PROFIT MAXIMIZATION

[See Chap 11]

Profit Maximization

• A profit-maximizing firm chooses both its inputs and its outputs with the goal of achieving maximum economic profits

Model

• Firm has inputs \((z_1, z_2)\). Prices \((r_1, r_2)\).
  – Price taker on input market.
• Firm has output \(q = f(z_1, z_2)\). Price \(p\).
  – Price taker in output market.
• Firm’s problem:
  – Choose output \(q\) and inputs \((z_1, z_2)\) to maximise profits. Where:
    \[
    \pi = pq - r_1z_1 - r_2z_2
    \]
One-Step Solution

- Choose \((z_1, z_2)\) to maximise
  \[ \pi = pf(z_1, z_2) - r_1 z_1 - r_2 z_2 \]
- This is unconstrained maximization problem.
- FOCs are
  \[ \frac{\partial f(z_1, z_2)}{\partial z_1} = r_1 \quad \text{and} \quad \frac{\partial f(z_1, z_2)}{\partial z_2} = r_2 \]
- Together these yield optimal inputs \(z_1^*\) and \(z_2^*\).
- Output is \(q^*(p, r_1, r_2) = f(z_1^*, z_2^*)\). This is usually called the supply function.
- Profit is \(\pi(p, r_1, r_2) = pq - r_1 z_1^* - r_2 z_2^*\)

Example: \(f(z_1, z_2) = z_1^{1/3} z_2^{1/3}\)

- Profit is \(\pi = pz_1^{1/3} z_2^{1/3} - r_1 z_1 - r_2 z_2\)
- FOCs are
  \[ \frac{1}{3} p z_1^{2/3} z_2^{1/3} = r_1 \quad \text{and} \quad \frac{1}{3} p z_1^{1/3} z_2^{2/3} = r_2 \]
- Solving these two eqns, optimal inputs are
  \[ c(p, r_1, r_2) \quad \text{and} \quad c(p, r_1, r_2) \]
- Optimal output
  \[ q^*(p, r_1, r_2) = (c_1)^{1/3} (c_2)^{1/3} = \frac{p}{3 r_1} \]
- Profits
  \[ \pi^*(p, r_1, r_2) = pq - c(p, r_1, r_2) = \frac{p^2}{3 r_1 r_2} \]

Two-Step Solution

**Step 1:** Find cheapest way to obtain output \(q\).

\[ c(r_1, r_2, q) = \min_{z_1, z_2} r_1 z_1 + r_2 z_2 \quad \text{s.t} \quad f(z_1, z_2) \geq q \]

**Step 2:** Find profit maximizing output.

\[ \pi(p, r_1, r_2) = \max_q \quad pq - c(r_1, r_2, q) \]

This is unconstrained maximization problem.

- Solving yields optimal output \(q^*(r_1, r_2, p)\).
- Profit is \(\pi(p, r_1, r_2) = pq^* - c(r_1, r_2, q^*)\).
Step 2: Output Choice

- We wish to maximize $pq - c(r_1, r_2, q)$
- The FOC is $p = dc(r_1, r_2, q)/dq$
- That is, $p = MC(q)$
- Intuition: produce more if revenue from unit exceeds the cost from the unit.
- SOC: $MC'(q) \geq 0$, so MC curve must be upward sloping at optimum.

Example: $f(z_1, z_2) = z_1^{1/3}z_2^{1/3}$

- From cost slides (p18), $c(r_1, r_2, q) = 2(r_1r_2)^{1/2}q^{3/2}$
- We wish to maximize $\pi = pq - 2(r_1r_2)^{1/2}q^{3/2}$
- FOC is $p = 3(r_1r_2)^{1/2}q^{1/2}$
- Rearranging, optimal output is $q^*(p, r_1, r_2) = \frac{1}{9} \frac{\rho''}{\rho'}$
- Profits are $\pi^*(p, r_1, r_2, q) = pq - c(r_1, r_2, q) = \frac{1}{27} \rho'^2$

Profit Function

- Profits are given by $\pi = pq^* - c(q^*)$
- We can write this as $\pi = pq^* - AC(q^*)q^* = [p - AC(q^*)]q^*$
- We can also write this as $\pi = pq^* - \int_{\pi}^{} MC(x)dx - F = \int_{\pi}^{} (p - MC(x))dx - F$
  where $F$ is fixed cost
Profit Function

Left: Profit is distance between two lines.
Right: Max profit equals A+B+C. If no fixed cost, this equals A+B+D+E.

Supply Functions

Supply with Fixed Cost

Maximum profit occurs where $p = MC$.
Since $p > AC$, we have $\pi > 0$.

If the price rises to $p^*$, the firm will produce $q^*$ and $\pi > 0$.

If the price falls to $p^{**}$, we might think the firm chooses $q^{**}$. But $\pi < 0$ so firm prefers $q=0$. 
Supply Curve

- We can use the marginal cost curve to show how much the firm will produce at every possible market price.
- The firm can always choose q=0
  - Firm only operates if revenue covers costs.
  - Firm chooses q=0 if pq<c or p < AC.

Supply Function #1

- Increasing marginal costs.
- No fixed cost

Supply Function #2

- Increasing marginal costs.
- Fixed cost
Supply Function #3
- U-shaped marginal costs.
- No fixed cost.

Supply Function #4
- Wiggly marginal costs.
- No fixed cost

Sunk Costs
- In the short run, there may be sunk costs (i.e. unavoidable costs).
- Let AVC = average variable cost (excluding sunk cost).
- Then supply function is given by MC curve if p>AVC.
- Firm shuts down if p<AVC.
Short-Run Supply by a Price-Taking Firm

- The firm’s short-run supply curve is the SMC curve that is above SAVC.

Cost Function: Properties

1. \( \pi(p,r_1,r_2) \) is homogenous of degree 1 in \((p,r_1,r_2)\)
   - If prices double, profit equation scales up so optimal choices unaffected and profit doubles.
2. \( \pi(p,r_1,r_2) \) increases in \( p \) and decreases in \((r_1,r_2)\)
3. Hotelling’s Lemma:
   \[ \frac{\partial}{\partial p} \pi(p,r_1,r_2) = q^*(p,r_1,r_2) \]
   - If \( p \) rises by \( \Delta p \), then \( \pi(.) \) rises by \( \Delta p \times q^*(.) \)
   - Optimal output also changes, but effect second order.
4. \( \pi(p,r_1,r_2) \) is convex in \( p \).

Convexity and Hotelling’s Lemma

- This shows the pseudo-profit functions, when output is fixed, and real profit function.
Convexity and Hotelling’s Lemma

- Hotelling Lemma: \( B+C \approx B \) when \( \Delta p \) small.
- Convexity: \( C \) means profit increases more than linearly.

Supply Functions: Properties

1. \( q^*(p,r_1,r_2) \) is homogenous of degree 0 in \((p,r_1,r_2)\)
   - If prices double profit equation scales up, so optimal output unaffected.

2. Law of supply
   \[ \frac{\partial q}{\partial p} = \frac{\partial}{\partial p} \left[ \frac{\partial}{\partial p} \pi \right] \leq 0 \]
   - Uses Hotelling’s Lemma and convexity of \( \pi(.) \)
   - Hence supply curve is upward sloping!