Practice Problems 2: Asymmetric Information

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1. Nonlinear Pricing with Two Types

Suppose a seller of wine faces two types of customers, θ_1 and θ_2, where $\theta_2 > \theta_1$. The proportion of type θ_1 agents is $\pi \in [0, 1]$. Let q be the quality of the wine and t the price.

Let type θ_1 buy contract (q_1, t_1) and type θ_2 buy (q_2, t_2). The cost of production is zero, $c(q) = 0$, and the seller maximises profit $\pi t_1 + (1 - \pi)t_2$

(a) Suppose agent θ_i has utility

$$u(\theta_i) = \theta_i q - \frac{1}{2}q^2 - t$$

Derive the first–best and profit–maximising qualities.

(b) Suppose agent θ_i has utility

$$u(\theta_i) = \theta_i(q - \frac{1}{2}q^2) - t$$

Derive the first–best and profit–maximising qualities.

2. Nonlinear Pricing with Three Types

Consider the nonlinear pricing model with three types, $\theta_3 > \theta_2 > \theta_1$. The utility of agent θ_i is

$$u(\theta_i) = \theta_i q - t$$

Denote the bundle assigned to agent θ_i by (q_i, t_i). We now have six (IC) constraint and three (IR) constraints. For example, (IC_1^2) says that θ_1 must not want to copy θ_2, i.e.

$$\theta_1 q_1 - t_1 \geq \theta_1 q_2 - t_2 \qquad (\text{IC}_1^2)$$

The firm's profit is

$$\sum_{i=1}^{3} \pi_i [t_i - c(q_i)]$$

where π_i is the proportion of type θ_i agents and $c(q)$ is increasing and convex.

(a) Show that (IR_2) and (IR_3) can be ignored.
(b) Show that $q_3 \geq q_2 \geq q_1$.
(c) Using (IC_2^1) and (IC_3^2) show that we can ignore (IC_3^1). Using (IC_2^3) and (IC_1^2) show that we can ignore (IC_1^3).
(d) Show that (IR_1) will bind.
(e) Show that (IC_2^1) will bind.
(f) Show that (IC_3^2) will bind.
(g) Assume that $q_3 \geq q_2 \geq q_1$. Show that (IC_1^2) and (IC_2^3) can be ignored.

3. Optimal Taxation

There are two types of agents, $\theta_H > \theta_L$. Proportion β have productivity θ_L. An agent of type θ who exerts effort e produces output $q = \theta e$. The utility of an agent who produces quantity q with effort e is then

$$u(q - t - g(e))$$

where t is the net tax. Assume $g(e)$ is increasing and strictly convex, and $u(\cdot)$ is strictly concave.

Suppose that output is contractible so that a mechanism consists of a pair $(q(\theta), t(\theta))$. The government's problem is to maximise

$$\beta u\left(q_L - t_L - g\left(\frac{q_L}{\theta_L}\right)\right) + (1-\beta)u\left(q_H - t_H - g\left(\frac{q_H}{\theta_H}\right)\right)$$

subject to budget balance (BB), $\beta t_L + (1-\beta)t_H \geq 0$. Notice that there are no (IR) constraints here.

(a) First, suppose the government can observe agents' types. Solve for the first–best contract. Which type puts in the most effort?

Now suppose the government cannot observe agent's types. The incentive constraint for type L, for example, is

$$u\left(q_L - t_L - g\left(\frac{q_L}{\theta_L}\right)\right) \geq u\left(q_H - t_H - g\left(\frac{q_H}{\theta_L}\right)\right)$$

(b) Show that at the optimum (BB) binds.

(c) Show that at the optimum $u'_L \geq u'_H$, where u'_i is the marginal utility of type i.

(d) Show that at the optimum (IC_H) binds.

(e) Consider the government's relaxed problem of maximising welfare subject to (BB) and (IC_H), ignoring (IC_L). Show the optimal contract satisfies:

$$1 - \frac{1}{\theta_H} g'\left(\frac{q_H}{\theta_H}\right) = 0 \tag{1}$$

$$1 - \frac{1}{\theta_L} g'\left(\frac{q_L}{\theta_L}\right) = \frac{u'_L - u'_H}{u'_L}(1-\beta)\left(1 - \frac{1}{\theta_H} g'\left(\frac{q_L}{\theta_H}\right)\right) \tag{2}$$

(f) Show that (2) implies

$$1 - \frac{1}{\theta_L} g'\left(\frac{q_L}{\theta_L}\right) \geq 0 \tag{3}$$

(g) Using equations (1) and (3) show that $q_H \geq q_L$. Use this and the fact that (IC_H) binds, to show that (IC_L) holds.

