Homework 1: Basic Moral Hazard

1. Normal–Linear Model

The following normal–linear model is regularly used in applied models. Given action $a \in \Re$, output is $q = a + x$, where $x \sim N(0, \sigma^2)$. The cost of effort is $g(a)$ is increasing and convex. The agent's utility equals $u(w(q) - g(a))$, while the principals is $q - w(q)$. Suppose the agent's outside option is $u(0)$.

We make two large assumptions. First, the principal uses a linear contract:

$$w(q) = \alpha + \beta q$$

Second, the agent's utility is CARA, i.e., $u(w) = -e^{-w}$.

(a) Suppose $w \sim N(\mu, \sigma^2)$. Denote the certainty equivalent of w by $\bar{w}$, where

$$u(\bar{w}) = E[u(w)]$$

Show that $\bar{w} = \mu - \sigma^2/2$.

(b) Suppose effort is unobservable. The principal's problem is

$$\begin{aligned} &\max_{w(q),a} E[q - w(q)] \\ &\text{s.t.} \qquad E[u(w(q) - g(a))|a] \geq u(0) \\ &\qquad\qquad a \in \text{argmax}_{a' \in \Re} E[u(w(q) - g(a'))|a'] \end{aligned}$$

Using the first order approach, characterise the optimal contract (α, β, a). [Hint: write utilities in terms of their certainty equivalent.]

(c) How would the solution change if the agent knows x before choosing his action (but after signing the contract)?

2. Signal of Effort

Consider the same normal–linear model as in Question 1. After a is chosen, the principal observes output q and a signal y that is correlated with x. For example, if the agent is selling cars, then y could be the sales of the dealer next door. Let

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix} \right)$$

Suppose the principal uses the linear wage function:

$$w = \alpha + \beta(q + \gamma y)$$

(a) Using the same approach as above, solve for the optimal value of γ. How does γ vary with σ_{xy}? Provide an intuition.

(b) Now, suppose the principal employs N agents. The performance of agent i is given by

$$q_i = e_i + x_i + x_c$$

where $(x_1, \ldots, x_N, x_c)$ are independent and normally distributed with variances $(\sigma_1^2, \ldots, \sigma_N^2, \sigma_c^2)$. Assume the principal offers a linear contract

$$w_i = \alpha_i + \beta_i(q_i - \sum_{j \neq i} \gamma_j^i q_j)$$

The principal's profit is given by $E[\sum_i (q_i - w_i)]$.

Solve for the optimal $\{\gamma_j^i\}_{j,i}$. Interpret these coefficients. What implications does this have for the incentives in teams?

3. Insurance

An agent has increasing, concave utility $u(\cdot)$. They start with wealth W_0 and may have an accident costing x of their wealth. Assume x is publicly observable. The agent has access to a perfectly competitive market of risk–neutral insurers who offer payments $R(x)$ net of any insurance premium. The distribution of x is as follows

$$f(0, a) = 1 - p(a) \tag{1}$$

$$f(x, a) = p(a)g(x) \quad \text{for} \quad x > 0 \tag{2}$$

where $\int g(x)dx = 1$. The agent can affect the probability of an accident through their choice of a. The cost is given by increasing convex function, $\psi(a)$. The function $p(a)$ is decreasing and convex. Utility is then given by $u(W_0 - x + R(x)) - \psi(a)$.

(a) Suppose there is no insurance market. What action $\hat{a}$ does the agent take?

(b) Suppose a is contractible. Describe the first–best payment schedule $R(x)$ and the effort choice, a^*.

(c) Suppose a is not contractible. Describe the second–best payment schedule $R(x)$.

(d) Interpret the second–best payment schedule. Would the agent ever have an incentive to hide an accident? (i.e. report $x = 0$ when $x > 0$).

4. Private Evaluations with Limited Liability

A principal employs an agent. The game is as follows.

