Preferences and Utility

Simon Board*

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These lectures examine the preferences of a single agent. In Section 1 we analyse how the agent chooses among a number of competing alternatives, investigating when preferences can be represented by a utility function. In Section 2 we discuss two attractive properties of preferences: monotonicity and convexity. In Section 3 we analyse the agent's indifference curves and ask how she makes tradeoffs between different goods. Finally, in Section 4 we look at some examples of preferences, applying the insights of the earlier theory.

1 The Foundation of Utility Functions

1.1 A Basic Representation Theorem

Suppose an agent chooses from a set of goods $X = \{a, b, c, \ldots\}$. For example, one can think of these goods as different TV sets or cars.

Given two goods, x and y, the agent **weakly prefers** x over y if x is at least as good as y. To avoid us having to write "weakly prefers" repeatedly, we simply write $x \succeq y$. We now put some basic structure on the agent's preferences by adopting two axioms.[1]

Completeness Axiom: For every pair $x, y \in X$, either $x \succeq y$, $y \succeq x$, or both.

*Department of Economics, UCLA. http://www.econ.ucla.edu/sboard/. Please email suggestions and typos to sboard@econ.ucla.edu.

[1] An axiom is a foundational assumption.

Transitivity Axiom: For every triple $x, y, z \in X$, if $x \succcurlyeq y$ and $y \succcurlyeq z$ then $x \succcurlyeq z$.

An agent has complete preferences if she can compare any two objects. An agent has transitive preferences if her preferences are internally consistent. Let's consider some examples.

First, suppose that, given any two cars, the agent prefers the faster one. These preferences are complete: given any two cars x and y, then either x is faster, y is faster or they have the same speed. These preferences are also transitive: if x is faster than y and y is faster than z, then x is faster than z.

Second, suppose that, given any two cars, the agent prefers x to y if it is both faster and bigger. These preferences are transitive: if x is faster and bigger than y and y is faster and bigger than z, then x is faster and bigger than z. However, these preferences are not complete: an SUV is bigger and slower than a BMW, so it is unclear which the agent prefers. The completeness axiom says these preferences are unreasonable: after examining the SUV and BMW, the agent will have a preference between the two.

Third, suppose that the agent prefers a BMW over a Prius because it is faster, an SUV over a BMW because it is bigger, and a Prius over an SUV, because it is more environmentally friendly. In this case, the agent's preferences cycle and are therefore intransitive. The transitivity axiom says these preferences are unreasonable: if environmental concerns are so important to the agent, then she should also take them into account when choosing between the Prius and BMW, and the BMW and the SUV.

While it is natural to think about preferences, it is often more convenient to associate different numbers to different goods, and have the agent choose the good with the highest number. These numbers are called **utilities**. In turn, a **utility function** tells us the utility associated with each good $x \in X$, and is denoted by $u(x) \in \Re$. We say a utility function $u(x)$ **represents** an agent's preferences if

$$u(x) \geq u(y) \quad \text{if and only if} \quad x \succcurlyeq y \tag{1.1}$$

This means than an agent makes the same choices whether she uses her preference relation, $\succcurlyeq$, or her utility function $u(x)$.

Theorem 1 (Utility Representation Theorem)**.** *Suppose the agent's preferences, $\succcurlyeq$, are complete and transitive, and that X is finite. Then there exists a utility function $u(x) : X \to \Re$ which represents $\succcurlyeq$.*

Theorem 1 says that if an agent has complete and transitive preferences then we can associate these preferences with a utility function. Intuitively, the two axioms allow us to rank the goods under consideration. For example, if there are 10 goods, then we can say the best has a utility $u(x) = 9$, the second best has $u(x) = 8$, the third best has $u(x) = 7$ and so on. For a formal proof, see Section 1.2.

1.2 A Proof of Theorem 1[2]

The idea behind the proof is simple. For any good x, let $NBT(x) = \{y \in X | x \succcurlyeq y\}$ be the goods that are "no better than" x. The utility of x is simply given by the number of items in $NBT(x)$. That is,

$$u(x) = |NBT(x)|. \tag{1.2}$$

If there are 10 goods, then the worst has a "no better than" set which is empty, so that $u(x) = 0$. The second worst has a has a "no better than" set which has one element, so $u(x) = 1$. And so on.

We now have to verify that this utility function represents the agent's preferences. We do this in two steps: first, we show that $x \succcurlyeq y$ implies $u(x) \geq u(y)$; second, we show that $u(x) \geq u(y)$ implies $x \succcurlyeq y$.

Step 1: Suppose $x \succcurlyeq y$. Pick any $z \in NBT(y)$;[3] by the definition of $NBT(y)$, we have $y \succcurlyeq z$. Since preferences are complete, we know that z is comparable to x. Transitivity then tells us that $x \succcurlyeq z$, so $z \in NBT(x)$. We have therefore shown that every element of $NBT(y)$ is also an element of $NBT(x)$; that is, $NBT(y) \subseteq NBT(x)$. As a result,

$$u(x) = |NBT(x)| \geq |NBT(y)| = u(y)$$

as required.

Step 2: Suppose $u(x) \geq u(y)$. By completeness, we know that either $x \succcurlyeq y$ or $y \succcurlyeq x$. Using Step 1, it must then be the case that either $NBT(y) \subseteq NBT(x)$ or $NBT(x) \subseteq NBT(y)$, so the "no better than" sets cannot partially overlap or be disjoint. By the definition of utilities (1.2) we know that there are more elements in $NBT(x)$ than in $NBT(y)$, which implies that

[2] More advanced.

[3] $z \in NBT(y)$ means that z is an element of $NBT(y)$.

$NBT(y) \subseteq NBT(x)$. Completeness means that a good is weakly preferred to itself, so that $y \in NBT(y)$. Since $NBT(y) \subseteq NBT(x)$, we conclude $y \in NBT(x)$. Using the definition of the "no better than" set, this implies that $x \succcurlyeq y$, as required.

1.3 Increasing Transformations

A number system is **ordinal** if we only care about the ranking of the numbers. It is **cardinal** if we also care about the magnitude of the numbers. To illustrate, Usain Bolt and Richard Johnson came 1st and 2nd in the 2008 Olympic final of the 100m sprint. The numbers 1 and 2 are ordinal: they tell us that Bolt beat Johnson, but do not tell us that he was 1% faster or 10% faster. The actual finishing times were 9.69 for Bolt and 9.89 for Johnson. These numbers are cardinal: the ranking tells us who won, and the magnitudes tells us about the margin of the win.

Theorem 1 is ordinal: when comparing two goods, all that matters is the ranking of the utilities; the actual numbers themselves carry no significance. This is obvious from the construction: when there are 10 goods, it is clearly arbitrary that we give utility 9 to the best good, 8 to the second best, and so on. This idea can be formalised by the following result:

Theorem 2. *Suppose $u(x)$ represents the agent's preferences, $\succcurlyeq$, and $f : \Re \to \Re$ is a strictly increasing function. Then the new utility function $v(x) = f(u(x))$ also represents the agent's preferences $\succcurlyeq$.*

The proof of Theorem 2 is simply a rewriting of definitions. Suppose $u(x)$ represents the agent's preferences, so that equation (1.1) holds. If $x \succcurlyeq y$ then $u(x) \geq u(y)$ and $f(u(x)) \geq f(u(y))$, so that $v(x) \geq v(y)$. Conversely, if $v(x) \geq v(y)$ then, since $f(\cdot)$ is strictly increasing, $u(x) \geq u(y)$ and $x \succcurlyeq y$. Hence

$$v(x) \geq v(y) \quad \text{if and only if} \quad x \succcurlyeq y$$

and $v(x)$ represents $\succcurlyeq$.

Theorem 2 is important when solving problems. Suppose an agent has utility function

$$u(x) = -\frac{15}{(x_1^{1/2} + x_2^{1/2} + 10)^3}$$

Solving the agent's problem with this utility function may be be algebraically messy. Using

Theorem 2, we can rewrite the agent's utility as

$$v(x) = x_1^{1/2} + x_2^{1/2}$$

Since $u(x)$ and $v(x)$ preserve the rankings of the goods, they represent the same preferences. As a result, the agent will make the same choices with utility $u(x)$ and $v(x)$. This is useful since it is much simpler to solve the agent's choice problem using $v(x)$ than $u(x)$.

Theorem 2 is also useful for cocktail parties. For example, some people dislike the way I rank movies of a 1-10 scale. They claim that a movie is a rich artistic experience, and cannot be summarised by a number. However, Theorem 1 tells us that, if my preferences are complete and transitive, then I can represent my preferences over movies by a number. Moreover, Theorem 2 tells us that I can rescale the numbers to put them on a 1-10 scale.

Choosing from Budget Sets

Theorem 1 assumes that the consumer chooses from a finite number of goods. While this is realistic, it is more mathematically convenient to allow consumers to choose from a continuum of goods. For example, if the agent has \$10 and a hamburger costs \$2, it is easier to allow the consumer to any number between 0 and 5, rather than forcing her to choose an integer.

Suppose the choice set is given by $X \subseteq \Re^n_+$. A typical element is $x = (x_1, \ldots, x_n)$, where x_i is the number of the i^{th} good the agent consumes. In order to prove a representation theorem for this larger set of choices, we need one more (rather technical) axiom.

Continuity Axiom: Suppose $x^1, x^2, x^3, \ldots$ is a sequence of feasible choices, so that $x^i \in X$ for each i, and suppose the sequence converges to $x \in X$. If $x^i \succcurlyeq y$ for each i, then $x \succcurlyeq y$.

Theorem 3 (Representation Theorem for Budget Sets)**.** *Suppose the agent's preferences, $\succcurlyeq$, are complete, transitive and continuous, and that $X \subseteq \Re^n_+$. Then there exists a continuous utility function $u(x) : X \to \Re$ which represents $\succcurlyeq$.*

We will not prove this result. The following example examines a case where the continuity axiom does not hold and no utility representation exists.

Suppose there are two goods and the agent has **lexicographic preferences:** when faced with two bundles the agent prefers the bundle with the most of x_1; if the two bundles have the same

x_1 then she prefers the bundle with the most of x_2. To verify that this does not satisfy the continuity axiom, consider a sequence of bundles $x^i = (1 + \frac{1}{i}, 1)$ which converges to $x = (1, 1)$ as $i \to \infty$, and let $y = (1, 2)$. For each i, x^i is preferred to y since x^i contains more of good 1. However, in the limit, the agent prefers y to x since they have the same quantity of good 1, but y has more of good 2. One can also show that there exists no utility function that represents lexicographic preferences, but this is a little tricky.

2 Properties of Preferences

In this Section we introduce two key properties of preferences: monotonicity and convexity. Throughout, we suppose $X \subseteq \Re^n_+$.

First we need a couple of definitions. If the agent weakly prefers x to y (i.e. $x \succcurlyeq y$) and weakly prefers y to x (i.e. $y \succcurlyeq x$) then she is **indifferent** between x and y and we write $x \sim y$. In terms of utilities, an agent is indifferent between x and y if and only if $u(x) = u(y)$.

If the agent weakly prefers x to y ($x \succcurlyeq y$) and is not indifferent between x and y, then she **strictly prefers** x to y and we write $x \succ y$. In terms of utilities, an agent strictly prefers x to y if and only if $u(x) > u(y)$.

2.1 Monotonicity

Preferences are **monotone** if for any two bundles $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$,

$$\left.\begin{array}{l} x_i \geq y_i \text{ for each } i \\ x_i > y_i \text{ for some } i \end{array}\right\} \text{ implies } x \succ y.$$

In words: preferences are monotone if more of any good makes the agent strictly better off.

While monotonicity is stated in terms of preferences, we can rewrite it in terms of utilities. Preferences are monotone if for any two bundles x and y,

$$\left.\begin{array}{l} x_i \geq y_i \text{ for each } i \\ x_i > y_i \text{ for some } i \end{array}\right\} \text{ implies } u(x) > u(y).$$

As we will see, the assumption of monotonicity is very useful. It implies that indifference curves are thin and downwards sloping. It implies that an agent will always spend her budget. A slightly stronger version of monotonicity also rules out inflexion points in the agent's utility function which is useful when we analyse the agent's utility maximisation problem.

2.2 Convexity

Preferences are **convex** if whenever $x \succcurlyeq y$ then

$$tx + (1-t)y \succcurlyeq y \qquad \text{for all } t \in [0,1]$$

Convexity says that the agent prefers averages to extremes: if the agent is indifferent between x and y then she prefers the average $tx + (1-t)y$ to either x or y.

We can write this assumption in terms of utilities. Preferences are convex if whenever $u(x) \geq u(y)$ then

$$u(tx + (1-t)y) \geq u(y) \qquad \text{for all } t \in [0,1] \tag{2.1}$$

Slightly confusingly, a utility function that satisfies (2.1) is called **quasi–concave**.

The assumption of convexity if important when we analyse the consumers utility maximisation problem. Along with monotonicity, it means that any solution to the agent's first–order conditions solve the agent's problem.

3 Indifference Curves

An agent's **indifference curve** is the set of bundles which yield a constant level of utility. That is,

$$\text{Indifferent Curve} = \{x \in X | u(x) = \text{const.}\}$$

An agent has a collection of indifference curves, each one corresponding to a different level of utility. By varying this level, we can trace out the agent's entire preferences.

To illustrate, suppose an agent has utility

$$u(x_1, x_2) = x_1 x_2$$

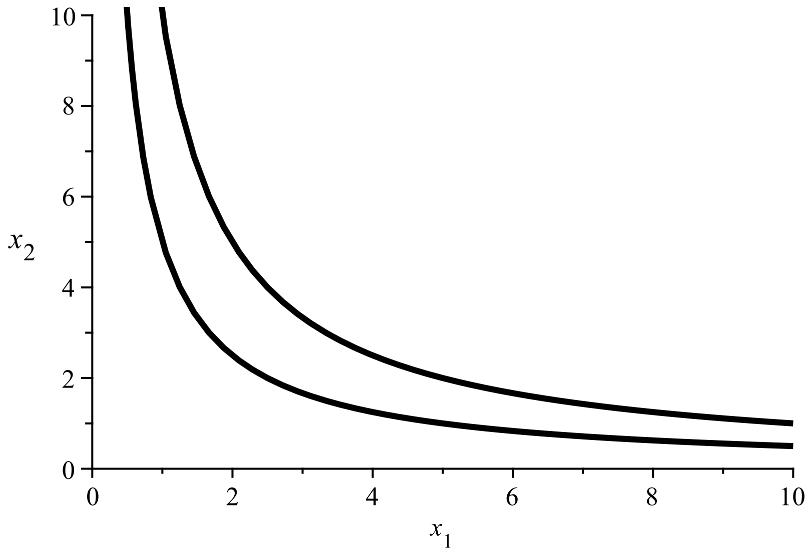

Figure 1: **Indifference Curves.** This figure shows two indifference curves. Each curve depicts the bundles that yield constant utility.

Then the indifference curve satisfies the equation $x_1 x_2 = k$. Rearranging, we can solve for x_2, yielding

$$x_2 = \frac{k}{x_1} \tag{3.1}$$

which is the equation of a hyperbola. This function is plotted in figure 1.

In Section 3.1 we derive five important properties of indifference curves. In Section 3.2 we introduce the idea of the marginal rate of substitution. For simplicity, we assume there are only two goods.

3.1 Properties of Indifference Curves

We now describe five important properties of indifference curves. Throughout, we assume that preferences satisfy completeness, transitivity and continuity, so a utility function exists. We also assume monotonicity.

1. Indifference curves are thin. We say an indifference curve is thick if it contains two points x and y such that $x_i > y_i$ for all i. This is illustrated in figure 2. Monotonicity says that y must

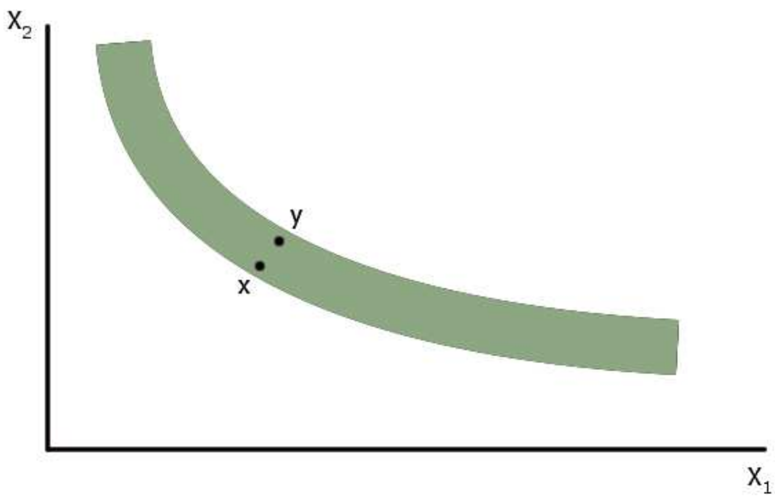

Figure 2: **A Thick Indifference Curve.** This figure shows a thick indifference curve containing points x and y.

be strictly preferred to x and therefore rules out thick indifference curves.

2. Indifference curves never cross. Suppose, by contradiction, that two indifference curves cross, as shown in figure 3. Since they lie on the same indifference curve, the agent is indifferent between points A and D, and indifferent between points B and C. In addition, by monotonicity, the agent strictly prefers A to B, and strictly prefers C to D. Putting all this together,

$$A \succ B \sim C \succ D \sim A$$

We conclude that A is strictly preferred to itself, which is false. Intuitively, two indifference curves describe the bundles that yield two different utility levels. By monotonicity, one indifference curve must always lie to the northeast of the other.

3. Indifference curves are strictly downward sloping. If an indifference curve is not strictly downward sloping, then we can find points x and y on the same indifference curve such that $y_i \geq x_i$ for all i, and $y_i > x_i$ for some i, as shown in figure 4. This contradicts monotonicity, which says the agent strictly prefers y to x.

4. Indifference curves are continuous, with no gaps. We cannot have gaps in the indifference curve, as shown in figure 5. This follows from preferences being continuous which, by Theorem 3, implies that the utility function is continuous.

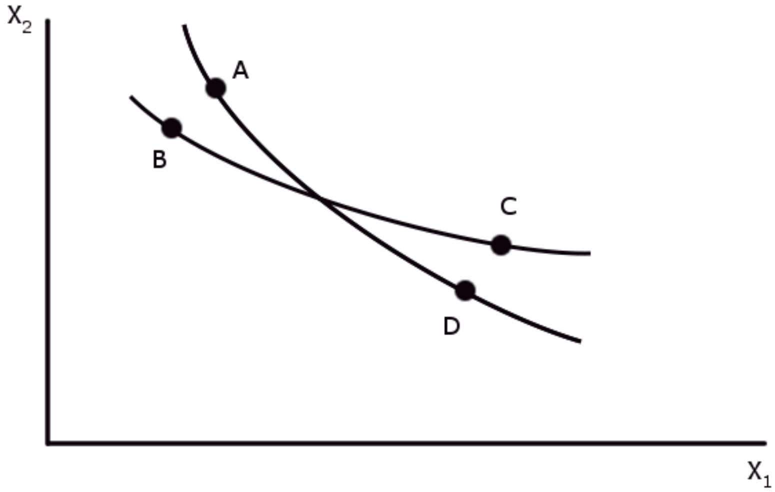

Figure 3: **Indifference Curves that Cross.**

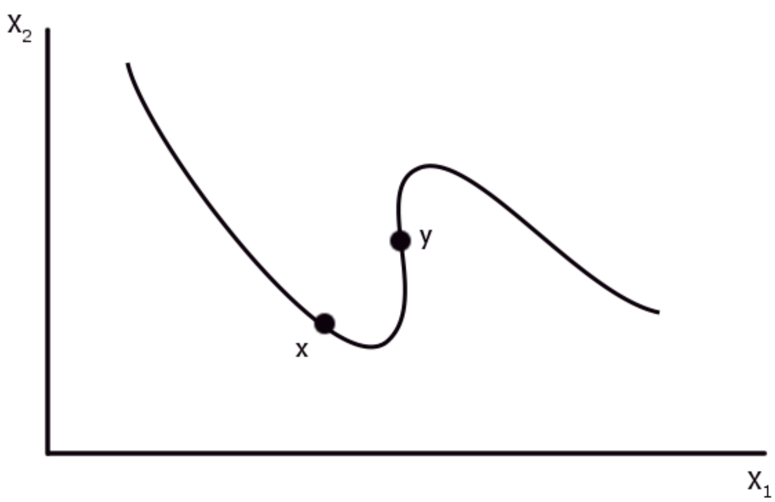

Figure 4: **An Upward Sloping Indifference Curve.**

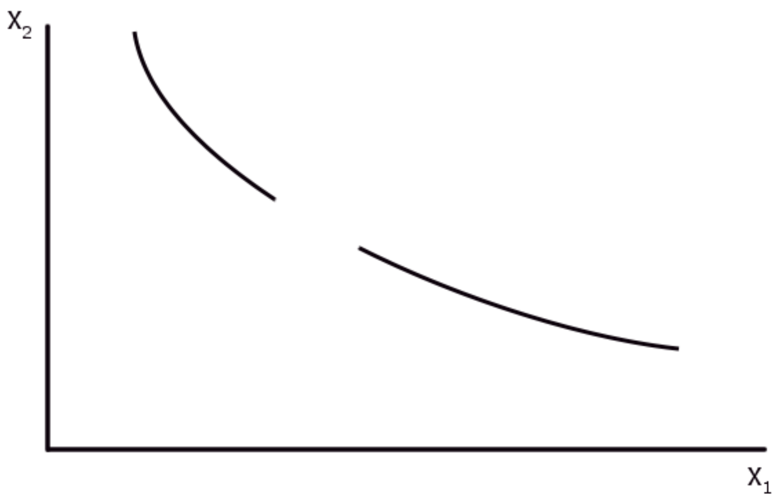

Figure 5: **An Indifference Curve with a Gap.**

5. If preferences are convex then indifference curves are convex to the origin. Suppose x and y lie on an indifference curve. By convexity, $tx + (1-t)y$ lies on a higher indifference curve, for $t \in [0,1]$. By monotonicity, this higher indifference curve lies to the northeast of the original indifference curve. Hence the indifference curve is convex, as shown in figure 6.

3.2 Marginal Rate of Substitution

The slope of the indifference curve measures the rate at which the agent is willing to substitute one good for another. This slope is called the **marginal rate of substitution** or **MRS**. Mathematically,

$$MRS = -\frac{dx_2}{dx_1}\Big|_{u(x_1,x_2)=\text{const.}} \tag{3.2}$$

We can rephrase this definition in words: the MRS equals the number of x_2 the agent is willing to give up in order to obtain one more x_1. This is shown in figure 7.

The MRS can be related to the agent's utility function. First, we need to introduce the idea of **marginal utility**

$$MU_i(x_1, x_2) = \frac{\partial u(x_1, x_2)}{\partial x_i}$$

which equals the gain in utility from one extra unit of good i.

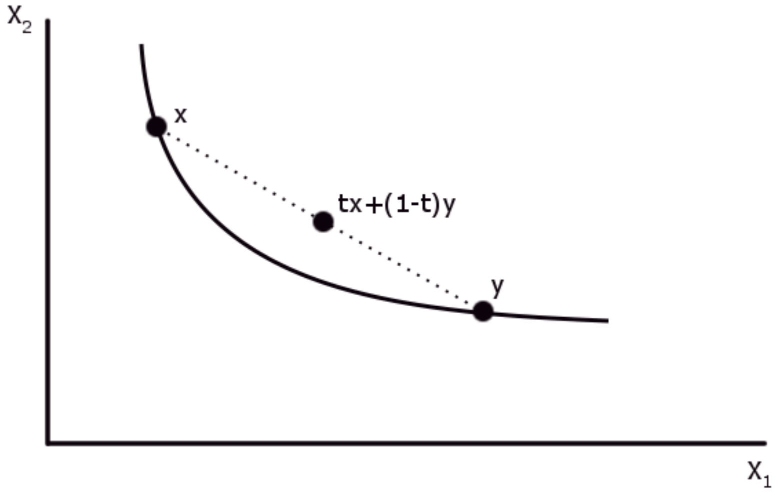

Figure 6: **Convex Indifference Curves.**

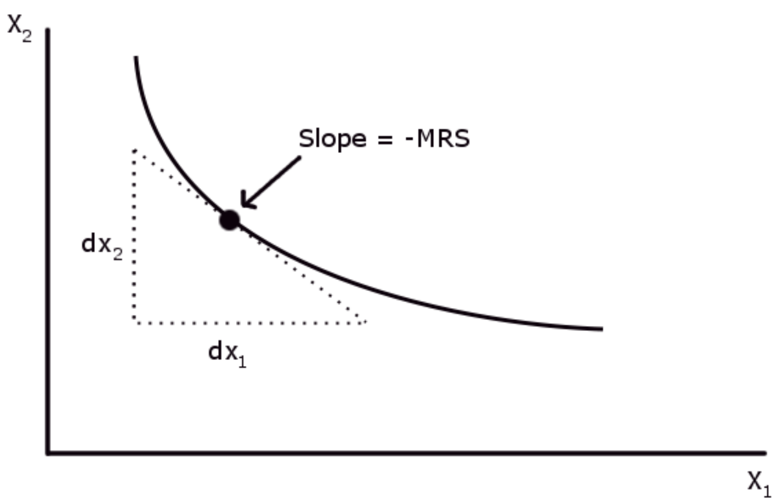

Figure 7: **Marginal Rate of Substitution.**

Let us consider the effect of a small change in the agent's bundle. Totally differentiating the utility $u(x_1, x_2)$ we obtain

$$du = \frac{\partial u(x_1, x_2)}{\partial x_1} dx_1 + \frac{\partial u(x_1, x_2)}{\partial x_2} dx_2 \tag{3.3}$$

Equation (3.3) says that the agent's utility increases by her marginal utility from good 1 times the increase in good 1 plus the marginal utility from good 1 times the increase in good 2. Along an indifference curve $du = 0$, so equation (3.3) becomes

$$\frac{\partial u(x_1, x_2)}{\partial x_1} dx_1 + \frac{\partial u(x_1, x_2)}{\partial x_2} dx_2 = 0$$

Rearranging,

$$-\frac{dx_2}{dx_1} = \frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2}$$

Equation (3.2) therefore implies that

$$MRS = \frac{MU_1}{MU_2} \tag{3.4}$$

The intuition behind equation (3.4) is as follows. Using the definition of MRS, one unit of x_1 is worth MRS units of x_2. That is, $MU_1 = MRS \times MU_2$. Rewriting this equation we obtain (3.4).

We can relate MRS to our earlier concepts of monotonicity and convexity. Monotonicity says that the indifference curve is downward sloping. Using equation (3.2), this means that MRS is positive.

Under the assumption of monotonicity, convexity says that the indifference curve is convex. This means that the MRS decreasing in x_1 along the indifference curve. Formally, an indifference curve defines an implicit relationship between x_1 and x_2,

$$u(x_1, x_2(x_1)) = k$$

Convexity then implies that $MRS(x_1, x_2(x_1))$ is decreasing in x_1. This is illustrated in the next section.

Finally, we can relate the MRS to the ordinal nature of the utility representation. In Theorem 2 we showed that the choices made under $u(x)$ and $v(x) = f(u(x))$ are the same, where $f : \Re \to \Re$ is strictly increasing. One way to understand this result is through the MRS. Under utility function $u(x)$ the MRS is given by equation (3.4). Under utility function $v(x)$, the MRS is given by

$$MRS^v = \frac{\partial v/\partial x_1}{\partial v/\partial x_2} = \frac{f'(u)\partial u/\partial x_1}{f'(u)\partial u/\partial x_2} = \frac{\partial u/\partial x_1}{\partial u/\partial x_2} = MRS^u$$

where the second equality uses the chain rule. This means that the agent faces the same tradeoffs under the two utility functions, has identical indifference curves, and therefore makes the same decisions.

3.3 Example: Symmetric Cobb Douglas

Suppose $u(x_1, x_2) = x_1 x_2$. We calculate the marginal rate of substitution two ways.

First, we can use equation (3.2) to derive MRS. As in equation (3.1), the equation of an indifference curve is

$$x_2 = \frac{k}{x_1} \tag{3.5}$$

Differentiating,

$$MRS = -\frac{dx_2}{dx_1} = \frac{k}{x_1^2} \tag{3.6}$$

We can now verify preferences are convex. Differentiating (3.6) with respect to x_1,

$$\frac{d}{dx_1} MRS = -2\frac{k}{x_1^3}$$

which is negative, as required.

Alternatively, we can use equation (3.4) to derive MRS. Differentiating the utility function

$$MRS = \frac{MU_1}{MU_2} = \frac{x_2}{x_1} \tag{3.7}$$

We now want to express MRS purely in terms of x_1. Using (3.5) to substitute for x_2, equation (3.7) becomes (3.6).

4 Examples of Preferences

4.1 Cobb Douglas

The Cobb–Douglas utility function is given by

$$u(x_1, x_2) = x_1^\alpha x_2^\beta \qquad \text{for } \alpha > 0,\ \beta > 0$$

A special case if the symmetric Cobb–Douglas, when $\alpha = \beta$. Using Theorem 2, we can then normalise the symmetric Cobb–Douglas to $\alpha = \beta = 1$.

The Cobb–Douglas indifference curve has equation $x_1^\alpha x_2^\beta = k$. Rearranging,

$$x_2 = k^{1/\beta} x_1^{-\alpha/\beta}$$

These indifference curves look like those in figure 1.

The marginal utilities are

$$MU_1 = \alpha x_1^{\alpha-1} x_2^\beta$$
$$MU_1 = \beta x_1^\alpha x_2^{\beta-1}$$

As a result the MRS is,

$$MRS = \frac{MU_1}{MU_2} = \frac{\alpha x_2}{\beta x_1}$$

4.2 Perfect Complements

Suppose an agent always consumes a hamburger patty with two slices of bread. If she has 5 patties and 15 slices of bread, then the last 5 slices are worthless. Similarly, if she has 7 patties and 10 slices of bread, then the last 2 patties are worthless. In this case, the agent's preferences can be represented by the utility function

$$u(x_1, x_2) = \min\{2x_1, x_2\}$$

where x_1 are patties and x_2 are slices of bread. Note the 2 goes in front of the number of patties because, intuitively speaking, each patty is twice as valuable as a piece of bread.

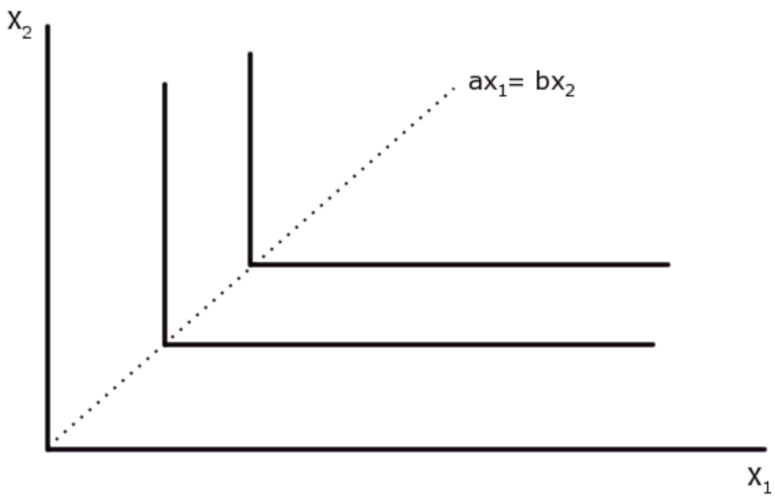

Figure 8: **Perfect Complements.** These indifference curves are L–shaped with the kink where $\alpha x_1 = \beta x_2$.

In general, preferences are perfect complements when they can be represented by a utility function of the form

$$u(x_1, x_2) = \min\{\alpha x_1, \beta x_2\}$$

The resulting indifference curves are L–shaped, as shown in figure 8, with the kink along the line $\alpha x_1 = \beta x_2$. Note that the indifference curve is not strictly decreasing along the bottom of the L. This is because these preferences do not quite obey the monotonicity condition: when the agent has 7 patties and 10 slices of bread, an extra patty does not strictly increase her utility.

The MRS in this example is a little odd. When $\alpha x_1 > \beta x_2$,

$$MRS = \frac{MU_1}{MU_2} = \frac{0}{\beta} = 0$$

When $\alpha x_1 < \beta x_2$,

$$MRS = \frac{MU_1}{MU_2} = \frac{\alpha}{0} = \infty$$

At the kink, when $\alpha x_1 = \beta x_2$, then MRS is not defined because the indifference curve is not differentiable.

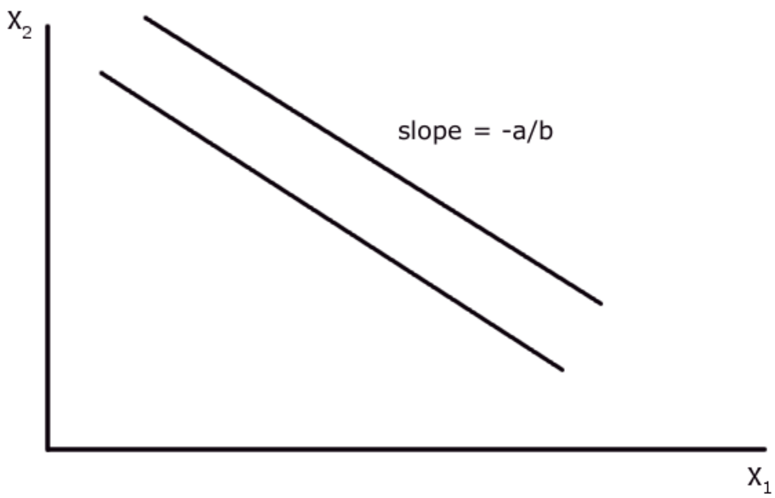

Figure 9: **Perfect Substitutes.** These indifference curves are linear with slope $-\alpha/\beta$.

4.3 Perfect Substitutes

Suppose an agent is buying food for a party. She wants enough food for her guests and considers 3 hamburgers to be equivalent to one pizza. Since each pizza is three times as valuable as a hamburger, her preferences can be represented by the utility function

$$u(x_h, x_p) = x_1 + 3x_2$$

where x_1 are hamburgers and x_2 are pizzas.

In general, preferences are perfect substitutes when they can be represented by a utility function of the form

$$u(x_1, x_2) = \alpha x_1 + \beta x_2$$

The resulting indifference curves are straight lines, as shown in figure 10. As a result, preferences are only weakly convex. The marginal rate of substitution is

$$MRS = \frac{MU_1}{MU_2} = \frac{\alpha}{\beta}$$

That is, the MRS is independent of the number of goods consumed.

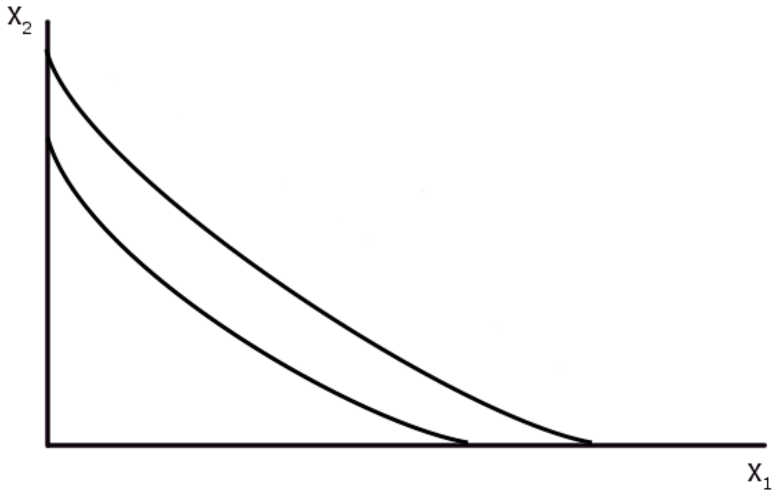

Figure 10: **CES Preferences.** In this picture $\delta > 0$ since the indifference curve intersects with the axes.

4.4 Constant Elasticity of Substitution (CES) Preferences

CES preferences have the form

$$u(x_1, x_2) = \frac{x_1^\delta}{\delta} + \frac{x_2^\delta}{\delta}$$

where $\delta \neq 0$ and $\delta < 1$.

This utility function can approximate the above examples. As $\delta \to 0$ the limit of the above utility function becomes

$$u(x_1, x_2) = \ln x_1 + \ln x_2$$

which is the same as Cobb-Douglas with equal exponents. As $\delta \to 1$, the preferences approximate perfect substitutes. As $\delta \to -\infty$, the preferences approximate perfect complements.

The MRS is

$$MRS = \frac{MU_1}{MU_2} = \frac{\delta x_1^{\delta-1}}{\delta x_2^{\delta-1}} = \frac{x_2^{1-\delta}}{x_1^{1-\delta}}.$$

The last expression is convenient since $1 - \delta > 0$. Substituting for x_2 in this equation and differentiating, one can show that MRS is decreasing in x_1, so the preferences are convex.

4.5 Additive Preferences

Additive preferences are represented by a utility function of the form

$$u(x_1, x_2) = v_1(x_1) + v_2(x_2)$$

The key property of additive preferences is that the marginal utility of x_i only depends on the amount of x_i consumed. As a result, the marginal rate of substitution is

$$MRS = \frac{MU_1}{MU_2} = \frac{v_1'(x_1)}{v_2'(x_2)}$$

For example, suppose we have

$$u(x_1, x_2) = x_1^2 + x_1^2$$

Differentiating, $MU_i = 2x_i$, so the marginal utility of each good is increasing in the amount of the good consumed. For example, one could imagine the agent becomes addicted to either good.

As shown in Figure 11, these preferences are concave. One can see this formally by showing the MRS is increasing in x_1 along an indifference curve. Differentiating,

$$MRS = \frac{MU_1}{MU_2} = \frac{2x_1}{2x_2} = \frac{x_1}{x_2} \tag{4.1}$$

The equation of an indifference curve is $x_1^2 + x_2^2 = k$. Rearranging, $x_2 = (k - x_1^2)^{1/2}$. Substituting into (4.1),

$$MRS = \frac{x_1}{(k - x_1^2)^{1/2}}$$

which is increasing in x_1.

4.6 Bliss Points

Suppose preferences are represented by the utility function

$$u(x_1, x_2) = -\frac{1}{2}(x_1 - 10)^2 - \frac{1}{2}(x_2 - 10)^2$$

Figure 12 plots the resulting indifference curves which are concentric circles around the bliss point of $(x_1, x_2) = (10, 10)$. These preferences violate monotonicity; as a result the indifference

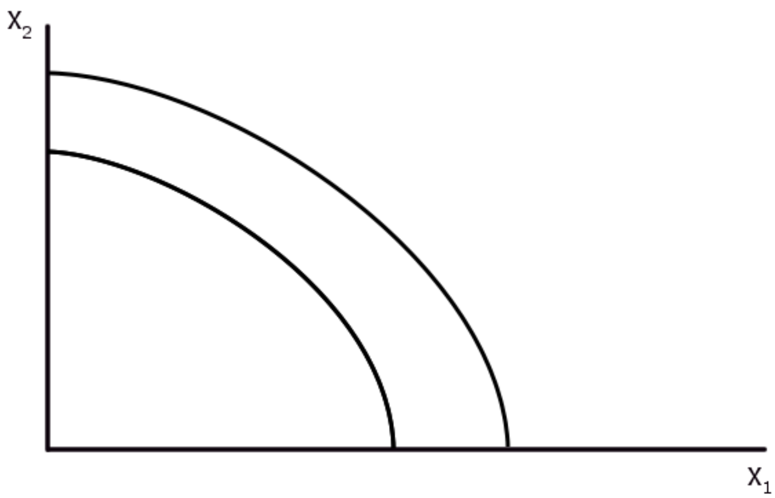

Figure 11: **Addiction Preferences.** These indifference curves are concave.

curves are sometimes upward sloping.

The marginal rate of substitution is

$$MRS = \frac{MU_1}{MU_2} = \frac{10 - x_1}{10 - x_2}$$

Hence the MRS is positive in the northeast and southwest quadrants, and is negative in the northwest and southeast quadrants. From figure 12 one can also see that preferences are convex. This is also possible to see from the MRS, but is a little tricky since monotonicity does not hold.

4.7 Quasilinear Preferences

An agent has quasilinear preferences if they can be represented by a utility function of the form

$$u(x_1, x_2) = v(x_1) + x_2$$

Quasilinear preferences are linear in x_2, so the marginal utility is constant. These preferences are often used to analyse goods which constitute a small part of an agent's income; good x_2 can then be thought of as "general consumption".

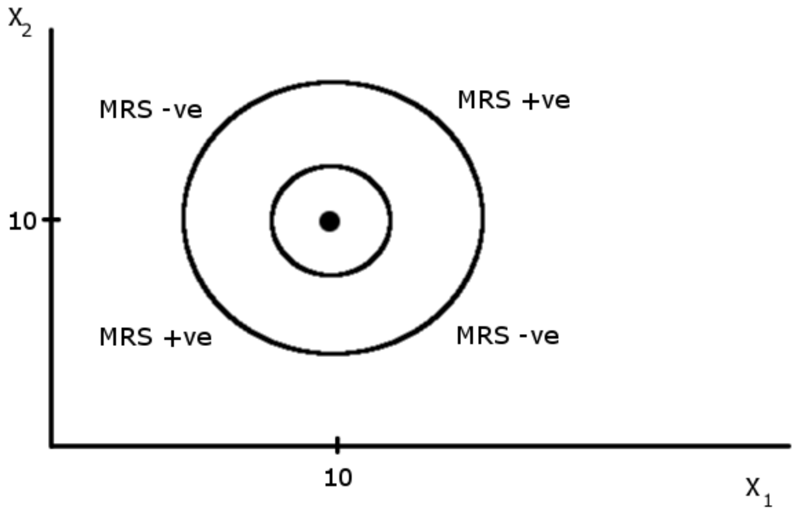

Figure 12: **Bliss Point.** Utility is maximised at $(10, 10)$. Indifference curves are circles around this bliss point.

The marginal rate of substitution equals

$$MRS = \frac{MU_1}{MU_2} = \frac{v'(x_1)}{1} = v'(x_1)$$

Observe that MRS only depends on x_1, and not x_2. This means that the indifference curves are vertical parallel shifts of each other, as shown in figure 13. As a consequence, preferences are convex if and only if $v(x_1)$ is a concave function, so the marginal utility of x_1 decreases in x_1.

As we will see later, quasilinear preferences have the attractive property that the consumption of x_1 is independent of the agent's income (ignoring boundary constraints). This makes the consumer's problem simple to analyse and provides an easy way to calculate consumer surplus.

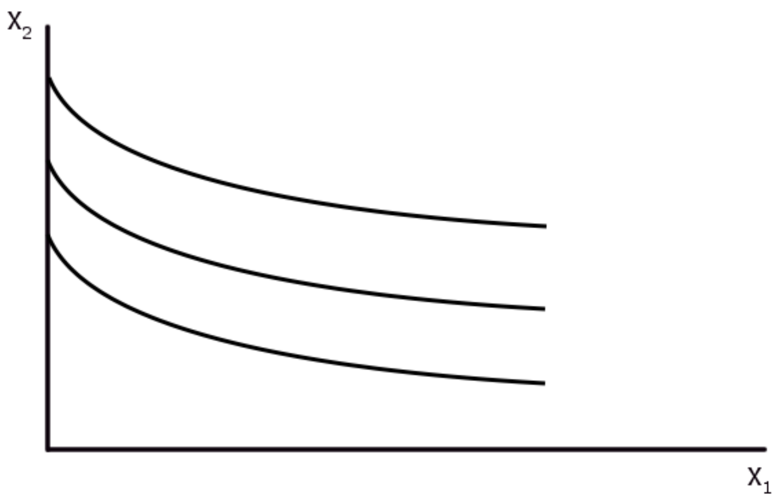

Figure 13: **Quasilinear Preferences.** These indifference curves are parallel shifts of each other.