Economics 11: Homework 1

30 September, 2008

Due date: Tuesday 14th October.

Instructions: You are required to write up your solution separately and independently, although you are encouraged to discuss and work in groups. Please write your name, student ID number, and the name of your TA on the front page of the assignment that you hand in. Also, please put boxes around your final answer to each part.

1. Utility Functions

Let \( X = \{ x \in \mathbb{R} | 0 \leq x \leq 4 \} \). Suppose an agent receives utility

\[
  u(x) = \begin{cases} 
    x & \text{if } 0 < x \leq 2 \\
    4 - x & \text{if } 2 < x \leq 4 
  \end{cases}
\]

from consuming \( x \) slices of pizza.

Which of the following utility functions represent the same preferences as above:

(a) \( v(x) = 2x \) if \( 0 \leq x \leq 2 \), and \( v(x) = 4 - x \) if \( 2 < x \leq 4 \).

(b) \( v(x) = 2x \) if \( 0 \leq x \leq 2 \), and \( v(x) = 8 - 2x \) if \( 2 < x \leq 4 \).

(c) \( v(x) = 4 - (2 - x)^2 \) if \( 0 \leq x \leq 4 \).

(d) \( v(x) = 4 - (2 - x)^3 \) if \( 0 \leq x \leq 4 \).

(e) \( v(x) = 4 - (2 - x)^4 \) if \( 0 \leq x \leq 4 \).

(f) \( v(x) = -(2 - x)^2 \) if \( 0 \leq x \leq 4 \).

2. Demography

This exercise considers the effect of changes in survival rates on the rate of population growth.
An agent chooses to divide her income $M$ between general expenditure, $x$, and the number of births $b$. A birth has survival rate $s \in [0,1]$. The cost of a birth is $c$. The cost of raising a child is $k$. The agent’s utility is

$$u(b, x) = 2 \sqrt{sb} + x$$

and her budget constraint is

$$M \geq x + cb + ksb$$

(a) Write down the Lagrangian for the agent’s problem. Assuming there is an internal optimum, whereby $x \geq 0$ and $b \geq 0$, solve for the optimal number of births.

(b) Show that if $M$ is sufficiently high, there is indeed an internal optimum.

(c) Suppose $c = 0$. How does the number of births depend on $s$? Provide an intuition.

(d) Suppose $k = 0$. How does the number of births depend on $s$? Provide an intuition.

(e) Suppose $k = 5$ and $c = 1$. For which value of $s$ is the number of births maximised?

3. Consumption with Three Goods

An agent consumes three goods, $\{x_1, x_2, x_3\}$. Her utility is

$$\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}$$

The agent has income $M = 1$.

(a) Suppose the prices are $p_1 = 1$, $p_2 = 1$ and $p_3 = 1$. Solve for the agent’s optimal consumption.

(b) Suppose the prices are $p_1 = 3$, $p_2 = 1$ and $p_3 = 1$. Solve for the agent’s optimal consumption.

4. Consumer choice problem

An agent chooses a consumption bundle $x$ to maximise her utility. Her utility function is

$$u(x_1, x_2) = \left(x_1 - \frac{1}{2} x_1^2\right) + \left(x_2 - \frac{1}{2} x_2^2\right)$$
where we assume $0 \leq x_1 \leq 1$ and $0 \leq x_2 \leq 1$. The agent has income $M = 1$.

(a) Suppose the prices are $p_1 = 2$ and $p_2 = 2$. Solve for the agent’s optimal consumption.

(b) Suppose the prices are $p_1 = 3$ and $p_2 = 2$. Solve for the agent’s optimal consumption.

(c) Suppose the prices are $p_1 = 6$ and $p_2 = 2$. Solve for the agent’s optimal consumption.
Economics 11: Solutions to Homework 1

October 16, 2008

Due date: Tuesday 14th October.

Instructions: You are required to write up your solution separately and independently, although you are encouraged to discuss and work in groups. Please write your name, student ID number, and the name of your TA on the front page of the assignment that you hand in. Also, please put boxes around your final answer to each part.

1. Utility Functions

Let $X = \{x \in \mathbb{R} | 0 \leq x \leq 4 \}$. Suppose an agent receives utility

$$u(x) = \begin{cases} x & \text{if } 0 < x \leq 2 \\ 4 - x & \text{if } 2 < x \leq 4 \end{cases}$$

from consuming $x$ slices of pizza.

Which of the following utility functions represent the same preferences as above:

(a) $v(x) = 2x$ if $0 \leq x \leq 2$, and $v(x) = 4 - x$ if $2 < x \leq 4$.

(b) $v(x) = 2x$ if $0 \leq x \leq 2$, and $v(x) = 8 - 2x$ if $2 < x \leq 4$.

(c) $v(x) = 4 - (2 - x)^2$ if $0 \leq x \leq 4$.

(d) $v(x) = 4 - (2 - x)^3$ if $0 \leq x \leq 4$.

(e) $v(x) = 4 - (2 - x)^4$ if $0 \leq x \leq 4$.

(f) $v(x) = -(2 - x)^2$ if $0 \leq x \leq 4$.

Solution

(a) This is different since $v(1) = 2 > 1 = v(3)$. 
(b) This is the same since \( v(x) = 2u(x) \).

(c) This is the same since \( v(x) \) is increasing for \( x < 2 \), decreasing for \( x > 2 \), and is symmetric around 2.

(d) This is different since \( v(x) \) is increasing for all \( x \).

(e) This is the same since \( v(x) \) is the square of part (c).

(f) This is the same since \( v(x) \) is a vertical shift of part (c).

2. Demography

This exercise considers the effect of changes in survival rates on the rate of population growth.

An agent chooses to divide her income \( M \) between general expenditure, \( x \), and the number of births \( b \). A birth has survival rate \( s \in [0, 1] \). The cost of a birth is \( c \). The cost of raising a child is \( k \). The agent’s utility is

\[
u(b, x) = 2\sqrt{sb} + x
\]

and her budget constraint is

\[
M \geq x + cb + ksb
\]

(a) Write down the Lagrangian for the agent’s problem. Assuming there is an internal optimum, whereby \( x \geq 0 \) and \( b \geq 0 \), solve for the optimal number of births.

(b) Show that if \( M \) is sufficiently high, there is indeed an internal optimum.

(c) Suppose \( c = 0 \). How does the number of births depend on \( s \)? Provide an intuition.

(d) Suppose \( k = 0 \). How does the number of births depend on \( s \)? Provide an intuition.

(e) Suppose \( k = 5 \) and \( c = 1 \). For which value of \( s \) is the number of births maximised?
Solution

(a) The Lagrangian is

\[ \mathcal{L} = 2\sqrt{sb} + x + \lambda [M - x - cb - ksb] \]

The FOCs are

\[ s^{1/2}b^{-1/2} = \lambda (c + ks) \]
\[ 1 = \lambda \]

Solving, we find

\[ b^* = \frac{s}{(c + ks)^2} \]

This is independent of income (since utility is quasi-linear in \( x \)).

(b) Clearly \( b^* \geq 0 \). Using the budget constraint

\[ x^* = M - (c + ks)b^* = M - \frac{s}{c + ks} \]

Hence there is an internal optimum if \( M \geq s/(c + ks) \).

(c) \( b^* = 1/sk^2 \) is decreasing in \( s \). Intuitively, the agent only cares about the number of children she has, hence \( sb^* = 1/k^2 \) is independent of \( s \). Thus an increase in \( s \) leads to an equal percentage reduction in births.

(d) \( b^* = s/c \) is increasing in \( s \). An increase in \( s \) raises the productivity of each birth but does not affect the cost. Hence the agent chooses to have more births.

(e) Using the quotient rule,

\[ \frac{db^*}{ds} = \frac{(c + ks)^2 - 2sk(c + ks)}{(c + ks)^4} = \frac{c - sk}{(c + ks)^4} \]

Setting this to zero, \( b^* \) is maximised at \( s = c/k = 1/5 \). By checking the SOC, one can also verify this is a maximum.
3. Consumption with Three Goods

Suppose an agent consumes three goods, \( \{x_1, x_2, x_3\} \). Her utility is

\[
\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}
\]

The agent has income \( M = 1 \).

(a) Suppose the prices are \( p_1 = 1, p_2 = 1 \) and \( p_3 = 1 \). Solve for the agent’s optimal consumption.

(b) Suppose the prices are \( p_1 = 3, p_2 = 1 \) and \( p_3 = 1 \). Solve for the agent’s optimal consumption.

Solution

(a) The Lagrangian is

\[
\mathcal{L} = \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \lambda[M - x_1 - x_2 - x_3]
\]

The FOCs are

\[
\frac{1}{2}x_1^{-1/2} = \lambda \\
\frac{1}{2}x_2^{-1/2} = \lambda \\
\frac{1}{2}x_3^{-1/2} = \lambda
\]

The equations imply \( x_1 = x_2 = x_3 \). Using the budget constraint, \( x_1^* = 1/3, x_2^* = 1/3, x_3^* = 1/3 \).

