\}$. The principal cannot observe an agent's type but believes the probability of either type is 1/2. The agents are identical except for their cost of taking the action: for agent $i \in \{1, 2\}$ the cost of $a \in \{L, H\}$ is $\{0, c^i\}$, where $c^2 > c^1$.

(b) What are the optimal wages if the principal wishes to implement $\{a^1, a^2\} = \{L, L\}$?

(c) What are the optimal wages if the principal wishes to implement $\{a^1, a^2\} = \{H, H\}$?

(d) What are the optimal wages if the principal wishes to implement $\{a^1, a^2\} = \{L, H\}$?

(e) What are the optimal wages if the principal wishes to implement $\{a^1, a^2\} = \{H, L\}$?

8. Random Pricing

Consider the pricing problem of a monopolist who has 300 units to sell and is only allowed to choose a price p per unit (i.e. no first degree price discrimination). There are 100 agents who are identical and have the following demand:

$$\begin{aligned} D(p) &= 0 \quad \text{if} \quad p > 2 \\ &= 1 \quad \text{if} \quad p \in (1, 2] \\ &= 5 \quad \text{if} \quad p \in [0, 1] \end{aligned}$$

(a) Suppose the firm can charge a single price, p, per unit. What is the best they can do?

(b) Suppose the firm can separate the agents into two groups. The first group of N are charged price p_1 per unit. The second are charged p_2 per unit. What is the best they can do?

(c) Agents are identical so, intuitively, how can splitting them into two groups help? Does this relate to anything we covered in class?

9. Nonlinear Pricing and Outside Options

Consider a second degree price discriminating firm facing customers with two possible types $\theta \in \{3, 4\}$ with equal probability. An agent with type θ gains utility $u(\theta) = \theta q - p$ from quality q supplied at price p. If the agent does not purchase they gain utility 0. The cost of quality q is $c(q) = q^2/2$.

(a) Suppose the firm could observe each agents type θ. What quantity would she choose for each type?

For the next two parts assume the firm cannot observe agents' types. She can choose two quantity–price bundles $\{q(\theta), p(\theta)\}$ for $\theta \in \{3, 4\}$.

(b) Suppose there is a single outside good of quality $q^* = 1$ and price $p^* = 1$. What quantity would the firm choose for each type?

(c) Now suppose the outside good has quality $q^* = 6$ and price $p^* = 18$. What quantity would the firm choose for each type?

10. Optimal Taxation

There are two types of agents, $\theta_H > \theta_L$. Proportion β have productivity θ_L. An agent of type θ who exerts effort e produces output $q = \theta e$. The utility of an agent who produces quantity q with effort e is then

$$u(q - t - g(e))$$

where t is the net tax. Assume $g(e)$ is increasing and strictly convex, and $u(\cdot)$ is strictly concave.

Suppose that output is contractible so that a mechanism consists of a pair $(q(\theta), t(\theta))$. The government's problem is to maximise

$$\beta u\left(q_L - t_L - g\left(\frac{q_L}{\theta_L}\right)\right) + (1-\beta)u\left(q_H - t_H - g\left(\frac{q_H}{\theta_H}\right)\right)$$

subject to budget balance (BB), $\beta t_L + (1-\beta)t_H \geq 0$. Notice that there are no (IR) constraints here.

(a) First, suppose the government can observe agents' types. Solve for the first–best contract. Which type puts in the most effort?

Now suppose the government cannot observe agent's types. The incentive constraint for type L, for example, is

$$u\left(q_L - t_L - g\left(\frac{q_L}{\theta_L}\right)\right) \geq u\left(q_H - t_H - g\left(\frac{q_H}{\theta_L}\right)\right)$$

(b) Show that at the optimum (BB) binds.

(c) Show that at the optimum $u'_L \geq u'_H$, where u'_i is the marginal utility of type i.

(d) Show that at the optimum (IC_H) binds.