1. The agent privately chooses an action $a \in \{L, H\}$. The cost of this action is $g(a)$.
2. The principal *privately* observes output $q \sim f(q|a)$ on $[\underline{q}, \overline{q}]$. Assume this distribution function satisfies strict MLRP. That is,

$$\frac{f(q|H)}{f(q|L)}$$

   is strictly increasing in q.
3. Suppose the principal reports that output is $\tilde{q}$. The principal then pays out $t(\tilde{q})$, while the agent receives $w(\tilde{q})$, where $w(\tilde{q}) \le t(\tilde{q})$. The difference is burned. The payments $\langle t, q \rangle$ are contractible.

Payoffs are as follows. The principal obtains

$$q - t$$

The agent obtains

$$u(w) - g(a)$$

where $u(\cdot)$ is strictly increasing and concave, and $g(\cdot)$ is increasing and convex. The agent has no (IR) constraint, but does have limited liability. That is, $w(q) \geq 0$ for all q.

First, assume the principal wishes to implement $a = L$.

(a) Characterise the optimal contract.

Second, assume the principal wishes to implement $a = H$.

(b) Write down the principal's problem as maximising expected profits subject to the agent's (IC) constraint, the principal's (IC) constraint, the limited liability constraint and the constraint that $w(q) \leq t(q)$.

(c) Argue that $t(q)$ is independent of q.

(d) Characterise the optimal contract. How does the wage vary with q?

5. Debt Contracts

A risk neutral agent seeks funding from a risk neutral principal. The game is as follows:

1. The project requires investment I from the principal.
2. The agent chooses effort $a \in \{L, H\}$ at cost $c(a)$. Assume $c(H) > c(L)$.
3. Output q is realised. Assume q takes values $\{q_1, \ldots, q_N\}$, where $q_{i+1} > q_i$. Output is distributed according to $f(q_i|a)$.
4. If q_i is realised, the principal obtains payment B_i and the agent obtains $q_i - B_i$. The agent's utility is $u = q_i - B_i - c(a)$; the principal's profit is $\pi = B_i - I$.

A contract specifies the payment to the principal as a function of the output $\langle B_i \rangle$. Assume the principal has outside option 0 and the agent makes a TIOLI offer to the principal. We also assume the contract satisfies feasibility (FE):

$$0 \leq B_i \leq q_i$$

and monotonicity (MON):

$$B_i \text{ is increasing in } i$$

Finally assume that $f(q_i|a)$ satisfies the monotone hazard rate principle (MHRP):

$$\frac{f(q_i|L)}{1-F(q_i|L)} \geq \frac{f(q_i|H)}{1-F(q_i|H)} \qquad \text{for each } q_i.$$

Assume the agent wishes to implement the high action. Show that a debt contract is optimal.

[Aside: In class we showed that MLRP implies debt contracts are optimal. The key insight is that, if we use the (MON) condition, we can use the weaker MHRP assumption.]

6. Bargaining Power

Suppose a risk neutral principal employs a risk averse agent. The two parties both sign a contract stating wage profile $w(q)$. The agent then chooses action $a \in A$ at cost $g(a)$.

Payoffs are as follows. The agent gets

$$u(w - g(a))$$

where $g(a)$ is increasing and convex. Utility is strictly increasing and strictly concave. The principal gets

$$q - w$$

The principal has reservation profit 0; the agent has reservation utility $u(0)$.

First, suppose the principal makes a TIOLI offer to the agent.

(a) Assume the effort a is observable. Set up and solve the principal's optimal contract.

(b) Assume effort a is not observable. Set up the principal's problem.

Next, suppose the agent makes a TIOLI offer to the principal.

(c) Assume the effort a is observable. Show that the optimal contract induces the same effort as when the principal proposes the contract.

(d) Assume effort a is not observable. Set up the agent's problem. Next, suppose that utility is CARA, i.e. $u(w) = -\exp(-w)$, which implies that $u(w+x) = u(w)e^{-x}$. Show that the optimal contract induces the same effort as when the principal proposes the contract (part (b)).