(b) The Lagrangian is

\[
\mathcal{L} = \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \lambda[M - 3x_1 - x_2 - x_3]
\]

The FOCs are

\[
\frac{1}{2}x_1^{-1/2} = \lambda \\
\frac{1}{2}x_2^{-1/2} = \lambda \\
\frac{1}{2}x_3^{-1/2} = 3\lambda
\]
The equations imply $9x_1 = x_2 = x_3$. Using the budget constraint, $x_1^* = 1/21$, $x_2^* = 9/21$, $x_3^* = 9/12$.

4. Consumer choice problem

An agent chooses a consumption bundle $x$ to maximise her utility. Her utility function is

\[ u(x_1, x_2) = \left( x_1 - \frac{1}{2}x_1^2 \right) + \left( x_2 - \frac{1}{2}x_2^2 \right) \]

where we assume $0 \leq x_1 \leq 1$ and $0 \leq x_2 \leq 1$. The agent has income $M = 1$.

(a) Suppose the prices are $p_1 = 2$ and $p_2 = 2$. Solve for the agent’s optimal consumption.

(b) Suppose the prices are $p_1 = 3$ and $p_2 = 2$. Solve for the agent’s optimal consumption.

(c) Suppose the prices are $p_1 = 6$ and $p_2 = 2$. Solve for the agent’s optimal consumption.

Solution

(a) The Lagrangian is

\[ \mathcal{L} = (x_1 - x_1^2/2) + (x_2 - x_2^2/2) + \lambda [M - 2x_1 - 2x_2] \]

The FOCs are

\[
\begin{align*}
(1 - x_1) &= 2\lambda \\
(1 - x_2) &= 2\lambda 
\end{align*}
\]

These equations imply that $x_1 = x_2$. Using the budget constraint, $x_1^* = 1/4$ and $x_2^* = 1/4$.

(b) The Lagrangian is

\[ \mathcal{L} = (x_1 - x_1^2/2) + (x_2 - x_2^2/2) + \lambda [M - 3x_1 - 2x_2] \]
The FOCs are

\[(1 - x_1) = 3\lambda\]
\[(1 - x_2) = 2\lambda\]

These equations imply that \(x_1 = \frac{3}{2} x_2 - \frac{1}{2}\). Using the budget constraint, \(x_1^* = 1/13\) and \(x_2^* = 5/13\).

(c) The Lagrangian is

\[\mathcal{L} = (x_1 - x_1^2/2) + (x_2 - x_2^2/2) + \lambda[M - 6x_1 - 2x_2]\]

The FOCs are

\[(1 - x_1) = 6\lambda\]
\[(1 - x_2) = 2\lambda\]

These equations imply that \(x_1 = 3x_2 - 2\). Using the budget constraint, \(x_1^* = -1/20\) and \(x_2^* = 13/20\). Since \(x_1^*\) is negative, we must have \(x_1^* = 0\), which implies \(x_2^* = 1/2\).
Economics 11: Homework 2

August 6, 2018

Due date: Tuesday 28th October.

Instructions: You are required to write up your solution separately and independently, although you are encouraged to discuss and work in groups. Please write your name, student ID number, and the name of your TA on the front page of the assignment that you hand in. Also, please put boxes around your final answer to each part.

1. Budget Sets with Price Discounts

There are two goods: $x_1$ and $x_2$. The seller of $x_2$ charges $p_2 = 2$. The seller of $x_1$ offers price discounts. If the agent buys $x_1 < 10$ she pays $p_1 = 2$ for every unit. If the agent buys $x_1 \geq 10$ she pays $p_1 = 1$ for every unit (including the first 10).

(a) Suppose $m = 30$. Draw the agent’s budget set.

(b) Assume the agent has monotone preferences. Do preferences exist such that the agent chooses (i) $x_1 = 3$, (ii) $x_1 = 7$ and (iii) $x_1 = 12$? Explain your answers.

2. Intertemporal Choice with Differential Interest Rates

An agent allocates consumption across two periods. Let the consumption in period $t$ be $x_t$, and the income in period $t$ be $m_t$. The agent’s utility is

$$u(x_1, x_2) = \ln(x_1) + \frac{3}{4} \ln(x_2)$$

The agent is poor in period 1 but wealthy in period 2. In particular, she has income $m_1 = 3$ in period 1 and $m_2 = 4$ in period 2.

(a) Suppose the agent can borrow and save at interest rate $r = 1/3$, so $\$1$ in period 1 is worth $\$(1 + 1/3)$ in period 2. Sketch the agent’s budget constraint. Solve for her optimal consumption.
(b) Suppose the agent can still save at \( r = 1/3 \), but can only borrow at \( r = 1/2 \). Sketch the agent’s budget constraint. Solve for her optimal consumption.

(c) Suppose the agent can still save at \( r = 1/3 \), but can only borrow at \( r = 1 \). Sketch the agent’s budget constraint. Solve for her optimal consumption. [Hint: Beware of kinks.]

3. Consumer Problem

An agent has utility \( u(x_1, x_2) = (x_1^{-1} + x_2^{-1})^{-1} \) for goods \( x_1 \) and \( x_2 \). The prices of the goods are \( p_1 \) and \( p_2 \). The agent has income \( m \).

First, we show that these preferences are convex. We do this three ways.

(a) (i) Solve for equation of a typical indifference curve, specifying \( x_2 \) in terms of \( x_1 \). (ii) Using the fact that

\[
MRS = \frac{dx_2}{dx_1}
\]

find the MRS. (iii) Show that MRS is decreasing in \( x_1 \).

(b) (i) Using the fact that

\[
MRS = \frac{MU_1}{MU_2}
\]

find the MRS. (ii) Using the equation for the indifference curve (see part (a)), substitute for \( x_2 \) and write MRS in terms of \( x_1 \). (iii) Show that MRS is decreasing in \( x_1 \).

(c) (i) Sketch a typical indifference curve. (ii) State the definition of convexity. (iii) Graphically verify the indifference curve you have drawn satisfies convexity.

(d) Write down the agent’s budget constraint.

(e) Solve for the agent’s optimal choice of \((x_1, x_2)\).

(f) Show the agent’s indirect utility function is given by

\[
v = \frac{m}{(p_1^{1/2} + p_2^{1/2})^2}
\]

(g) Solve for the agent’s Hicksian demand.
(h) Solve for the expenditure function.

(i) Verify the Slutsky equation for good $x_1$.

4. Labour Supply

Suppose an agent has the same utility function as in question 3

$$u(x_1, x_2) = (x_1^{-1} + x_2^{-1})^{-1}$$

Suppose that $x_1$ is hours of leisure and $x_2$ is quantity of food. The agent is endowed with $T$ hours to divide between work and leisure. Her wage rate is $w$, while the price of $x_2$ is $p_2 = 1$. The agent has no outside income, $m = 0$.

(a) Derive the budget constraint of the agent.

(b) Solve for the agent’s optimal choice of $(x_1, x_2)$. [Hint: the answer is very similar to part (e) of question 3.]

(c) How does the agent’s leisure consumption change as a function of her wage? Explain this intuitively in terms of income and substitution effects.

For your own interest...

The following questions are optional.

(d) Derive the Hicksian demands.

(e) Verify the Slutsky equation with endowments. That is,

$$\frac{\partial x_1^*}{\partial w} = \frac{\partial h_1}{\partial w} - (x_1^* - T) \frac{\partial x_1^*}{\partial m}$$

Note: The final term $\partial x_1^*/\partial m$ involves ‘m’. For this term, you should use the demand calculated in part (e) of Question 3.
Economics 11: Solutions to Homework 2

October 28, 2008

Due date: Tuesday 28th October.

Instructions: You are required to write up your solution separately and independently, although you are encouraged to discuss and work in groups. Please write your name, student ID number, and the name of your TA on the front page of the assignment that you hand in. Also, please put boxes around your final answer to each part.

1. Budget Sets with Price Discounts

There are two goods: $x_1$ and $x_2$. The seller of $x_2$ charges $p_2 = 2$. The seller of $x_1$ offers price discounts. If the agent buys $x_1 < 10$ she pays $p_1 = 2$ for every unit. If the agent buys $x_1 \geq 10$ she pays $p_1 = 1$ for every unit (including the first 10).

(a) Suppose $m = 30$. Draw the agent’s budget set.

(b) Assume the agent has monotone preferences. Do preferences exist such that the agent chooses (i) $x_1 = 3$, (ii) $x_1 = 7$ and (iii) $x_1 = 12$? Explain your answers.

Solution

(a) The budget line is given by

$$x_2 = 15 - x_1 \quad \text{for } x_1 < 10$$

$$= 15 - \frac{1}{2}x_1 \quad \text{for } x_1 \geq 10$$

As a result, there is a discontinuity at $x_1 = 10$.

(b) (i) Yes.
(ii) No. Because of the discontinuity, the agent can obtain more of both goods.
(iii) Yes.
2. Intertemporal Choice with Differential Interest Rates

An agent allocates consumption across two periods. Let the consumption in period $t$ be $x_t$, and the income in period $t$ be $m_t$. The agent’s utility is

$$u(x_1, x_2) = \ln(x_1) + \frac{3}{4} \ln(x_2)$$

The agent is poor in period 1 but wealthy in period 2. In particular, she has income $m_1 = 3$ in period 1 and $m_2 = 4$ in period 2.