(e) Consider the government's relaxed problem of maximising welfare subject to (BB) and (IC_H), ignoring (IC_L). Show the optimal contract satisfies:

$$\begin{aligned} 1 - \frac{1}{\theta_H} g'\left(\frac{q_H}{\theta_H}\right) &= 0 && (1) \\ 1 - \frac{1}{\theta_L} g'\left(\frac{q_L}{\theta_L}\right) &= \frac{u'_L - u'_H}{u'_L}(1-\beta)\left(1 - \frac{1}{\theta_H} g'\left(\frac{q_L}{\theta_H}\right)\right) && (2) \end{aligned}$$

(f) Show that (2) implies

$$1 - \frac{1}{\theta_L} g'\left(\frac{q_L}{\theta_L}\right) \geq 0 \tag{3}$$

(g) Using equations (1) and (3) show that $q_H \geq q_L$. Use this and the fact that (IC_H) binds, to show that (IC_L) holds.

(h) What does (3) imply about the level of work performed by the low type. Provide an intuition for this distortion.

11. Durable Goods with Varying Demand

A monopolist sells a good over two periods, $t \in \{1, 2\}$, with zero marginal cost. The discount rate is δ for both the firm and the agents. The firm chooses prices $\{p_1, p_2\}$.

Each period a demand curve enters the market $f_t(\theta)$ with support $[0, 1]$ and cumulative distribution $F_t(\theta)$. Once they enter, an agent has value θ for one unit of the good. If an agent buys in period $s \in \{1, 2\}$, his utility is

$$u = (\theta - p_s)\delta^s$$

Agents who enter in period 1 can thus buy in either period 1 or 2 (or never). An agent who enters in period 2 can only buy in period 2 (or never).

(a) An agent with value θ who is born in period t has equilibrium utility

$$u(\theta, t) = \max_{\tau \geq t} (\theta - p_\tau)\delta^\tau$$

Show that aggregate equilibrium utility for cohort t is given by

$$E[u(\theta, t)] = \int_0^1 \delta^{\tau(\theta,t)}[1 - F_t(\theta)]d\theta$$

where $\tau(\theta, t)$ is the time agent (θ, t) buys.

(b) Show that the firm's aggregate profits are given by

$$\Pi = \int_0^1 \delta^{\tau(\theta,1)} m_1(\theta)d\theta + \int_0^1 \delta^{\tau(\theta,2)} m_2(\theta)d\theta$$

where

$$m_t(\theta) := \theta f_t(\theta) - [1 - F_t(\theta)]$$

Assume that $m_t(\theta)$ is increasing in θ for each cohort.

(c) Given the firm charges prices $\{p_1, p_2\}$, argue that we can characterise the allocations by cutoffs $\{\theta_1^*, \theta_2^*\}$, where θ_t^* is the lowest type that buys at time t. That is, an agent buys in time s if their value is above θ_s^*, independent of the cohort to which they belong.

(d) Suppose demand is identical in the two periods, $m_1(\theta) = m_2(\theta)$. What are the optimal cutoffs and prices?

(e) Suppose demand is rising over time, $m_1(\theta) \geq m_2(\theta)$. What are the optimal cutoffs and prices? [Note: the demand is rising here because the static monopoly price is rising].

(f) Suppose demand is falling over time, $m_1(\theta) \leq m_2(\theta)$. What are the optimal cutoffs and prices?

12. Naïve Price Discrimination.

Agents have values $v \sim F$, where the lowest value is zero. Assume the marginal revenue curve is strictly increasing and linear, i.e.

$$MR(v) := v - \frac{1 - F(v)}{f(v)} = av - b$$

Agents' utility is given by

$$u = vq - p$$

where q is quality and p is price.

(a) Give two examples of distributions of values that satisfy the linearity assumption.

(b) Suppose the firm offers a single good of quality q at cost $c(q)$ (e.g. VW is selling a Golf). What is the optimal price?

(c) Suppose the firm can choose to sell any good of quality $q \in [0, \infty]$ and cost $c(q)$, where $c(\cdot)$ is increasing and convex with $c(0) = 0$ and $\lim_{q \to \infty} c'(q) = \infty$ (e.g. VW is selling a Golf, Jetta, Passat etc.). Show the optimal prices are the same as in part (b).

2 Many-Agent Problems

1. Auctions with Correlated Values

A seller wants to sell a good to one of two symmetric buyers. Buyer i gains utility $v_i x_i - t_i$, where v_i is his valuation, x_i is the probability he gets the good and t_i is his payment to the seller. The seller wishes to maximise expected payments.

