Cost Functions
[See Chap 10]

Definitions of Costs
• Economic costs include both implicit and explicit costs.
• Explicit costs include wages paid to employees and the costs of raw materials.
• Implicit costs include the opportunity cost of the entrepreneur and the capital used for production.

Economic Cost
• The economic cost of any input is its opportunity cost:
  – the remuneration the input would receive in its best alternative employment
Model
• Firm produces single output, q
• Firm has N inputs \( \{z_1, \ldots, z_N\} \).
• Production function \( q = f(z_1, \ldots, z_N) \)
  – Monotone and quasi-concave.
• Prices of inputs \( \{r_1, \ldots, r_N\} \).
• Price of output \( p \).

Firm’s Payoffs
• Total costs for the firm are given by
  total costs = \( C = r_1z_1 + r_2z_2 \)
• Total revenue for the firm is given by
  total revenue = \( pq = p(f(z_1, z_2)) \)
• Economic profits (\( \pi \)) are equal to
  \( \pi = \text{total revenue} - \text{total cost} \)
  \( \pi = pq - r_1z_1 - r_2z_2 \)

Firm’s Problem
• We suppose the firm maximizes profits.
• One-step solution
  – Choose \( (q, z_1, z_2) \) to maximize \( \pi \)
• Two-step solution
  – Minimize costs for given output level.
  – Choose output to maximize revenue minus costs.
• We first analyze two-step method
  – Where do cost functions come from?
Cost-Minimization Problem (CMP)

• The cost minimization problem is

\[ \min r_1 z_1 + r_2 z_2 \quad \text{s.t.} \quad f(z_1, z_2) \geq q \quad \text{and} \quad z_1, z_2 \geq 0 \]

• Denote the optimal demands by \( z_1^*(r_1, r_2, q) \)

• Denote cost function by

\[ C(r_1, r_2, q) = r_1 z_1^*(r_1, r_2, q) + r_2 z_2^*(r_1, r_2, q) \]

• Problem very similar to EMP.

• Output constraint binds if \( f(.) \) is monotone.

CMP: Graphical Solution

Given output \( q \), we wish to find the lowest cost point on the isoquant

Isocost line are parallel with a slope of \(-r_1/r_2\):

\[ C_1 < C_2 < C_3 \]
The minimum cost of producing \( q \) is \( C_2 \)

This occurs at the tangency between the isoquant and the total cost curve

The optimal choice is \((z_1^*, z_2^*)\)

### CMP: Lagrangian Method

• Set up the Lagrangian:
  \[
  L = r_1 z_1 + r_2 z_2 + \lambda [q - f(z_1, z_2)]
  \]

• Find the first order conditions:
  \[
  \frac{\partial L}{\partial z_1} = r_1 - \lambda \left( \frac{\partial f}{\partial z_1} \right) = 0 \\
  \frac{\partial L}{\partial z_2} = r_2 - \lambda \left( \frac{\partial f}{\partial z_2} \right) = 0 \\
  \frac{\partial L}{\partial \lambda} = q - f(z_1, z_2) = 0
  \]

### Cost-Minimizing Input Choices

• Dividing the first two conditions we get:
  \[
  \frac{r_1}{r_2} = \frac{\frac{\partial f}{\partial z_1}}{\frac{\partial f}{\partial z_2}} = MRTS
  \]

• The cost-minimizing firm equates the MRTS for the two inputs to the ratio of their prices.

• Equivalently, the firm equates the bang-per-buck from each input
  \[
  \frac{\frac{\partial f}{\partial z_1}}{r_1} = \frac{\frac{\partial f}{\partial z_2}}{r_2}
  \]
Interpretation of Multiplier

• Note that the first order conditions imply the following:
  \[
  \frac{r_1}{f_1} = \frac{r_2}{f_2} = \lambda
  \]

• The Lagrange multiplier describes how much total costs would increase if output \( q \) would increase by a small amount.

The Firm’s Expansion Path

• The firm can determine the cost-minimizing combinations of \( z_1 \) and \( z_2 \) for every level of output.
• The set of combinations of optimal amount of \( z_1 \) and \( z_2 \) is called the firm’s expansion path.

The expansion path is the locus of cost-minimizing tangencies.

The curve shows how inputs increase as output increases.
The Firm’s Expansion Path

- The expansion path does not have to be a straight line
  - the use of some inputs may increase faster than others as output expands
  - depends on the shape of the isoquants
- The expansion path does not have to be upward sloping.

