# Production Functions

[See Chap 9]

1

## Production Function

- The firm's production function for a particular good ($q$) shows the maximum amount of the good that can be produced using alternative combinations of inputs.

$$q = f(z_1, \dots, z_N)$$

- Examples (with N=2):
  - $z_1$ = capital, $z_2$ = labor.
  - $z_1$ = skilled labor, $z_2$ = unskilled labor
  - $z_1$ = capital, $z_2$ = land.

2

## Marginal Product

- The marginal product is the additional output that can be produced by employing one more unit of the input while holding other inputs constant:

$$\text{marginal product of } z_1 = MP_1 = \frac{\partial q}{\partial z_1} = f_1$$

3

## Average Product

- Input productivity can be measured by average product

$$AP_1 = \frac{q}{z_1} = \frac{f(z_1, z_2)}{z_1}$$

4

## ISOQUANTS

5

## Isoquants

- An isoquant shows the combinations of $z_1$ and $z_2$ that can produce a given level of output ($q_0$)

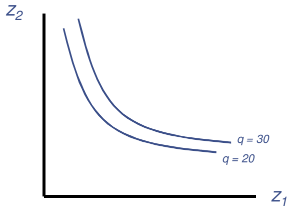
$$f(z_1, z_2) = q_0$$

- Like indifference curves for technology.

6

## Isoquants

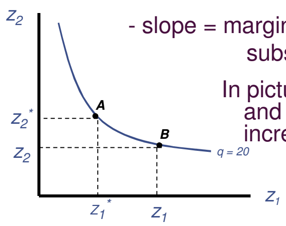
- Each isoquant represents a different level of output.




7

## Marginal Rate of Technical Substitution

- The slope of an isoquant shows the rate at which $z_2$ can be substituted for $z_1$
- MRTS = number of $z_2$ the firm gives up to get 1 unit of $z_1$, if she wishes to hold output constant.




- slope = marginal rate of technical substitution (M*RTS*)

In picture, *MRTS* is positive and is diminishing for increasing inputs of labor

8

## MRTS

- The <u>marginal rate of technical substitution (*MRTS*)</u> shows the rate at which labor can be substituted for capital while holding output constant along an isoquant

$$MRTS = -\left.\frac{dz_2}{dz_1}\right|_{q=q_0}$$

9

## *MRTS* and Marginal Products

- Take the total differential of the production function:

$$dq = \frac{\partial f}{\partial z_2} \cdot dz_2 + \frac{\partial f}{\partial z_1} \cdot dz_1 = MP_2 \cdot dz_2 + MP_1 \cdot dz_1$$

- Along an isoquant $dq = 0$, so

$$MRTS = \left.\frac{-dz_2}{dz_1}\right|_{q=q_0} = \frac{MP_1}{MP_2}$$

10

## Properties of Technology

11

## 1. Monotonicity

- A production function is monotone if $f(z_1,z_2)$ is strictly increasing in both inputs.

$$\frac{\partial f}{\partial z_i} = f_i > 0$$

- This implies that
  - isoquants are thin
  - isoquants do not cross
  - isoquants are downward sloping.

12

## 2. Quasi-Concavity

- Suppose $z=(z_1,z_2)$ and $z'=(z_1',z_2')$ are two input bundles.
- f(.) is quasi-concave in z if whenever $f(z) \geq f(z')$ then

  $f(tz+(1-t)z') \geq f(z') \quad 1>t>0.$
- Implications
  - Isoquants are convex.
  - MRTS decreases in $z_1$, as move along isoquant.

13

## 3. Concavity

- Suppose $z=(z_1,z_2)$ and $z'=(z_1',z_2')$ are two input bundles.
- f(.) is concave in z if

  $f(tz+(1-t)z') \geq tf(z) + (1-t)f(z') \quad 1>t>0.$
- Implies quasi-concavity (convex isoquants).
- Implies diminishing marginal productivity:

$$\frac{\partial MP_1}{\partial z_1} = \frac{\partial^2 f}{\partial z_1^{\,2}} = f_{11} \leq 0 \qquad \frac{\partial MP_2}{\partial z_2} = \frac{\partial^2 f}{\partial z_2^{\,2}} = f_{22} \leq 0$$

- Implies constant or decreasing returns to scale.