A seller designs a mechanism $(x_i(v_1, v_2), t_i(v_1, v_2))$, $i \in \{1, 2\}$, where the allocation probability and payments are a function of the agents' reports. The mechanism must allocate the good to the highest valuation buyer if valuations are different, and to each buyer with probability 1/2 if the valuations are the same. We consider only symmetric mechanisms, where payments depend on the agents' reports and not their identities. Denote $t_{ab} := t_1(v_a, v_b) = t_2(v_b, v_a)$.

Each buyer has one of two valuations, v_l or v_h, where $v_h > v_l$. The probability that the agents have valuations a, b is given by p_{ab}, where $a, b \in \{l, h\}$. We assume $p_{hh} p_{ll} > p_{hl}^2$, so valuations are positively correlated.

(a) The seller wants to design an ex–post individually rational (EPIR) and ex–post incentive compatible (EPIC) mechanism to maximise their expected revenue.[2] Determine the optimal transfers and the expected utility of a high and low type.

(b) The seller now drops the EPIR and EPIC requirement. The mechanism only has to be interim individually rational (IR) and interim incentive compatible (IC). Show that the seller can fully extract from the buyers. [Hint: Choose $t_{hh} = v_h/2$ and $t_{hl} = v_h$.] Intuitively, why can the seller fully extract the buyers' rent?

(c) The seller is concerned the buyers may collude. Suppose that if the buyers collude, they choose a pair of reports that minimises the sum of the transfers they pay. Show that if the buyers collude in the mechanism from part (a), they pay a total of v_l. Show that if the buyers collude in the mechanism from part (b), they pay less than v_l.

(d) Show that any (IR) and (IC) mechanism where buyers pay at least v_l by colluding, gives them at least as much rent as the mechanism from part (a).

[2] That is, every type should be happy to participate and reveal their type truthfully after knowing their opponent's type.

2. All Pay Auction

Assume all bidders have IID private valuations $v_i \sim F(v)$ with support $[0,1]$. Suppose the good is sold via an all–pay auction.

(a) Derive the symmetric equilibrium bidding strategy directly.

(b) Derive the symmetric equilibrium bidding strategy via revenue equivalence.

3. Asymmetric Auctions

(a) There is one bidder with value $v_1 \sim U[a, a+1]$, where $a \geq 0$. What is the optimal auction? Intuitively, why is the optimal reservation price increasing in a?

(b) Now there is a second bidder with value $v_2 \sim U[0,1]$, where agents' types are independent. What is the optimal auction?

4. Grants

Each of N agents have a project which needs funding. The value they place on funding is $\theta \sim F$ on $[0,1]$. The NSF wants to fund the most worthwhile project, but cannot observe θ. Agents write proposals which are time consuming: an agent who spends time t on a proposal gains utility $u_i(\theta_i) = P_i \cdot \theta - t_i$, where the project is funded with probability P_i. The NSF can only observe the time t_i each agent spends writing their proposal. Their aim is to maximise welfare which, since writing proposals is wasteful, is the same as maximising $\sum_i u_i$.

(a) Specify the problem as a DRM and write down the agents' utility.

(b) Characterise the agent's utility under incentive compatibility in terms of an integral equation and a monotonicity constraint.

(c) Suppose $(1 - F(x))/f(x)$ is strictly decreasing in x. Show the NSF's optimal policy is to allocate the grant randomly.

5. Auctions with Hidden Quality

The economics department is trying to procure teaching services from one of N potential assistant professors. Candidate i has an outside option of wage $\theta_i \in [0,1]$ with distribution function F. This wage is private information and can be thought of as the candidate's type. The department gets value $v(\theta_i)$ from type θ_i.

Consider a direct revelation mechanism consisting of an allocation function $P(\tilde{\theta}_1, \dots, \tilde{\theta}_N)$ and a transfer function $t(\tilde{\theta}_1, \dots, \tilde{\theta}_N)$. Suppose candidate i's utility is $u(\theta_i, \tilde{\theta}_i) = E_{-i}[t(\tilde{\theta}) - P(\tilde{\theta})\theta_i]$ and the department's profit is $\pi = E[P(\tilde{\theta})v(\theta_i) - t(\tilde{\theta})]$.

(a) Characterise the agent's utility under incentive compatibility in terms of an integral equation and a monotonicity constraint.

(b) Using (a), what is the department's profit?