(h) What does (3) imply about the level of work performed by the low type. Provide an intuition for this distortion.

4. All Pay Auction

Assume all bidders have IID private valuations $v_i \sim F(v)$ with support $[0, 1]$. Suppose the good is sold via an all–pay auction.

(a) Derive the symmetric equilibrium bidding strategy directly.

(b) Derive the symmetric equilibrium bidding strategy via revenue equivalence.

5. Asymmetric Auctions

(a) There is one bidder with value $v_1 \sim U[a, a + 1]$, where $a \geq 0$. What is the optimal auction? Intuitively, why is the optimal reservation price increasing in a?

(b) Now there is a second bidder with value $v_2 \sim U[0,1]$, where agents' types are independent. What is the optimal auction?

6. Grants

Each of N agents have a project which needs funding. The value they place on funding is $\theta \sim F$ on $[0,1]$. The NSF wants to fund the most worthwhile project, but cannot observe θ. Agents write proposals which are time consuming: an agent who spends time t on a proposal gains utility $u_i(\theta_i) = P_i \cdot \theta - t_i$, where the project is funded with probability P_i. The NSF can only observe the time t_i each agent spends writing their proposal. Their aim is to maximise welfare which, since writing proposals is wasteful, is the same as maximising $\sum_i u_i$.

(a) Specify the problem as a DRM and write down the agents' utility.

(b) Characterise the agent's utility under incentive compatibility in terms of an integral equation and a monotonicity constraint.

(c) Suppose $(1 - F(x))/f(x)$ is strictly decreasing in x. Show the NSF's optimal policy is to allocate the grant randomly.

7. Auctions with Endogenous Entry

This question studies optimal auction design with endogenous entry. There are a large number of potential bidders who must pay k in order to enter an auction. After the entry decision, each entering bidder learns their private value θ_i which are distributed independently and identically with positive density $f(\theta)$, distribution function $F(\theta)$ and support $[\underline{\theta}, \overline{\theta}]$. The auctioneer has known valuation θ_0.

Denote the direct mechanism by $\langle N, P_i, t_i \rangle$, which is common knowledge. The auctioneer first allows bidders in the set N to enter. Each entering bidder learns their type θ_i and reports $\tilde{\theta}_i$. If the other bidders report truthfully, bidder i wins the good with probability $P_i(\tilde{\theta}_i, \theta_{-i})$ and pays $t_i(\tilde{\theta}_i, \theta_{-i})$ yielding utility,

$$u_i(\theta_i, \tilde{\theta}_i) = E_{\theta_{-i}} \left[\theta_i P_i(\tilde{\theta}_i, \theta_{-i}) - t_i(\tilde{\theta}_i, \theta_{-i}) \right]$$

where the lowest type gets utility $u_i(\underline{\theta})$.

(a) Show that incentive compatibility (IC) implies that utility obeys an integral equation and a monotonicity constraint.

(b) Write down the ex–ante individual rationality (IR) constraint which ensures that each bidder is happy to pay the entry cost and participate.

(c) Write down the auctioneer's program or maximising revenue, equal to the sum of payments, subject to (IC) and (IR).

(d) Show that the (IR) constraint will bind at the optimum.

(e) Optimal allocation function. Show that the revenue maximising mechanism awards the object to the agent with the highest valuation if that value exceeds θ_0.

(f) Optimal entry policy. Define welfare with n bidders by

$$W(n) := E_\theta \max\{\theta_0, \theta_1, \ldots, \theta_n\}$$

Show that $W(n+1) - W(n)$ decreases in n. Use this to show that the optimal number of bidders, n^*, obeys $W(n^*) - W(n^* - 1) \geq k \geq W(n^* + 1) - W(n^*)$.

(g) Argue that the optimal mechanism can be implemented by a standard auction with reserve price, entry fee and having bidders make their entry decisions sequentially. What are the optimal entry fee and reserve price?

8. Negotiations and Auctions

Assume all bidders have IID private valuations $v_i \sim F(v)$ with support $[\underline{V}, \overline{V}]$. Define marginal revenue as

$$MR(v) = v - \frac{1 - F(v)}{f(v)}$$

(a) Show that $E[MR(v)] = \underline{V}$.

(b) In terms of marginal revenues, what is the revenue from 2 bidders with no reservation price?

(c) Let the sellers valuation be v_0. In terms of marginal revenue, what is the revenue from 1 bidder and a reservation price?

(d) Assume $\underline{V} \geq v_0$, i.e. all bidders are "serious". How is revenue affected if one bidder is swapped for a reservation price?