14

## 4. Returns to Scale

- How does output respond to increases in all inputs together?
  - suppose that all inputs are doubled, would output double?
- The effect of a proportional change in all inputs on output is called the <u>returns to scale</u>

15

## Returns to Scale

- If the production function is given by $q = f(z_1, z_2)$ and all inputs are multiplied by the same positive constant ($t > 1$), then

| Effect on Output | Returns to Scale |
|---|---|
| $f(tz_1, t\,z_2) = tf(z_1, z_2)$ | Constant |
| $f(tz_1, t\,z_2) < tf(z_1, z_2)$ | Decreasing |
| $f(tz_1, t\,z_2) > tf(z_1, z_2)$ | Increasing |

16

## Returns to Scale

- Why should there ever be DRS?
  - If expand all inputs then shouldn't output at least double (just recreate what firm was doing before).
  - May be able to do better due to specialization (leading to IRS).
- DRS can be seen as coming from omitted factor of production. For example, limited management time.

17

## EXAMPLES

18

## Perfect Substitutes

- Suppose that the production function is

$$q = f(z_1,z_2) = az_1 + bz_2$$

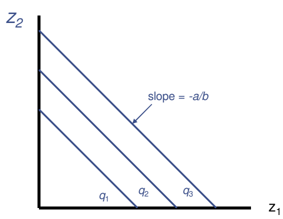
- Isoquants are straight lines.
  - MRTS is constant, since $MP_1$=a and $MP_2$=b.
- Production function exhibits constant returns to scale

$$f(tz_1,tz_2) = atz_1 + btz_2 = t(az_1 + bz_2) = tf(z_1, z_2)$$



## Perfect Substitutes

Capital and labor are perfect substitutes






## Perfect Complements

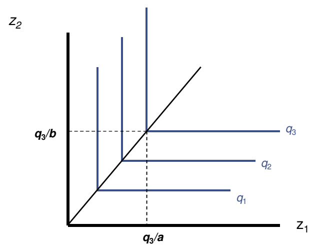
- Suppose that the production function is

$$q = \min(az_1,bz_2) \quad a,b > 0$$

- Capital and labor must always be used in a fixed ratio
  - the firm will always operate along a ray where $z_1/z_2$ is constant

## Perfect Complements

No substitution between labor and capital is possible




22

## Cobb-Douglas

- Suppose that the production function is

  $q = f(z_1,z_2) = z_1^a z_2^b \quad a,b > 0$

- Returns to scale

  $f(tz_1,tz_2) = (tz_1)^a(tz_1)^b = t^{a+b} z_1^a z_1^b = t^{a+b}f(z_1, z_1)$

  - if $a + b = 1 \Rightarrow$ constant returns to scale
  - if $a + b > 1 \Rightarrow$ increasing returns to scale
  - if $a + b < 1 \Rightarrow$ decreasing returns to scale

23

## Generalized Subs/Comps

- Generalized perfect substitutes

  $q = f(z_1,z_2) = (az_1 + bz_2)^\gamma$

- Generalized perfect complements

  $q = f(z_1,z_2) = (\min(az_1,bz_2))^\gamma$

- Constant returns if $\gamma$=1.
- Increasing returns if $\gamma$>1.
- Decreasing returns if $\gamma$<1.

24

## CES Production Function

- Suppose that the production function is

  $q = f(z_1,z_2) = [z_1^\rho + z_2^\rho]^{\gamma/\rho}$ $\rho \leq 1$, $\rho \neq 0$, $\gamma > 0$
  - If $\gamma > 1 \Rightarrow$ increasing returns to scale
  - If $\gamma < 1 \Rightarrow$ decreasing returns to scale
- Special cases
  - If $\rho = 1 \Rightarrow$ perfect substitutes
  - If $\rho = -\infty \Rightarrow$ perfect complements
  - If $\rho = 0 \Rightarrow$ Cobb-Douglas

25

## Example

- Suppose that the production function is

  $q = f(z_1,z_2) = z_1 + z_2 + 2(z_1z_2)^{0.5}$
- Marginal productivities are

  $f_1 = 1 + (z_2/z_1)^{0.5}$

  $f_2 = 1 + (z_1/z_2)^{0.5}$
- Thus,

$$MRTS = \frac{f_1}{f_2} = \frac{1+(z_2/z_1)^{0.5}}{1+(z_1/z_2)^{0.5}}$$

26

## Technical Progress

- Methods of production change over time
- Following the development of superior production techniques, the same level of output can be produced with fewer inputs:
  - In this case the isoquants shifts down.

27

## Technical Progress

- Suppose that the production function is

$$q = A(t)f(z_1, z_2)$$

where $A(t)$ represents all factors that affect the production of $q$ other than $z_1$ and $z_2$

  - Changes in $A$ over time represent technical progress
  - We would imagine that $dA/dt > 0$

28