(a) Suppose the agent can borrow and save at interest rate $r = 1/3$, so $1$ in period 1 is worth $(1+1/3)$ in period 2. Sketch the agent’s budget constraint. Solve for her optimal consumption.

(b) Suppose the agent can still save at $r = 1/3$, but can only borrow at $r = 1/2$. Sketch the agent’s budget constraint. Solve for her optimal consumption.

(c) Suppose the agent can still save at $r = 1/3$, but can only borrow at $r = 1$. Sketch the agent’s budget constraint. Solve for her optimal consumption. [Hint: Beware of kinks.]

Solution

(a) Writing everything in terms of period 1 money, the agent’s budget constraint is

$$m_1 + \frac{3}{4} m_2 = x_1 + \frac{3}{4} x_2$$

The tangency condition states that $MRS = p_1/p_2$. That is,

$$\frac{4 x_2}{3 x_1} = \frac{4}{3}$$

which implies $x_1 = x_2$. The LHS of the budget equation is 6. Using this, we obtain $x_1^* = x_2^* = 24/7$. Since $x_1^* > 3$, the agent borrow in period 1.

(b) To the left of $(x_1, x_2) = (3, 4)$, the agent’s budget constraint has a slope of $-4/3$. To the right it has a slope of $-3/2$. From part (a) we know the agent does not wish to save at $r = 1/3$. When the agent borrows, the equation for the budget line is

$$m_1 + \frac{2}{3} m_2 = x_1 + \frac{2}{3} x_2$$
Suppose the agent wishes to borrow a positive amount of money. The tangency condition states that
\[
\frac{4}{3} x_2 = \frac{3}{2} x_1
\]
which implies \( x_2 = \frac{9}{8} x_1 \), so the agent now wishes to consume more in the second period due to the high interest rate of borrowing. The LHS of the budget equation is \( \frac{17}{3} \). Using this, we obtain \( x_1^* = 68/21 \) and \( x_2^* = 153/32 \). Observe \( x_1^* > 3 \), so the agent still borrows.

(c) To the left of \((x_1, x_2) = (3, 4)\), the agent’s budget constraint has a slope of \(-4/3\). To the right it has a slope of \(-2\). As before, the agent will never wish to save. When she borrows, the equation for the budget line is
\[
m_1 + \frac{1}{2} m_2 = x_1 + \frac{1}{2} x_2
\]
Suppose the agent wishes to borrow a positive amount of money. The tangency condition states that
\[
\frac{4}{3} x_2 = 2
\]
which implies \( x_2 = \frac{3}{2} x_1 \). The LHS of the budget equation is \( 5 \). Using this, we obtain \( x_1^* = 20/7 \) and \( x_2^* = 30/7 \). Observe \( x_1^* < 3 \) so the agent is not willing to borrow at this interest rate. Since they are not willing to save at \( r = 1/3 \), she must choose to consume her endowment: \((x_1^*, x_2^*) = (3, 4)\).

Another way of seeing the same result is to note that the agent prefers to consume at the kink if
\[
\frac{p_1}{p_2} \bigg|_{x_1 < 3} \geq \frac{MU_1}{MU_2} \bigg|_{x_1 = 3} \geq \frac{p_1}{p_2} \bigg|_{x_1 > 3}
\]
Plugging in the numbers,
\[
\frac{4}{3} \geq \frac{16}{9} \geq 2
\]
we see that the agent neither wishes to save or borrow.

3. Consumer Problem

An agent has utility \( u(x_1, x_2) = (x_1^{-1} + x_2^{-1})^{-1} \) for goods \( x_1 \) and \( x_2 \). The prices of the goods are \( p_1 \) and \( p_2 \). The agent has income \( m \).

First, we show that these preferences are convex. We do this three ways.

(a) (i) Solve for equation of a typical indifference curve, specifying \( x_2 \) in terms of \( x_1 \). (ii) Using
the fact that

\[ \text{MRS} = -\frac{dx_2}{dx_1} \]

find the MRS. (iii) Show that MRS is decreasing in \( x_1 \).

(b) (i) Using the fact that

\[ \text{MRS} = \frac{MU_1}{MU_2} \]

find the MRS. (ii) Using the equation for the indifference curve (see part (a)), substitute for \( x_2 \) and write MRS in terms of \( x_1 \). (iii) Show that MRS is decreasing in \( x_1 \).

(c) (i) Sketch a typical indifference curve. (ii) State the definition of convexity. (iii) Graphically verify the indifference curve you have drawn satisfies convexity.

(d) Write down the agent’s budget constraint.

(e) Solve for the agent’s optimal choice of \((x_1, x_2)\).

(f) Show the agent’s indirect utility function is given by

\[ v = \frac{m}{(p_1^{1/2} + p_2^{1/2})^2} \]

(g) Solve for the agent’s Hicksian demand.

(h) Solve for the expenditure function.

(i) Verify the Slutsky equation for good \( x_1 \).

**Solution**

(a) The indifference curve is given by

\[ (x_1^{-1} + x_2^{-1})^{-1} = \overline{u} \]

Solving for \( x_2 \),

\[ x_2 = \frac{x_1 \overline{u}}{x_1 - \overline{u}} \] (1)
Differentiating,
\[ \text{MRS} = -\frac{dx_2}{dx_1} = \frac{u^2}{(x_1 - \bar{u})^2} \]

Differentiating again,
\[ \frac{d}{dx_1} \text{MRS} = \frac{-2u^2}{(x_1 - \bar{u})^3} \]

Hence MRS is decreasing in \( x_1 \) and preferences are convex. Economically, this means that when \( x_1 \) is higher, the consumer is willing to give up fewer units of \( x_2 \) for a unit of \( x_1 \).

(b) The MRS is
\[ \frac{MU_1}{MU_2} = \frac{(x_1^{-1} + x_2^{-1})^{-2}x_1^{-2}}{(x_1^{-1} + x_2^{-1})^{-2}x_2^{-2}} = \frac{x_2^2}{x_1^2} \]

Using (1) we obtain
\[ \text{MRS} = \frac{\bar{u}^2}{(x_1 - \bar{u})^2} \]

As in part (a).

(c) Preferences are convex if the convex combination of any two points above the indifference curve are also above the indifference curve. Plotting (1), one can easily verify this condition.

(d) The budget is
\[ p_1x_1 + p_2x_2 = m \]

(e) The tangency condition says that \( MRS = p_1/p_2 \). That is,\(^1\)
\[ \frac{x_2^2}{x_1^2} = \frac{p_1}{p_2} \]

Rearranging, \( x_1 p_1^{1/2} = x_2 p_2^{1/2} \). Using the budget constraint,
\[ x_1^* = \frac{m}{p_1^{1/2}(p_1^{1/2} + p_2^{1/2})} \quad \text{and} \quad x_2^* = \frac{m}{p_2^{1/2}(p_1^{1/2} + p_2^{1/2})} \]

(f) Substituting,
\[ v = \frac{m}{(p_1^{1/2} + p_2^{1/2})^2} \]

\(^1\)One can derive the same equation using a Lagrangian or the substitution method.
For future reference, it is useful to invert this to find the expenditure function:

\[ e = \bar{u}(p_1^{1/2} + p_2^{1/2})^2 \]

(g) For Hicksian demand, we attain the same tangency condition: \( p_1^{1/2} x_1 = p_2^{1/2} x_2 \). Using the constraint,

\[ (x_1^{-1} + x_2^{-1})^{-1} = \bar{u} \]

we obtain

\[ h_1 = \bar{u}(1 + p_1^{-1/2} p_1^{1/2}) \quad \text{and} \quad h_2 = \bar{u}(1 + p_1^{1/2} p_2^{-1/2}) \]

(h) Plugging in, the expenditure function is

\[ e = p_1 h_1 + p_2 h_2 = \bar{u}(p_1^{1/2} + p_2^{1/2})^2 \]

Just to check this is correct, one can verify this is the same as the answer in part (f).

(i) The LHS of the Slutsky equation is

\[ \frac{\partial x_1^*}{\partial p_1} = -\frac{1}{2} p_1^{-3/2} m \frac{p_1^{1/2} + p_2^{1/2}}{(p_1^{1/2} + p_2^{1/2})^2} - \frac{1}{2} p_1^{-1} m \frac{p_1^{1/2} + p_2^{1/2}}{(p_1^{1/2} + p_2^{1/2})^2} \]

\[ = - \left[ p_1^{-1} - \frac{1}{2} p_1^{-3/2} \frac{p_1^{1/2} + p_2^{1/2}}{(p_1^{1/2} + p_2^{1/2})^2} \right] \frac{m}{(p_1^{1/2} + p_2^{1/2})^2} \]

The RHS of the slutsky equation is

\[ \frac{\partial x_1}{\partial p_1} + -x_1^* \frac{\partial x_1^*}{\partial m} = \frac{1}{2} \bar{u} p_1^{-3/2} \frac{p_1^{1/2} + p_2^{1/2}}{(p_1^{1/2} + p_2^{1/2})^2} \frac{m p_1^{1/2} - m}{(p_1^{1/2} + p_2^{1/2})^2} \frac{p_1^{-1/2}}{(p_1^{1/2} + p_2^{1/2})^2} \]

\[ = \frac{1}{2} p_1^{-3/2} \frac{p_1^{1/2} + p_2^{1/2}}{(p_1^{1/2} + p_2^{1/2})^2} \frac{m}{(p_1^{1/2} + p_2^{1/2})^2} - p_1^{-1} m \frac{p_1^{1/2} + p_2^{1/2}}{(p_1^{1/2} + p_2^{1/2})^2} \]

\[ = - \left[ p_1^{-1} + \frac{1}{2} p_1^{-3/2} \frac{p_1^{1/2}}{(p_1^{1/2} + p_2^{1/2})^2} \right] \frac{m}{(p_1^{1/2} + p_2^{1/2})^2} \]

which is the same as the LHS.
4. Labour Supply

Suppose an agent has the same utility function as in question 3

\[ u(x_1, x_2) = (x_1^{-1} + x_2^{-1})^{-1} \]

Suppose that \( x_1 \) is hours of leisure and \( x_2 \) is quantity of food. The agent is endowed with \( T \) hours to divide between work and leisure. Her wage rate is \( w \), while the price of \( x_2 \) is \( p_2 = 1 \). The agent has no outside income, \( m = 0 \).

(a) Derive the budget constraint of the agent.

(b) Solve for the agent’s optimal choice of \((x_1, x_2)\). [Hint: the answer is very similar to part (e) of question 3.]

(c) How does the agent’s leisure consumption change as a function of her wage? Explain this intuitively in terms of income and substitution effects.

For your own interest...

The following questions are optional.

(d) Derive the Hicksian demands.

(e) Verify the Slutsky equation with endowments. That is,

\[ \frac{\partial x_1^*}{\partial w} = \frac{\partial h_1}{\partial w} - (x_1^* - T) \frac{\partial x_1^*}{\partial m} \]

Note: The final term \( \partial x_1^*/\partial m \) involves ‘m’. For this term, you should use the demand calculated in part (e) of Question 3.

Solution

(a) The budget is

\[ (T - x_2)w = x_1 \]
Equivalently,
\[ x_1 + wx_2 = wT \]
We can thus think of \( wT \) as the income of the agent.

(b) The tangency condition says that \( x_1 p_1^{1/2} = x_2 p_2^{1/2} \). Using the budget constraint,
\[ x_1^* = \frac{wT}{w^{1/2}(1 + w^{1/2})} \quad \text{and} \quad x_2^* = \frac{wT}{(1 + w^{1/2})} \]
Note: This is the same as (e) in Question 3, where we have replaced \( m \) with \( wT \).

(c) Leisure is increasing in \( w \). Intuitively, as \( w \) increase the value of the agent’s endowment \( wT \) increases, leading to an increase in the demand for leisure. This income effect dominates the substitution effect.

(d) For Hicksian demand, we attain the same tangency condition: \( p_1^{1/2} x_1 = p_2^{1/2} x_2 \). Using the constraint,
\[ (x_1^{-1} + x_2^{-1})^{-1} = \bar{u} \]
we obtain
\[ h_1 = \bar{u}(1 + p_1^{-1/2} p_2^{1/2}) \quad \text{and} \quad h_2 = \bar{u}(1 + p_1^{1/2} p_2^{-1/2}) \]

(e) The LHS of the Slutsky equation is
\[ \frac{\partial x_1^*}{\partial w} = \frac{1}{2} w^{-1/2} \frac{T}{(1 + w^{1/2})^2} \]
The RHS of the Slutsky equation is
\[ \frac{\partial h_1}{\partial w} + (T - x_1^*) \frac{\partial x_1^*}{\partial m} = -\frac{1}{2} w^{-3/2} \bar{u} + \frac{T}{1 + w^{1/2}} \frac{1}{w^{1/2}(1 + w^{1/2})} \]
\[ = -\frac{1}{2} w^{-1/2} \frac{T}{(1 + w^{1/2})^2} + w^{-1/2} \frac{T}{(1 + w^{1/2})^2} \]
We thus see the decomposition into the substitution effect and income effect. The substitution effect states that as leisure becomes more expensive the agent substitutes away from leisure. The income effect states that as the wage increases the value of the agent’s endowment rises, and the agent consumes more leisure. Note that leisure is a normal good; the reason demand of leisure increases in the price is because an increase in the wage increases the agents effective income by \( (T - x_1^*)dw \). In comparison, in the model without endowments, an increase in a price always reduces effective income.
Economics 11: Homework 3

November 13, 2008

Due date: Tuesday 25th November.

Instructions: You are required to write up your solution separately and independently, although you are encouraged to discuss and work in groups. Please write your name, student ID number, and the name of your TA on the front page of the assignment that you hand in. Also, please put boxes around your final answer to each part.

1. Consumer Surplus with CES utility

An agent has utility \( u(x_1, x_2) = x_1^{1/2} + x_2^{1/2} \), income \( m = 4 \) and faces prices \( p_1 = 2 \) and \( p_2 = 2 \). Suppose the price of good 1 rises to \( p_1' = 3 \). Calculate the increase in the agent’s income required to compensate her for this price rise.

2. Cost Minimisation with Quasilinear Production

A firm has production function

\[
f(z_1, z_2) = 2z_1^{1/2} + z_2
\]

The firm’s inputs cost \( r_1 \) and \( r_2 \).

(a) Does this production function exhibit decreasing, constant or increasing returns to scale?

(b) Derive the MRTS.

(c) Show that the isoquants are convex.

(d) Solve the firm’s cost minimisation problem. You may find it easy to do this in two parts. First, assume

\[
q \geq \frac{2r_2}{r_1}
\]

so there is an internal solution and solve for the resulting cost function. Second, assume

\[
q \leq \frac{2r_2}{r_1}
\]
so there is a boundary solution and solve for the resulting cost function.

(e) Is the cost function concave or convex in \( q \)? How does this relate to part (a)?

3. Firm’s Problem with CES Production

A firm has production function

\[
f(z_1, z_2) = z_1^{1/2} + z_2^{1/2}
\]

The firm’s inputs cost \( r_1 \) and \( r_2 \). The output price is \( p \).

(a) Suppose \( z_2 = 1 \), and assume \( q \geq 1 \) to avoid boundary problems. Derive the firm’s short–run cost and short–run marginal cost.

(b) Calculate the long–run cost function, when both factors are flexible.

(c) Using the cost function in (b), solve for the firm’s profit maximising output, an their maximal profit.

(d) Solve the firm’s profit maximisation problem directly, whereby the firm chooses \((z_1, z_2)\) to maximise

\[
\pi = pf(z_1, z_2) - r_1 z_1 - r_2 z_2
\]

What are the optimal inputs? What is the firm’s output and profit?

4. Profit Maximisation

A firm has cost curve

\[
c(q) = 100 + q^2
\]

The firm can also shut down and make profits \( \pi = 0 \).

(a) Suppose the firm faces price \( p = 30 \). What is the profit maximising quantity? What is the maximal profit? Will the firm shut down?

(b) Suppose \( p = 10 \). What is the profit maximising quantity? What is the maximal profit? Will the firm shut down?
(c) For which price levels will the firm shut down?

(d) Draw the supply function $q^*(p)$ as a function of the output price $p$. 
Economics 11: Solutions to Homework 3

November 25, 2008

Due date: Tuesday 25th November.

Instructions: You are required to write up your solution separately and independently, although you are encouraged to discuss and work in groups. Please write your name, student ID number, and the name of your TA on the front page of the assignment that you hand in. Also, please put boxes around your final answer to each part.

1. Consumer Surplus with CES utility

An agent has utility \( u(x_1, x_2) = x_1^{1/2} + x_2^{1/2} \), income \( m = 4 \) and faces prices \( p_1 = 2 \) and \( p_2 = 2 \). Suppose the price of good 1 rises to \( p_1' = 3 \). Calculate the increase in the agent’s income required to compensate her for this price rise.

Solution

We derived the Marshallian demand and indirect utility function in PS4. The indirect utility function is

\[
v(p_1, p_2, m) = \left[ \frac{m(p_1 + p_2)}{p_1 p_2} \right]^{1/2}
\]

Plugging in the parameters, we see \( v(p_1, p_2, m) = 2 \). Inverting the indirect utility function we obtain the expenditure function,

\[
e(p_1, p_2, \bar{u}) = u^2 \frac{p_1 p_2}{p_1 + p_2}
\]

Letting, \( \bar{u} = 2 \), the compensating variation is

\[
CV = e(p_1', p_2, \bar{u}) - e(p_1, p_2, \bar{u}) = u^2 \frac{p_1' p_2}{p_1' + p_2} - u^2 \frac{p_1 p_2}{p_1 + p_2} = 4 \frac{5}{5} = \frac{4}{5}
\]
2. Cost Minimisation with Quasilinear Production

A firm has production function

\[ f(z_1, z_2) = 2z_1^{1/2} + z_2 \]

The firm’s inputs cost \( r_1 \) and \( r_2 \).

(a) Does this production function exhibit decreasing, constant or increasing returns to scale?

(b) Derive the MRTS.

(c) Show that the isoquants are convex.

(d) Solve the firm’s cost minimisation problem. You may find it easy to do this in two parts. First, assume

\[ q \geq \frac{r_2}{r_1} \]

so there is an internal solution and solve for the resulting cost function. Second, assume

\[ q \leq \frac{r_2}{r_1} \]

so there is a boundary solution and solve for the resulting cost function.

(e) Is the cost function concave or convex in \( q \)? How does this relate to part (a)?

Solution

(a) Decreasing returns. For \( t > 1 \),

\[ f(tz_1, tz_2) = 2t^{1/2}z_1^{1/2} + tz_2 \leq 2tz_1^{1/2} + tz_2 = tf(z_1, z_2) \]

(b) We have

\[ MRTS = \frac{MP_1}{MP_2} = \frac{z_1^{-1/2}}{1} = z_1^{-1/2} \]

(c) From part (b), MRTS is clearly decreasing in \( z_1 \).
(d) The firm minimises the Lagrangian

\[ \mathcal{L} = r_1 z_1 + r_2 z_2 + \lambda [q - 2 z_1^{1/2} - z_2] \]

Assuming an internal optimum, the FOCs are

\[ r_1 = \lambda z_1^{-1/2} \]
\[ r_2 = \lambda \]

These yield

\[ z_1^* = \frac{r_2}{r_1^2} \quad \text{and} \quad z_2^* = q - 2 \frac{r_2}{r_1} \]

The cost function is

\[ c(q) = r_1 z_1^* + r_2 z_2^* = qr_2 - 2 \frac{r_2}{r_1} \]

For \( q \leq 2 \frac{r_2}{r_1} \), we have \( z_2^* = 0 \). The production function is therefore \( q = 2 z_1^{1/2} \). Inverting,

\[ z_1^* = \frac{1}{4} q^2 \quad \text{and} \quad z_2^* = 0 \]

The cost function is

\[ c(q) = r_1 z_1^* + r_2 z_2^* = r_1^2 q^2 \]

(e) This cost function is convex in \( q \). This also follows from part (a), since DRS implies the cost function is convex.

3. Firm’s Problem with CES Production

A firm has production function

\[ f(z_1, z_2) = z_1^{1/2} + z_2^{1/2} \]

The firm’s inputs cost \( r_1 \) and \( r_2 \). The output price is \( p \).

(a) Suppose \( z_2 = 1 \), and assume \( q \geq 1 \) to avoid boundary problems. Derive the firm’s short–run cost and short–run marginal cost.

(b) Calculate the long–run cost function, when both factors are flexible.
(c) Using the cost function in (b), solve for the firm’s profit maximising output, an their maximal profit.

(d) Solve the firm’s profit maximisation problem directly, whereby the firm chooses \((z_1, z_2)\) to maximise

\[
\pi = pf(z_1, z_2) - r_1z_1 - r_2z_2
\]

What are the optimal inputs? What is the firm’s output and profit?

Solution

(a) The firm’s short run production function is

\[
f(z_1, z_2) = z_1^{1/2} + 1
\]

Hence the inputs needed to produce \(q\) are given by

\[
z^*_1 = (q - 1)^2
\]

where we assumed that \(q \geq 1\). The cost function is given by

\[
SRTC(q) = r_1z_1 + r_2z_2 = r_1(q - 1)^2 + r_2
\]

The marginal cost is

\[
SRMC(q) = 2r_1(q - 1)
\]

(b) The firm minimises the Lagrangian

\[
\mathcal{L} = r_1z_1 + r_2z_2 + \lambda[q - z_1^{1/2} - z_2^{1/2}]
\]

The FOCs are

\[
r_1 = \lambda \frac{1}{2} z_1^{-1/2}
\]
\[
r_2 = \lambda \frac{1}{2} z_2^{-1/2}
\]

Rearranging, we have

\[
\frac{r_1^2}{r_2^2} = \frac{z_2}{z_1}
\]
Substituting into the production function, we obtain

\[ z_1^* = \left( \frac{q r_2}{r_1 + r_2} \right)^2 \quad \text{and} \quad z_2^* = \left( \frac{q r_1}{r_1 + r_2} \right)^2 \]

The cost function is given by

\[ c(q) = r_1 z_1^* + r_2 z_2^* = \frac{r_1 r_2}{r_1 + r_2} q^2 \]

[Note the similarity between this and the expenditure function used in the first question.]

(c) The firm chooses \( q \) to maximise

\[ \pi = pq - \frac{r_1 r_2}{r_1 + r_2} q^2 \]

This is a concave problem. The FOC is

\[ p = 2 \frac{r_1 r_2}{r_1 + r_2} q \]

Rearranging,

\[ q^* = \frac{p r_1 + r_2}{2 r_1 r_2} \]

Profit is

\[ \pi^* = pq^* - c(q^*) = \frac{p^2}{2} \frac{r_1 + r_2}{r_1 r_2} - \frac{p^2}{4} \frac{r_1 + r_2}{r_1 r_2} = \frac{p^2}{4} \frac{r_1 + r_2}{r_1 r_2} \]

(d) The firm maximises

\[ \pi = p\left( z_1^{1/2} + z_2^{1/2} \right) - r_1 z_1 - r_2 z_2 \]

The FOCs are

\[ \frac{1}{2} p z_1^{-1/2} = r_1 \]
\[ \frac{1}{2} p z_2^{-1/2} = r_2 \]

Rearranging,

\[ z_1^* = \frac{p^2}{4 r_1^2} \quad \text{and} \quad z_2^* = \frac{p^2}{4 r_2^2} \]

Output is

\[ q^* = \left( z_1^* \right)^{1/2} + \left( z_2^* \right)^{1/2} = \frac{p r_1 + r_2}{2 r_1 r_2} \]
Profit is
\[ \pi^* = p\left(\left(z_1^*\right)^{1/2} + \left(z_2^*\right)^{1/2}\right) - r_1z_1^* - r_2z_2^* = \frac{p^2}{2} \frac{r_1 + r_2}{r_1r_2} - \frac{p^2}{4} \frac{r_1 + r_2}{r_1r_2} = \frac{p^2}{4} \frac{r_1 + r_2}{r_1r_2} \]

4. Profit Maximisation

A firm has cost curve
\[ c(q) = 100 + q^2 \]
The firm can also shut down and make profits \( \pi = 0 \).

(a) Suppose the firm faces price \( p = 30 \). What is the profit maximising quantity? What is the maximal profit? Will the firm shut down?

(b) Suppose \( p = 10 \). What is the profit maximising quantity? What is the maximal profit? Will the firm shut down?

(c) For which price levels will the firm shut down?

(d) Draw the supply function \( q^*(p) \) as a function of the output price \( p \).

Solution

The firm maximises
\[ \pi = pq - 100 - q^2 \]
Assuming an internal optimum, the FOC implies
\[ q^* = \frac{p}{2} \]
Maximal profits are
\[ \pi^* = pq^* - 100 - (q^*)^2 = \frac{p^2}{4} - 100 \]
Recall, the firm can also shut down and make \( \pi = 0 \).

(a) If \( p = 30 \), then \( q^* = 15 \) and \( \pi^* = 125 \). This is positive, so the firm should not shut down.
(b) If \( p = 10 \), then the internal optimum yields \( q^* = 5 \) and \( \pi^* = -75 \). Hence the firm is better off by shutting down.

(c) Setting \( \pi^* = 0 \), we find that \( p = 20 \). Hence the firm shuts down if \( p < 20 \), operates if \( p > 20 \), and is indifferent at \( p = 20 \).

(d) The supply function is \( q^*(p) = 0 \) for \( p \leq 20 \) and \( q^*(p) = p/2 \) for \( p \geq 20 \).
Economics 11: Homework 4

November 26, 2008

Due date: Thursday 4th December.

Instructions: You are required to write up your solution separately and independently, although you are encouraged to discuss and work in groups. Please write your name, student ID number, and the name of your TA on the front page of the assignment that you hand in. Also, please put boxes around your final answer to each part.

1. Deriving Supply Functions

(a) A firm has cost function

\[ c(q) = \begin{cases} q^2 & \text{for } q < 2 \\ q^2 + q - 2 & \text{for } q \geq 2 \end{cases} \]

Derive the firm’s supply curve, \( q^*(p) \).

(b) A firm has cost function

\[ c(q) = \begin{cases} q^2 & \text{for } q < 2 \\ q^2 - q + 2 & \text{for } q \geq 2 \end{cases} \]

Derive the firm’s supply curve, \( q^*(p) \).

2. Equilibrium

There is an economy with 50 agents. Of these agents, ten have income \( m = 10 \), ten have \( m = 20 \), ten have \( m = 30 \), ten have \( m = 40 \) and ten have \( m = 50 \). Each agent has utility function

\[ u(x_1, x_2) = x_1^{1/2} + x_2^{1/2} \]

over goods \( x_1 \) and \( x_2 \). The price of good 2 equals 1. The price of good 1 is to be determined.
(a) Derive each agent’s demand curve for good 1.

(b) Derive the market demand for good 1.

There are $J = 40$ firms. Each has production function

$$q = (z_1 - 1)^{1/4}(z_2 - 1)^{1/4}$$

The cost of the inputs is $r_1 = 1$ and $r_2 = 1$.

(c) Derive each firm’s supply curve.

(d) Derive the market supply curve.

(e) Verify the equilibrium price is $p = 5$. Show that, at this price, new entrants will wish to enter.

(f) Find the long–run free–entry equilibrium price, assuming all potential entrants have the same production technology. In addition, find the output of each firm ($q$), the number of firms in the industry ($J$) and the total industry output ($Q$).

3. Perfect Competition

A perfectly competitive industry has a large number of potential entrants. Each firm has an identical production function given by $f(z) = (z - 100)^{1/2}$. Total market demand is $X = 1800 - 10p$. The input price is $r = 4$.

(a) Derive the long–run equilibrium price, the output of each firm ($q$), the number of firms in the industry ($J$), the total industry output ($Q$), and the profits of each firm ($\pi$).

Suppose the input price falls to $r = 1$.

(b) In the very short–run, firms cannot change their output. What is the new price?

(c) In the short–run, firms cannot exit or enter. Their fixed cost is also sunk, so the firms cannot produce 0 for $c(0) = 0$. Calculate the short–run equilibrium price, the total industry output, and the output of each firm. Show that, taking into account the sunk cost, firms would like to enter.
(d) In the long-run, firms can enter and exit. Derive the long-run equilibrium price, the output of each firm (q), the number of firms in the industry (J), the total industry output (Q), and the profits of each firm (π).
Economics 11: Solutions to Homework 4

December 4, 2008

Due date: Thursday 4th December.

Instructions: You are required to write up your solution separately and independently, although you are encouraged to discuss and work in groups. Please write your name, student ID number, and the name of your TA on the front page of the assignment that you hand in. Also, please put boxes around your final answer to each part.

1. Deriving Supply Functions

(a) A firm has cost function

\[ c(q) = \begin{cases} q^2 & \text{for } q < 2 \\ &= q^2 + q - 2 & \text{for } q \geq 2 \end{cases} \]

Derive the firm’s supply curve, \( q^*(p) \).

(b) A firm has cost function

\[ c(q) = \begin{cases} q^2 & \text{for } q < 2 \\ &= q^2 - q + 2 & \text{for } q \geq 2 \end{cases} \]

Derive the firm’s supply curve, \( q^*(p) \).

Solution

(a) First suppose \( q < 2 \). The FOC is

\[ p = 2q \]

Hence \( q^*(p) = p/2 \). This assumes that \( q < 2 \), and hence \( p < 4 \).

Next suppose \( q \geq 2 \). The FOC is

\[ p = 2q + 1 \]
Figure 1: **Convex Cost Function.** The black line shows the MC curve. The red line shows the supply curve.

Hence \( q^*(p) = (p - 1)/2 \). This assumes that \( q \geq 2 \) and hence \( p \geq 5 \).

For \( p \in [4,5] \), the firm does not want to produce more than \( q = 2 \) at \( MC = 2q + 1 \). It also does not want to produce less that \( q = 2 \) at \( MC = 2q \). Hence the firm will produce \( q = 2 \). See figure 1.

(a) First suppose \( q < 2 \). The FOC is

\[
p = 2q
\]

Hence \( q^*(p) = p/2 \). This assumes that \( q < 2 \), and hence \( p < 4 \).

Next suppose \( q \geq 2 \). The FOC is

\[
p = 2q - 1
\]

Hence \( q^*(p) = (p + 1)/2 \). This assumes that \( q \geq 2 \) and hence \( p \geq 3 \).

For \( p \in [3,4] \) we therefore have two locally optimal solutions. To see which is the global optimum, we can calculate the profits. If \( q^*(p) = p/2 \) then profit equals

\[
\pi = pq^* - c(q^*) = \frac{p^2}{2} - \frac{p^2}{4} = \frac{p^2}{4}
\]
If \( q^*(p) = (p + 1)/2 \) then profit equals
\[
\pi = pq^* - c(q^*) = p \cdot \frac{p + 1}{2} - \frac{(p + 1)^2}{4} + \frac{(p + 1)}{2} - 2 = \frac{p^2}{4} + \frac{p}{2} - \frac{7}{4} \tag{2}
\]
Comparing (1) and (2), we see that the optimal supply is \( q^*(p) = p/2 \) if \( p < 7/2 \) and \( q^*(p) = (p + 1)/2 \) if \( p \geq 7/2 \). See figure 2.

2. Equilibrium

There is an economy with 50 agents. Of these agents, ten have income \( m = 10 \), ten have \( m = 20 \), ten have \( m = 30 \), ten have \( m = 40 \) and ten have \( m = 50 \). Each agent has utility function
\[
u(x_1, x_2) = x_1^{1/2} + x_2^{1/2}
\]
over goods \( x_1 \) and \( x_2 \). The price of good 2 equals 1. The price of good 1 is to be determined.

(a) Derive each agent’s demand curve for good 1.

(b) Derive the market demand for good 1.
There are $J = 40$ firms who produce good 1. Each has production function

$$q = (z_1 - 1)^{1/4}(z_2 - 1)^{1/4}$$

The cost of the inputs is $r_1 = 1$ and $r_2 = 1$.

(c) Derive each firm’s supply curve.

(d) Derive the market supply curve.

(e) Verify the equilibrium price for good 1 is $p = 5$. Show that, at this price, new entrants will wish to enter.

(f) Find the long–run free–entry equilibrium price for good 1, assuming all potential entrants have the same production technology. In addition, find the output of each firm ($q$), the number of firms in the industry ($J$) and the total industry output ($Q$).

Solution

(a) As in Practice Problem Set 4, the demand for each agent is

$$x_{i,1}^* = \frac{m_i}{p_1} \frac{p_2}{p_1 + p_2} = \frac{m_i}{p_1(p_1 + 1)}$$

(b) Summing up, market demand is

$$X_1 = \sum_i \frac{m_i}{p_1(p_1 + 1)} = \frac{1500}{p_1(p_1 + 1)}$$

(b) The cost minimisation problem is

$$\min_{z_1, z_2} \mathcal{L} = r_1 z_1 + r_2 z_2 + \lambda[q - (z_1 - 1)^{1/4}(z_2 - 1)^{1/4}]$$

The FOCs imply that $r_1(z_1 - 1) = r_2(z_2 - 1)$. The optimal input demands are

$$z_1^* = \left( \frac{r_2}{r_1} \right)^{1/2} q^2 + 1 \quad \text{and} \quad z_2^* = \left( \frac{r_1}{r_2} \right)^{1/2} q^2 + 1$$
The resulting cost function is

\[ c(q; r_1, r_2) = 2q^2(r_1 r_2)^{1/2} + (r_1 + r_2) \]

The firm’s profit maximisation problem implies \( p = MC \). That is,

\[ p = 4q(r_1 r_2)^{1/2} \]

Hence the firm’s supply curve is

\[ q^*(p) = \frac{p}{4(r_1 r_2)^{1/2}} \]

(d) There are 40 firms, so the market supply is

\[ Q^*(p) = 10\frac{p}{(r_1 r_2)^{1/2}} \]

Using \( r_1 = 1 \) and \( r_2 = 1 \), we have \( Q^*(p) = 10p \).

(e) Equilibrium is characterised by

\[ \frac{1500}{p(p + 1)} = 10p \]

Rearranging, \( p^2(p + 1) = 150 \). This is solved by \( p = 5 \). The total supply is \( Q = 50 \). Each firm produces \( q = 5/4 \). A firm’s profit is

\[ \pi = pq - c(q) = \frac{25}{4} - 2 \frac{25}{16} - 2 = \frac{9}{8} > 0 \]

Hence entry is profitable.

(f) The average cost is

\[ AC = 2q + 2q^{-1} \]

This is minimised at \( q = 1 \). The average cost is \( AC = 4 \). Hence the long run price is \( p = 4 \). Demand is \( X = 1500/20 = 75 \). Hence there are \( J = 75 \) firms.

3. Perfect Competition

A perfectly competitive industry has a large number of potential entrants. Each firm has an identical production function given by \( f(z) = (z - 100)^{1/2} \). Total market demand is \( X = 1800 - 10p \). The input price is \( r = 4 \).
(a) Derive the long–run equilibrium price, the output of each firm \(q\), the number of firms in the industry \(J\), the total industry output \(Q\), and the profits of each firm \(\pi\).

Suppose the input price falls to \(r = 1\).

(b) In the very short–run, firms cannot change their output. What is the new price?

(c) In the short–run, firms cannot exit or enter. Their fixed cost is also sunk, so the firms cannot produce 0 for \(c(0) = 0\). Calculate the short–run equilibrium price, the total industry output, and the output of each firm. Show that, taking into account the sunk cost, firms would like to enter.

(d) In the long–run, firms can enter and exit. Derive the the long–run equilibrium price, the output of each firm \(q\), the number of firms in the industry \(J\), the total industry output \(Q\), and the profits of each firm \(\pi\).

Solution

(a) Given an input price \(r\), the firm’s cost function is

\[ c(q) = rz = rf^{-1}(q) = r(q^2 + 100) \]

Average cost is

\[ AC(q) = rq + 100rq^{-1} \]

Differentiating, this is minimised at \(r = 100rq^{-2}\). Rearranging, \(q = 10\). The price therefore equals,

\[ p = AC(q) = 20r \]

When \(r = 4\), \(p = 80\). The output of each firm is 10. The total demand is \(X = 1800 - 800 = 1000\). As a result, there are \(J = 100\) firms. By construction, each firm makes zero profits.

(b) Output is \(Q = 1000\). The price is still \(p = 80\).

(b) The firm’s marginal cost is

\[ MC = 2rq = 2q \]

Profit maximisation implies \(p = MC\). Hence the short run supply curve is \(q^* = p/2\). Summing
over firms, industry supply is \( Q = 50p \). The equilibrium is

\[
50p = 1800 - 10p
\]

Rearranging, \( p = 30 \). The output of each firm is \( q = 15 \). The industry output is \( Q = 1500 \).

The profit of each firm is

\[
\pi = pq - c(q) = 30 \times 15 - (15)^2 - 100 = 125
\]

Which will induce entry.

(d) As in (a), average cost is minimised when \( q = 10 \). The price equals \( p = 20 \). Demand is \( X = 1800 - 200 = 1600 \). Hence there are \( J = 160 \) firms. By construction, each firm makes zero profits.
Economics 11: Practice Problems 1 (Week 2)

29 September, 2008

1. Utility Functions

Define the utility functions and draw the indifference curves for each of the following cases:

a) “Every time I consume one unit of $x_1$, I want to consume 2 units of $x_2$”.

b) “If the price of $x_1$ is bigger than the price of $x_2$, I will only consume $x_2$. If the price of $x_2$ is bigger than the price of $x_1$, I will only consume $x_1$”.

c) “I don’t care about the prices of $x_1$ and $x_2$, I only want to consume $x_2$”.

2. Consumption with Perfect Substitutes

Suppose John views butter and margarine as perfectly substitutable in the ratio of one to one for each other.

a) Draw a set of indifference curves that describes John’s preferences for butter and margarine.

b) If butter costs $2 per package, while margarine cost only $1, and John has a $20 budget to spend for the month, which butter-margarine combination will he choose?

(c) Can you show this graphically? What happens when the price of margarine increases to $3?

[Note: As you read this, we probably haven’t yet studied the utility maximisation problem formally. However, this is simple enough that you can use your intuition to solve it.]

3. Convexity and Monotonicity

For the following utility functions, sketch the indifference curves and explain whether the underlying preferences satisfy convexity and monotonicity. Throughout, assume $x_1, x_2 \geq 0$.

(a) $u(x_1, x_2) = x_1 + x_2$
(b) \( u(x_1, x_2) = x_1^2 + x_2^2 \)

(c) \( u(x_1, x_2) = \min\{x_1 + x_2, 1\} \)

(d) \( u(x_1, x_2) = I(x_1) + I(x_2) \), where \( I(x) \) is the integer component of \( x \). Hence \( I(3.64) = 3 \) and \( I(5.2) = 5 \).

(e) \( u(x_1, x_2) = -(x_1 - 1)^2 - (x_2 - 1)^2. \)

(f) \( u(x_1, x_2) = \min\{x_1, x_2\}. \)

4. Properties of Indifference Curves

Suppose an agent’s preferences satisfy transitivity, completeness, continuity and monotonicity. Suppose \( X = \mathbb{R}_+^2 \). Which of the following statements are correct? Explain your answer.

(a) Indifference curves are thin.

(b) Indifference curves are convex.

(c) There can be a bliss point, where utility is maximised.

(d) Indifference curves have no kinks.

(e) Indifference curves can be flat. [An indifference curve is flat if \( x \sim y, x_1 > y_1 \) and \( x_2 = y_2 \).]
Economics 11: Solutions to Practice Problems 1 (Week 2)

29 September, 2008

1. Utility Functions

Define the utility functions and draw the indifference curves for each of the following cases:

a) “Every time I consume one unit of $x_1$, I want to consume 2 units of $x_2$”.

b) “If the price of $x_1$ is bigger than the price of $x_2$, I will only consume $x_2$. If the price of $x_2$ is bigger than the price of $x_1$, I will only consume $x_1$”.

c) “I don’t care about the prices of $x_1$ and $x_2$, I only want to consume $x_2$”.

Solution

I won’t draw the pictures. I refer you to the lectures.

(a) Preferences are represented by $u(x_1, x_2) = \min\{2x_1, x_2\}$. That is, preferences are perfect complements.

(b) Preferences are represented by $u(x_1, x_2) = x_1 + x_2$. That is, preferences are perfect substitutes.

(c) Preferences are represented by $u(x_1, x_2) = x_2$.

2. Consumption with Perfect Substitutes

Suppose John views butter and margarine as perfectly substitutable in the ratio of one to one for each other.

a) Draw a set of indifference curves that describes John’s preferences for butter and margarine.

b) If butter costs $2 per package, while margarine cost only $1, and John has a $20 budget to spend for the month, which butter-margarine combination will he choose?
(c) Can you show this graphically? What happens when the price of margarine increases to $3?

[Note: As you read this, we probably haven’t yet studied the utility maximisation problem formally. However, this is simple enough that you can use your intuition to solve it.]

**Solution**

(a) Indifference curves are straight lines. See the lecture.

(b) The MRS between butter and margarine is equal to 1 at every consumption level. As the cost of margarine is less, John will choose to buy margarine with all his income and will buy no butter. So he will buy 20 units of margarine and 0 units of butter.

(c) If the price of margarine increases to 3 he will consume only butter (10 units).

3. **Convexity and Monotonicity**

For the following utility functions, sketch the indifference curves and explain whether the underlying preferences satisfy convexity and monotonicity. Throughout, assume $x_1, x_2 \geq 0$.

(a) $u(x_1, x_2) = x_1 + x_2$

(b) $u(x_1, x_2) = x_1^2 + x_2^2$

(c) $u(x_1, x_2) = \min\{x_1 + x_2, 1\}$

(d) $u(x_1, x_2) = I(x_1) + I(x_2)$, where $I(x)$ is the integer component of $x$. Hence $I(3.64) = 3$ and $I(5.2) = 5$.

(e) $u(x_1, x_2) = -(x_1 - 1)^2 - (x_2 - 1)^2$.

(f) $u(x_1, x_2) = \min\{x_1, x_2\}$. 
Solution

(a) The goods here are perfect substitutes. These preferences satisfy monotonicity and convexity.

(b) These preferences satisfy monotonicity but not convexity.

(c) The consumer has a maximum utility of 1. Hence these preferences satisfy convexity but not monotonicity.

(d) The goods here are indivisible. These preferences satisfy neither monotonicity or convexity.

(e) This has a bliss point at (1, 1). These preferences satisfy convexity but not monotonicity.

(f) Preferences are perfect complements. These preferences satisfy convexity but not monotonicity.

4. Properties of Indifference Curves

Suppose an agent’s preferences satisfy transitivity, completeness, continuity and monotonicity. Suppose $X = \mathbb{R}_{+}^2$. Which of the following statements are correct? Explain your answer.

(a) Indifference curves are thin.

(b) Indifference curves are convex.

(c) There can be a bliss point, where utility is maximised.

(d) Indifference curves have no kinks.

(e) Indifference curves can be flat. [An indifference curve is flat if $x \sim y, x_1 > y_1$ and $x_2 = y_2$.]

Solution

(a) True. Otherwise we can find $\{x, y\}$ such that $x \sim y$, and $y_1 > x_1$ and $y_2 > x_2$. This contradicts monotonicity.
(b) False. We need to assume convexity for this.

(c) False. If there is a bliss point, we can find \( \{x, y\} \) such that \( x \sim y \), and \( y_1 > x_1 \) and \( y_2 > x_2 \). This contradicts monotonicity.

(d) False. Nothing we have assumed rules out kinks.

(e) False. This does not satisfy monotonicity.
Economics 11: Practice Problems 2 (Week 3)

29 September, 2008

1. Consumer choice problem

Sam eats breakfast at Luvalle Commons everyday. She likes bagels and coffee and they provide her a utility of $U(b, c) = b^{2/3}c^{1/3}$. Bagels cost $2 and coffee costs $1 per cup. Her daily budget for breakfast is $12.

a) Suppose that she consumes 2 bagels and 8 coffees. Find the marginal rate of substitution. Is this an optimal choice? Why?

b) Suppose that she consumes 3 bagels and 3 coffees. Find the marginal rate of substitution. Is this an optimal choice? Why?

c) Find the optimal choice for Sam.

2. Consumer choice problem

Joe is a teenager who likes video games G and music M. Each week, he receives $60 to buy games and music CDs. His utility function for these two goods is $U(G, M) = G^{1/2} + M^{1/2}$. The price of a game is $20 and the price of a CD is $10.

a) Solve for Joe’s demand using the Lagrangian Method.

b) Solve the same problem as in (a) using the Substitution Method.

c) Joe is a bit disappointed with your calculations. He thinks he can buy more games and CDs if his utility function is logarithmic instead. Take the (natural) log of $U(G, M)$ and solve the optimisation problem again using the Lagrangian method. Compare the optimal levels with those found in part (a).
3. Consumer choice with three goods

Susan and Paul are college students whose parents give them monthly allowances to be spent on food (f), clothing (c) and books (b). Prices are as follows: one meal costs $3, one piece of clothing costs $40, and one book costs $5. Assume the goods are perfectly divisible. Susan is a senior and she receives $200 per month; Paul is a freshman and he receives $160 per month. Susan’s preferences are given by the following utility function:

\[ U(f, c, b) = 3 \ln(f) + 6 \ln(c) + \ln(b) \]

Paul’s preferences are given by the following utility function:

\[ U(f, c, b) = 3 \ln(f) + \ln(c) + 2 \ln(b) \]

a) Write down the budget constraint for each individual.

b) Suppose Susan buys 20 meals, 3 pieces of clothing and 4 books. What is her marginal rate of substitution between food and clothing. What is her marginal rate of substitution between clothing and books? Is this allocation optimal for her?

c) Suppose Paul buys 10 meals, 2 pieces of clothing and 10 books. What is his marginal rate of substitution between food and clothing. What is his marginal rate of substitution between clothing and books? Is this allocation optimal for him?

d) Using the Lagrangian method, find the optimal allocations for food, clothing, and books for Susan and Paul.

e) Suppose that the students parents decide that even though Susan is older, both children are facing the same prices and thus should have the same allowance. They increase Paul’s allowance to $200. Find his new optimal allocations. Which good has the largest percentage increase in consumption, if any?

4. Calculating Demand Functions

Find the demand functions for \( x_1 \) and \( x_2 \) in the following utility functions:

a) \( U(x_1, x_2) = x_1^{1/4} x_2^{3/4} \)
b) \( U(x_1, x_2) = \min\{2x_1, 3x_2\} \)

c) \( U(x_1, x_2) = 2x_1 + 3x_2 \)

5. Consumption with Endowments

Robinson is the only person in an island, and has an endowment of 100 units of \( x_1 \) and 120 units of \( x_2 \). His utility function is \( u(x_1, x_2) = \ln x_1 + \ln x_2 \).

a) Find the marginal utility of \( x_1 \) and \( x_2 \).

b) Find the marginal rate of substitution when Robinson consumes all his endowment.

c) Suppose now that Robinson finds that he is not alone on the Island, so now he can trade goods with other people. Find the relative prices \( p_1/p_2 \) for which Robinson decides to sell \( x_1 \) and to buy \( x_2 \).

[Note: In this problem the agent has endowments of goods rather than of income. Later in the course we’ll properly study this model.]
Economics 11: Practice Problems 2 (Week 3)

29 September, 2008

1. Consumer choice problem

Sam eats breakfast at Luvalle Commons everyday. She likes bagels and coffee and they provide her a utility of \( U(b, c) = b^{2/3}c^{1/3} \). Bagels cost $2 and coffee costs $1 per cup. Her daily budget for breakfast is $12.

a) Suppose that she consumes 2 bagels and 8 coffees. Find the marginal rate of substitution. Is this an optimal choice? Why?

b) Suppose that she consumes 3 bagels and 3 coffees. Find the marginal rate of substitution. Is this an optimal choice? Why?

c) Find the optimal choice for Sam.

Solution

(a) The MRS is

\[
MRS = \frac{MU_b}{MU_c} = \frac{\frac{2}{3}b^{1/3}}{\frac{1}{3}c^{2/3}} = \frac{2c}{b} = \frac{16}{2} = 8
\]

The price ratio is \( p_b/p_c = 2 \), so Sam would prefer to eat more bagels.

(b) Sam’s total expenditure is $9. Since her preferences are monotone, this cannot be optimal.

(c) The Lagrangian is:

\[
\mathcal{L} = b^{2/3}c^{1/3} + \lambda[M - p_b b - p_c c]
\]

The FOCs are:

\[
\frac{2}{3} \left( \frac{c}{b} \right)^{1/3} = \lambda p_b
\]

\[
\frac{1}{3} \left( \frac{b}{c} \right)^{2/3} = \lambda p_c
\]

These imply that \( 2p_c c = p_b b \). Using the budget equation, \( b^* = 4 \) and \( c^* = 4 \).
2. Consumer choice problem

Joe is a teenager who likes video games $G$ and music $M$. Each week, he receives $60 to buy games and music CDs. His utility function for these two goods is $U(G, M) = G^{1/2} + M^{1/2}$. The price of a game is $20 and the price of a CD is $10.

a) Solve for Joe’s demand using the Lagrangian Method.

b) Solve the same problem as in (a) using the Substitution Method.

c) Joe is a bit disappointed with your calculations. He thinks he can buy more games and CDs if his utility function is logarithmic instead. Take the (natural) log of $U(G, M)$ and solve the optimisation problem again using the Lagrangian method. Compare the optimal levels with those found in part (a).

Solution

(a) Joe’s budget constraint is $20G + 10M = 60$. The Lagrangian is:

$$\mathcal{L} = G^{1/2} + M^{1/2} + \lambda[60 - 20G - 10M]$$

The FOCs are

$$\frac{1}{2}G^{-1/2} = 20\lambda$$

$$\frac{1}{2}M^{-1/2} = 10\lambda$$

Rearranging, $M = 4G$. Using the budget constraint, $G^* = 1$ and $M^* = 4$.

(b) Using the budget constraint, $M = 6 - 2G$. The utility function is thus:

$$U = G^{1/2} + (6 - 2G)^{1/2}$$

The FOC is

$$\frac{1}{2}G^{-1/2} - (6 - 2G)^{-1/2} = 0$$

Rearranging, $G^* = 1$. Using the budget constraint, $M^* = 4$. 

2
(c) The Lagrangian is:

\[ \mathcal{L} = \ln(G^{1/2} + M^{1/2}) + \lambda[60 - 20G - 10M] \]

The FOCs are

\[
\frac{1}{2} \frac{G^{-1/2}}{G^{1/2} + M^{1/2}} = 20\lambda \\
\frac{1}{2} \frac{M^{-1/2}}{G^{1/2} + M^{1/2}} = 10\lambda
\]

Rearranging, \( M = 4G \). Using the budget constraint, \( G^* = 1 \) and \( M^* = 4 \). The solution is the same since we have just applied an increasing function to the utility function.

3. Consumer choice with three goods

Susan and Paul are college students whose parents give them monthly allowances to be spent on food (\( f \)), clothing (\( c \)) and books (\( b \)). Prices are as follows: one meal costs $3, one piece of clothing costs $40, and one book costs $5. Assume the goods are perfectly divisible. Susan is a senior and she receives $200 per month; Paul is a freshman and he receives $160 per month. Susan’s preferences are given by the following utility function:

\[ U(f, c, b) = 3\ln(f) + 6\ln(c) + \ln(b) \]

Paul’s preferences are given by the following utility function:

\[ U(f, c, b) = 3\ln(f) + \ln(c) + 2\ln(b) \]

a) Write down the budget constraint for each individual.

b) Suppose Susan buys 20 meals, 3 pieces of clothing and 4 books. What is her marginal rate of substitution between food and clothing. What is her marginal rate of substitution between clothing and books? Is this allocation optimal for her?

c) Suppose Paul buys 10 meals, 2 pieces of clothing and 10 books. What is his marginal rate of substitution between food and clothing. What is his marginal rate of substitution between clothing and books? Is this allocation optimal for him?
d) Using the Lagrangian method, find the optimal allocations for food, clothing, and books for Susan and Paul.

e) Suppose that the students parents decide that even though Susan is older, both children are facing the same prices and thus should have the same allowance. They increase Paul’s allowance to $200. Find his new optimal allocations. Which good has the largest percentage increase in consumption, if any?

Solution

(a) Susan’s budget is $3f + 40c + 5b = 200$. Paul’s is $3f + 40c + 5b = 160$.

(b) Susan’s MRSs are:

\[
MRS_{f,c} = \frac{MU_f}{MU_c} = \frac{3/f}{6/c} = \frac{c}{2f} = \frac{3}{40}
\]

\[
MRS_{c,b} = \frac{MU_c}{MU_b} = \frac{6/c}{1/b} = \frac{6b}{c} = 8
\]

For the allocation to be optimal we need to verify (a) Susan’s marginal rates of substitution equal the price ratios and (b) Susan spends all her budget. The price ratios are $p_c/p_b = 8$ and $p_f/p_c = 3/40$. Moreover, