Below, you will find the solutions to the Math Guide problems.

**Exercise 1a:** using the definition of \( \frac{dF}{dx} \), find the derivatives of the functions (i) \( F(x) = x^2 \); (ii) \( F(x) = \sqrt{x} \)

**Solution:**

\[
\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = 2x
\]

\[
\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
\]

**Exercise 1b:** Find the first and second derivatives of \( F(x) = \frac{1}{1+x} \); \( F(x) = e^{1+2x} \); \( F(x) = \frac{1+x^2}{1+x} \)

\[
\frac{d}{dx} \left( \frac{1}{1+x} \right) = -\frac{1}{(1+x)^2}
\]

\[
\frac{d}{dx} (e^{1+2x}) = 2e^{1+2x}
\]

\[
\frac{d}{dx} \left( \frac{1+x^2}{1+x} \right) = \frac{2x(1+x) - (1+x^2)}{(1+x)^2} = \frac{x^2 + 2x - 1}{(1+x)^2}
\]

**Exercise 2:** Find the first- and second-order partial derivatives, and the cross-partial derivatives of (i) \( F(x,y) = xy \), (ii) \( F(x,y) = x^2y^3 \), and (iii) \( F(x,y) = 2xy-x^2-2y^2-3x-y \)

**Solution:** (i) \( \frac{\partial F}{\partial x} = y \), \( \frac{\partial F}{\partial y} = x \), \( \frac{\partial^2 F}{\partial x^2} = 0 \), \( \frac{\partial^2 F}{\partial y^2} = 0 \), \( \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = 1 \).

(ii) \( \frac{\partial F}{\partial x} = 2xy^3 \), \( \frac{\partial F}{\partial y} = 3x^2y^2 \), \( \frac{\partial^2 F}{\partial x^2} = 2y^3 \), \( \frac{\partial^2 F}{\partial y^2} = 6x^2y \), \( \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = 6xy^2 \).

(iii) \( \frac{\partial F}{\partial x} = 2y - 2x - 3 \), \( \frac{\partial F}{\partial y} = 2x - 4y - 1 \), \( \frac{\partial^2 F}{\partial x^2} = -2 \), \( \frac{\partial^2 F}{\partial y^2} = -4 \), \( \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = 2 \).

**Exercise 3:** Let \( F(x,y) = xy \) and consider the implicit function \( F(x,y) = 5 \). Find \( \frac{dx}{dy} \) in two different ways, first by applying the implicit function theorem, then by rewriting \( x \) as an explicit function of \( y \) and taking derivatives.
Solution: \( \frac{\partial F}{\partial x} = y \) and \( \frac{\partial F}{\partial y} = x \), so

\[
\frac{dx}{dy} = -\frac{\partial F}{\partial y} = -\frac{x}{y} = -\frac{5}{y^2}, \text{ using the fact that } xy = 5
\]

Or: \( xy = 5 \), and hence \( x = \frac{5}{y} \), so that \( \frac{dx}{dy} = -\frac{5}{y^2} \).

Exercise 4: Solve for the optimal quantity \( q^* \) in the example of the profit maximizing firm, but change the cost function so that \( C(q) = \frac{1}{2}bq^2 \) for an arbitrary value of \( b \) (above, I considered the case where \( b = 1 \)). Check that the second-order condition is satisfied.

Solution: The firm’s problem is

\[
\max_q \left\{ pq - \frac{1}{2} bq^2 \right\}
\]

The corresponding FOC is

\[
p - bq = 0
\]

so that \( q^* = \frac{2}{b} \). The SOC is \( -b < 0 \), which is satisfied.

Exercise 5: Find the maximum of \( F(x, y) = 2xy - x^2 - 2y^2 - 3x - y \) by solving the pair of first-order conditions. Verify the second-order conditions to check that this is a maximum.

Solution: the pair of corresponding FOC’s is

\[
\begin{align*}
\frac{\partial F}{\partial x} &= 2y - 2x - 3 = 0 \\
\frac{\partial F}{\partial y} &= 2x - 4y - 1 = 0
\end{align*}
\]

Adding up the two FOC’s one finds: \( -2y - 4 = 0 \) or \( y = -2 \). The corresponding solution for \( x \) is \( 2x = -7 \) or \( x = -3.5 \). The corresponding second-order conditions are: \( \frac{\partial^2 F}{\partial x^2} = -2 < 0 \), \( \frac{\partial^2 F}{\partial y^2} = -4 < 0 \) and \( \frac{\partial^2 F}{\partial x \partial y} \cdot \left( \frac{\partial^2 F}{\partial x \partial y} \right)^2 = (-2) (-4) - 2^2 = 4 > 0 \), and hence the solution achieves a maximum.

Exercise 6: Solve the student’s time allocation problem using the substitution method, but suppose that each additional hour studied raises her grade by 4 points. The objective function then becomes \( F(x, y) = 11 + y (14 - y) + 4x \).

Solution: the student’s problem is

\[
\max_{x,y} \{11 + y (14 - y) + 4x\} \text{ subject to: } x + y \leq 12
\]
Substituting for \( x \leq 12 - y \), we find:
\[
\begin{align*}
\max_y \{11 + y (14 - y) + 4 (12 - y)\} &= \max_y \{11 + 14y - y^2 + 48 - 4y\} \\
&= \max_y \{59 + 10y - y^2\}
\end{align*}
\]

The corresponding FOC is: \( 10 - 2y = 0 \) or \( y^* = 5 \) and \( x^* = 7 \). The SOC is satisfied: \(-2 < 0\).

**Exercise 7:** Solve the student’s time allocation problem, when each hour studied raises the grade by 4 points. But this time, use the Lagrangian method.

**Solution:** The Lagrangian for the student’s problem is
\[
L (x, y; \lambda) = 11 + y (14 - y) + 4x + \lambda [12 - x - y]
\]

The corresponding FOC’s are:
\[
\begin{align*}
\frac{\partial L}{\partial x} &= 4 - \lambda = 0 \\
\frac{\partial L}{\partial y} &= 14 - 2y - \lambda = 0 \\
\frac{\partial L}{\partial \lambda} &= 12 - x - y = 0
\end{align*}
\]

The solution to these equations is \( \lambda = 4 \), \( 2y = 14 - \lambda = 10 \) or \( y = 5 \) and \( x = 12 - y = 7 \).

**Additional problem:** For each of the functions \( F (x, y) = \sqrt{xy} \), \( F (x, y) = \sqrt{x} + \sqrt{y} \) and \( F (x, y) = x^2 + y^2 \), carry out the following calculations:

(i) define the level curves, for arbitrary levels \( K \). Find the derivative of the level curve, and check whether the level curve is concave or convex.

(ii) compute the first-order partial derivatives with respect to \( x \) and \( y \).

(iii) Consider the constrained optimization problem \( \max_{x,y} F (x, y) \) subject to the constraint that \( x + y = 1 \).

**Solution for** \( F (x, y) = \sqrt{xy} \):
\[
\frac{\partial F}{\partial x} = \frac{\sqrt{y}}{2\sqrt{x}}; \quad \frac{\partial F}{\partial y} = \frac{\sqrt{x}}{2\sqrt{y}}; \quad \text{so}
\]
\[
\left( \frac{dy}{dx} \right)_{\sqrt{xy}=K} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{\sqrt{y}}{\sqrt{x}} = -\frac{y}{x}
\]

\(-\frac{y}{x}\) is increasing in \( x/y \) that is, as \( x \) increases, the slope of the level curve increases (the MRS decreases). The level curve is convex. Using a Lagrangian, I solve part (iii):
\[
L (x, y; \lambda) = \sqrt{xy} + \lambda [1 - x - y]
\]
The first-order conditions are

\[
\frac{\partial L}{\partial x} = \frac{\sqrt{y}}{2\sqrt{x}} - \lambda = 0 \\
\frac{\partial L}{\partial y} = \frac{\sqrt{x}}{2\sqrt{y}} - \lambda = 0 \\
\frac{\partial L}{\partial \lambda} = 1 - x - y = 0
\]

Combining the first two yields \( x = y = 1/2 \).

**Solution for** \( F(x, y) = \sqrt{x} + \sqrt{y} \): \( \frac{\partial F}{\partial x} = \frac{1}{2\sqrt{x}} \), \( \frac{\partial F}{\partial y} = \frac{1}{2\sqrt{y}} \), so

\[
\left. \frac{dy}{dx} \right|_{\sqrt{xy} = K} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{1}{\frac{2\sqrt{x}}{2\sqrt{y}}} = -\sqrt{\frac{x}{y}}
\]

Again, the slope of the level curve is increasing, i.e. the level curve is convex. For part (iii), I set up the Lagrangian:

\[
L(x, y; \lambda) = \sqrt{x} + \sqrt{y} + \lambda [1 - x - y]
\]

with first-order conditions:

\[
\frac{\partial L}{\partial x} = \frac{1}{2\sqrt{x}} - \lambda = 0 \\
\frac{\partial L}{\partial y} = \frac{1}{2\sqrt{y}} - \lambda = 0 \\
\frac{\partial L}{\partial \lambda} = 1 - x - y = 0
\]

Again, combining the first two yields \( x = y = 1/2 \).

**Solution for** \( F(x, y) = x^2 + y^2 \): \( \frac{\partial F}{\partial x} = 2x \), \( \frac{\partial F}{\partial y} = 2y \), so

\[
\left. \frac{dy}{dx} \right|_{\sqrt{xy} = K} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{2x}{2y} = -\frac{x}{y}
\]

This time the slope of the level curve is decreasing i.e. the level curve is concave. The corresponding Lagrangian will therefore deliver a minimum. In this case, extreme solutions are optimal: if the solution \( x, y \) can take on any negative value, then no solution exists. If the solution is restricted to be non-negative, then the solution is at a corner, and since \( x = 1, y = 0 \) and \( x = 0, y = 1 \) lead to the same value of \( x^2 + y^2 \), they both represent solutions.