Example: Symmetric CD

- Production function is symmetric cobb-douglas:
  \[ q = z_1^{\gamma} z_2^{\gamma} \]
- The Lagrangian for the CMP is
  \[ L = r_1 z_1 + r_2 z_2 + \lambda (q - z_1^{\gamma} z_2^{\gamma}) \]

Example: Symmetric CD

- FOCs for a minimum:
  \[ \frac{\partial L}{\partial z_1} = r_1 - \lambda z_1^{(\gamma - 1)} z_2^{\gamma} = 0 \]
  \[ \frac{\partial L}{\partial z_2} = r_2 - \lambda z_1^{\gamma} z_2^{(\gamma - 1)} = 0 \]
- Rearranging yields \( r_1 z_1 = r_2 z_2 \).
- Using the constraint \( q = z_1^{\gamma} z_2^{\gamma} \),
  \[ \zeta_1(\ell, r, q) = \left( \frac{r_1}{\ell} \right)^{\gamma} q^{1/\gamma} \]
  and
  \[ \zeta_2(\ell, r, q) = \left( \frac{r_2}{\ell} \right)^{\gamma} q^{1/\gamma} \]
- Substituting, the cost is
  \[ c(\ell, r, q) = \zeta_1^2 + \zeta_2^2 = 2(\ell r)^{1/\gamma} q^{1/\gamma} \]
Example: Perfect Complements

- Suppose
  \[ q = f(z_1, z_2) = \min(z_1, z_2) \]
- Production will occur at the vertex of the L-shaped isoquants, \( z_1 = z_2 \).
- Using constraint, \( z_1 = z_2 = q \)
- Hence cost function is
  \[ C(r_1, r_2, q) = r_1z_1 + r_2z_2 = (r_1+r_2)q \]

Cost Functions

Total Cost Function

- The cost function shows the minimum cost incurred by the firm is
  \[ C(r_1, r_2, q) = r_1z_1^*(r_1, r_2, q) + r_2z_2^*(r_1, r_2, q) \]
- Cost is a function of output and input prices.
- When prices fixed, sometimes write \( C(q) \)
Average Cost Function

• The average cost function (AC) is found by computing total costs per unit of output.

\[
\text{average cost} = AC(r_1, r_2, q) = \frac{C(r_1, r_2, q)}{q}
\]

Marginal Cost Function

• The marginal cost function (MC) equals the extra cost from one extra unit of output.

\[
\text{marginal cost} = MC(r_1, r_2, q) = \frac{\partial C(r_1, r_2, q)}{\partial q}
\]

Picture #1

• Concave production function.
Cost Function: Properties

1. \( c(r_1, r_2, q) \) is homogenous of degree 1 in \((r_1, r_2)\)
   - If prices double constraint unchanged, so cost doubles.

2. \( c(r_1, r_2, q) \) is increasing in \((r_1, r_2, q)\)

3. Shepard’s Lemma:
   \[
   \frac{\partial}{\partial n} c(r_1, r_2, q) = \zeta'(r_1, r_2, q)
   \]
   - If \( r_1 \) rises by \( \Delta r \), then \( c(.) \) rises by \( \Delta r \times \zeta'(.) \)
   - Input demand also changes, but effect second order.

4. \( c(r_1, r_2, q) \) is concave in \((r_1, r_2)\)
Since the firm's input mix will likely change, actual costs will be less than $C_{\text{pseudo}}$ such as $C(r_1, r_2, q)$.

If the firm continues to buy the same input mix as $r_1$ changes, its cost function would be $C_{\text{pseudo}}$.

Cost Function: Concavity and Shepard’s Lemma

At $r_1^*$, the cost is $c(r_1^*, \ldots, r_2^*, z_1^*, z_2^*)$.

Since the firm's input mix will likely change, actual costs will be less than $C_{\text{pseudo}}$ such as $C(r_1, r_2, q)$.

Cost Function: Properties

5. If $f(z_1, z_2)$ is concave then $c(r_1, r_2, q)$ is convex in $q$.
   - Hence $MC(q)$ increases in $q$.
   - Concavity implies decreasing returns.
   - More inputs needed for each unit of $q$, raising cost.

6. If $f(z_1, z_2)$ is exhibits decreasing (increasing) returns then $AC(q)$ increases (decreases) in $q$.
   - Under DRS, doubling inputs produces less than double output. Hence average cost rises.

7. $AC(q)$ is increasing when $MC(q) \geq AC(q)$, and decreasing when $MC(q) \leq AC(q)$.
   - If $MC(q) \geq AC(q)$ then cost being dragged up.
   - When $AC(q)$ minimized, $MC(q) = AC(q)$.