For the rest of the question assume that

$$1 \geq \frac{d}{d\theta_i} \frac{F(\theta_i)}{f(\theta_i)} \geq 0$$

(c) If $v'(\theta_i) \leq 1$ what is the department's optimal hiring policy (i.e. allocation function)? How can this be implemented?

(d) Suppose $v'(\theta_i) \geq 2$ and $E[v(\theta_i)] \geq 1$. What is the department's optimal hiring policy (i.e. allocation function)? How can this be implemented?

6. Double Auction

A seller and buyer participate in a double auction. The seller's cost, $c \in [0,1]$, is distributed according to F_S. The buyer's value, $v \in [0,1]$, is distributed according to F_B. The seller names a price s and the buyer a price b. If $b \geq s$ the agents trade at price $p = (s+b)/2$, the seller gains $p - c$ and the buyer gains $v - p$. If $s < b$ there is no trade and both gain 0.

(a)Write down the utilities of buyer and seller. Derive the FOCs for the optimal bidding strategies.

For the rest of the question assume $c \sim U[0,1]$ and $v \sim U[0,1]$.

(b) Show that $S(c) = \frac{2}{3}c + \frac{1}{4}$ and $B(v) = \frac{2}{3}v + \frac{1}{12}$ satisfy the FOCs.

(c) Under which conditions on (v, c) does trade occur?

7. Auctions with Endogenous Entry

This question studies optimal auction design with endogenous entry. There are a large number of potential bidders who must pay k in order to enter an auction. After the entry decision, each entering bidder learns their private value θ_i which are distributed independently and identically with positive density $f(\theta)$, distribution function $F(\theta)$ and support $[\underline{\theta}, \overline{\theta}]$. The auctioneer has known valuation θ_0.

Denote the direct mechanism by $\langle N, P_i, t_i \rangle$, which is common knowledge. The auctioneer first allows bidders in the set N to enter. Each entering bidder learns their type θ_i and reports $\tilde{\theta}_i$. If the other bidders report truthfully, bidder i wins the good with probability $P_i(\tilde{\theta}_i, \theta_{-i})$ and pays $t_i(\tilde{\theta}_i, \theta_{-i})$ yielding utility,

$$u_i(\theta_i, \tilde{\theta}_i) = E_{\theta_{-i}} \left[\theta_i P_i(\tilde{\theta}_i, \theta_{-i}) - t_i(\tilde{\theta}_i, \theta_{-i}) \right]$$

where the lowest type gets utility $u_i(\underline{\theta})$.

(a) Show that incentive compatibility (IC) implies that utility obeys an integral equation and a monotonicity constraint.

(b) Write down the ex–ante individual rationality (IR) constraint which ensures that each bidder is happy to pay the entry cost and participate.

(c) Write down the auctioneer's program or maximising revenue, equal to the sum of payments, subject to (IC) and (IR).

(d) Show that the (IR) constraint will bind at the optimum.

(e) Optimal allocation function. Show that the revenue maximising mechanism awards the object to the agent with the highest valuation if that value exceeds θ_0.

(f) Optimal entry policy. Define welfare with n bidders by

$$W(n) := E_\theta \max\{\theta_0, \theta_1, \dots, \theta_n\}$$

Show that $W(n+1) - W(n)$ decreases in n. Use this to show that the optimal number of bidders, n^*, obeys $W(n^*) - W(n^*-1) \geq k \geq W(n^*+1) - W(n^*)$.

(g) Argue that the optimal mechanism can be implemented by a standard auction with reserve price, entry fee and having bidders make their entry decisions sequentially. What are the optimal entry fee and reserve price?

8. Bilateral Trade

Suppose two agents wish to trade a single good. The seller has privately known cost $c \sim g(\cdot)$ on $[0,1]$. The buyer has privately known value $v \sim f(\cdot)$ on $[0,1]$. These random variables are independent of each other. The agents' payoffs are

$$U_S = t - cp$$
$$U_B = vp - t$$

where $t \in \Re$ is a transfer and $p \in [0,1]$ is the probability of trade. If an agent abstains from trade, they receive 0.

In class, we showed that it is impossible to implement the ex–post efficient allocation. We now wish to find the revenue and welfare maximising mechanisms.

(a) Consider the problem of a middleman who runs mechanism $\langle p(\tilde{v},\tilde{c}), t_B(\tilde{v},\tilde{c}), t_S(\tilde{v},\tilde{c})\rangle$ where t_B and t_S are the transfers from the buyer and to the seller respectively. Show that a middleman can make profit

$$\Pi = E\Big[[MR(v) - MC(c)]p(v,c)\Big] - U_B(\underline{v}) - U_S(\overline{c})$$

where

$$MR(v) = v - \frac{1-F(v)}{f(v)} \quad \text{and} \quad MC(c) = c + \frac{G(c)}{g(c)}$$

(b) Maximise the middleman's expected profits.

(c) Maximise expected welfare subject to $\Pi = 0$.

9. Revenue Management

A firm has one unit of a good to sell over T periods. One agent enters each period and has a value drawn IID from $F(\cdot)$ on [0,1]. Agents values are privately known. The discount rate is $\delta \in (0,1)$.

First, assume there is no recall, so agent t leaves at the end of period t if they do not buy. A mechanism $\langle P_t, Y_t \rangle$ gives the probability of allocating the object in period t as a function of the reports until time t, and the corresponding payment. Agent t's utility is then given by

$$u_t = v_t \delta^t P_t - Y_t$$

and the firm's profits equal $\Pi = \sum_t Y_t$. Assume $MR(v) = v - (1 - F(v))/f(v)$ is increasing.

(a) Argue that the firm's profit is given by

$$\Pi = E_0 \left[\sum_t \delta^t P_t MR(v_t) \right]$$

where E_0 is the expectation at time 0, over all sequences of values. [Note: you don't have to be formal]

(b) What is the optimal allocation in period T? [Hint: it is easy to think in terms of cutoffs v_T^*, where the firm is indifferent between allocating the object and not]

(c) Using backwards induction, characterize the optimal cutoff in period $t < T$? What happens to the cutoffs over time?

Next, suppose there is recall. Hence agent t can buy in any period $\tau \geq t$. A mechanism $\langle P_{t,\tau}, Y_t \rangle$ gives the probability of allocating the object to agent t in period τ as a function of the reports until time τ, and the corresponding payment. Agent t's utility is then given by

$$u_t = \sum_{\tau \geq t} v_t \delta^\tau P_{t,\tau} - Y_t$$

and the firm's profits equal $\Pi = \sum_t Y_t$.

(d) Argue that the firm's profit is given by

$$\Pi = E_0\left[\sum_t \sum_{\tau\geq t} \delta^\tau P_{t,\tau} MR(v_t)\right]$$

(e) When there is recall, what is the optimal allocation in period T?

(f) Using backwards induction, what is the optimal allocation in period $t<T$? What happens to the cutoffs over time? [Hint: try $t=T-1$ and $t=T-2$, and the general case only if you have time]

10. Set Asides and Subsidies

Two agents, H and L, compete in an auction. Agent H has value $v_H \sim f_H(\cdot)$, while agent L has value $v_L \sim f_L(\cdot)$. The support of both distributions is bounded away from zero, while the seller has value $v_0 = 0$.

A mechanism $\langle p_i(v), t_i(v)\rangle$ gives the probability of agent $i \in \{L, H\}$ winning the good, and their payment. The government wishes to maximize welfare, subject to giving the good to agent L with at least α probability, i.e.,

$$E_v[p_L(v_L, v_H)] \geq \alpha$$

To motivate this, one can view L as a disadvantaged bidder (e.g. a small firm) that the government wishes to keep in business. [Hint: You can solve the problem by ignoring the (IC) constraints and then verifying that there are payments that implement this allocation.]

(a) What is the government's optimal mechanism?

Under a set-aside policy the government sets β such that: with probability β only L competes; and with probability $1-\beta$ they run a welfare maximizing auction. Under a subsidy policy, the government subsidizes L's bid by a fixed amount (e.g. in a second-price auction).

(b) Does the government prefer a set-aside policy or a subsidy policy? Provide intuition.

11. Negotiations and Auctions

Assume all bidders have IID private valuations $v_i \sim F(v)$ with support $[\underline{V}, \overline{V}]$. Define marginal revenue as

$$MR(v) = v - \frac{1 - F(v)}{f(v)}$$

(a) Show that $E[MR(v)] = \underline{V}$.

(b) In terms of marginal revenues, what is the revenue from 2 bidders with no reservation price?

(c) Let the sellers valuation be v_0. In terms of marginal revenue, what is the revenue from 1 bidder and a reservation price?

(d) Assume $\underline{V} \geq v_0$, i.e. all bidders are "serious". How is revenue affected if one bidder is swapped for a reservation price?

12. Allocation via Queue

A pro-bono law firm has two lawyers, A and B. There are an infinite number of clients waiting. Each client is a good match with one partner and a bad match with the other; if the match is good (bad) the client receives value v_H (v_L). The client prefers A with probability 1/2; a client's preference is private information. Clients are also impatient, so a client who receives value v at time t obtains utility ve^{-rt}, where r is the interest rate.

It takes time for a lawyer to solve the problem of a client. The amount of time is Poisson distributed with arrival rate λ. Ideally we would like to allocate clients to the lawyer that matches their needs, but clients preferences are private information.

(a) Suppose lawyers are allocated via a queue. That is, we line up the clients according to, say, their name. Each time a lawyer becomes available, we approach clients in order. If a client accepts, he sees the lawyer; if he rejects, then he retains his place in the queue. Show that a client in n^{th} position will tell the truth about their type iff

$$v_H \left(\frac{\lambda}{r + \lambda} \right)^n \geq v_L.$$

[Hint: You may find it useful to note that the arrival time $t \geq 0$ of the n^{th} arrival from a Poisson arrival process obeys the Erlang distribution, $f_n(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$.]

(b) Suppose we instead use a "preferred group" allocation system. Suppose there are k clients in the "preferred group". Each time a lawyer becomes available, we ask clients in the preferred group in random order. If one of the clients accepts, we bring a new client into the group. If none of the clients accept, we allocate the lawyer to someone outside the group. Show that truth-telling is an ex-post equilibrium for clients in the preferred group if k satisfies

$$v_H \frac{1}{k}\left(\frac{\lambda}{r+\lambda}\right) \geq v_L$$

(c) Assume $r \geq \lambda$. Which system is better?

13. Auctions with Endogenous Quantity

There are N agents bidding for a procurement contract. The agents have constant marginal costs that are distributed iid, $c_i \sim f[0,1]$, where the hazard rate $f(c_i)/F(c_i)$ is decreasing in c_i. The principal has value function $V(q)$ for quantity q. Assume V is differentiable and concave, with $\lim_{q\to\infty} V'(q) = 0$.

Consider the mechanism design problem. An agent reports cost $\tilde{c}_i$, is paid $t_i(\tilde{c}_i, \tilde{c}_{-i})$ and is allocated quantity $q_i(\tilde{c}_i, \tilde{c}_{-i})$. It is assumed that only one agent can win the contract, so $q_i = 0$ for all but one agent. The winning agent obtains utility $u_i = t_i - q_i c_i$, while losing agents receive $u_j = t_j$. The principal obtains profit $\pi = V(q_i) - \sum_j t_j$. Aside: we are assuming the mechanism is deterministic; this is without loss here.

(a) What is the optimal mechanism? Suppose $c_i \sim U[0,1]$ and $V(q) = q - q^2/2$ for $q \in [0,1]$. What is the optimal allocation function?

For the rest of this question we suppose the agents compete via an auction and the principal then chooses the quantity q afterwards so that $V'(q) = p$, where p is the price from the auction.

(b) First, lets consider a first-price auction. As a benchmark, suppose agents bid as if quantity q is fixed (say $q = 1$). Characterize agents' symmetric bidding function, $\beta(c_i)$. What is the bidding function when $c_i \sim U[0,1]$ and $N = 2$? Fixing the bids, suppose $V(q) = q - q^2/2$ for

$q \in [0,1]$, and the quantity is determined so that $V'(q) = \beta_{(1)}$, where $\beta_{(1)}$ is the lowest bid. How does the allocation function differ from the optimal allocation in part (a)?

For the rest of the question, suppose when the agents bid they take into account the fact the demand function $q(p) = (V')^{-1}(p)$ is downward sloping.

(c) Considering a FPA, write down the FOC for the agent's bidding function, $b(c)$. Argue that the agent bids more aggressively than in part (b). That is, $b(c) \leq \beta(c)$. [Hint: one can do this by comparing the FOCs, or by integrating the bidding function up.]

(d) How would agents bid in a SPA? Using part (c), argue that the FPA yields lower expected prices than the SPA.