Average and Marginal Costs

$MC$ is the slope of the $C$ curve.

If $AC > MC$, $AC$ must be falling.

If $AC < MC$, $AC$ must be rising.
Can Costs Look Like This?

- Left: When AC minimized, MC=AC.
- Right: If no fixed costs AC=MC for first unit. If fixed costs, AC=∞ for first unit.

Input Demand: Properties
1. \( z^* (r_1, r_2, q) \) is homogenous of degree 0 in \( (r_1, r_2) \)
   - If prices double constraint unchanged, so demand unchanged.
2. Symmetry of cross derivatives
   \[
   \frac{\partial z^*}{\partial r_2} = \frac{\partial}{\partial r_1} \left[ \frac{\partial z^*}{\partial r_1} \right] = \frac{\partial}{\partial r_1} \left[ \frac{\partial z^*}{\partial r_2} \right] = \frac{\partial}{\partial r_2} \frac{\partial z^*}{\partial r_1}
   \]
   - Uses Shepard’s Lemma
3. Law of demand
   \[
   \frac{\partial z^*}{\partial r_1} = \frac{\partial}{\partial r_1} \left[ \frac{\partial z^*}{\partial r_2} \right] \leq 0
   \]
   - Uses Shepard’s Lemma and concavity of \( c(.) \)

SHORT-RUN VS. LONG-RUN
Short-Run, Long-Run Distinction

• Costs may differ in the short and long run.
• In the short run it is (relatively) easy to hire and fire workers but relatively difficult to change the level of the capital stock.
• Suppose firm wishes to raise production
  – Can’t change capital stock
  – Hires more workers.
  – Capital/Labor balance no longer optimal.
  – High production costs.

Time Frames

• In very short run, all inputs are fixed.
• In short run, some inputs fixed with others are flexible.
• In medium run, all inputs are flexible but firm cannot enter/exit.
  – Fixed costs are sunk.
• In long run, all factor are flexible and firm can exit without cost.

Example: \( f(z_1,z_2)=(z_1-1)^{1/3}(z_2-1)^{1/3} \)

• Cobb-Douglas production but first unit of each input is useless.
• In long run,
  \[ L = r_1z_1 + r_2z_2 + \lambda[q - (z_1-1)^{1/3}(z_2-1)^{1/3}] \]
• FOC becomes \( r_1(z_1-1)=r_2(z_2-1) \).
• Using constraint, demands are
  \[ z_1^* = q(r_1)q^{1/2} + 1 \]
  \[ z_2^* = q(r_2)q^{1/2} + 1 \]
• Long-run cost function
  \[ c(r_1,r_2,q) = r_1z_1^* + r_2z_2^* = 2q(r_2)q^{1/2} + (r_1 + r_2) \]
  with \( c(r_1,r_2,0)=0 \).
Example: \( f(z_1,z_2) = (z_1-1)^{1/3}(z_2-1)^{1/3} \)

- In medium run, startup cost of \((r_1+r_2)\) is sunk.

- Cost function is thus
  
  \[
  c(r_1,r_2,q) = r_1 z_1^0 + r_2 z_2^0 = 2(z_1 r_1)^{1/3} q^{1/3} + (r_1 + r_2) \]
  
  with \( c(r_1,r_2,0) = r_1 + r_2 \).

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Example: \( f(z_1,z_2) = (z_1-1)^{1/3}(z_2-1)^{1/3} \)

- In short run, \( z_2 \) is fixed at \( z_2^* \).

- The constraint in the CMP becomes
  
  \[
  q = (z_1-1)^{1/3}(z_2^*-1)^{1/3} \]

- Rearranging,
  
  \[
  z_1' = \frac{q}{z_2^-} + 1 \]

- Cost function is
  
  \[
  c(r_1,r_2,q) = r_1 z_1'^0 + r_2 z_2'^0 = \frac{q}{z_2^-} r_1 + r_2 z_2'^0 \]

- In very short run, \((z_1,z_2)\) fixed so output fixed.

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Short-Run Total Costs

When \( z_2 \) is fixed at \( z_2^* \), the firm cannot equate MRTS with the ratio of input prices.
Relationship between Short-Run and Long-Run Costs

The long-run C curve can be derived by varying the level of $z_2$.

Short-Run Marginal and Average Costs

- The short-run average total cost ($SAC$) function is:
  \[ SAC = \text{total costs/total output} = \frac{SC}{q} \]
- The short-run marginal cost ($SMC$) function is:
  \[ SMC = \text{change in } SC/\text{change in output} = \frac{\partial SC}{\partial q} \]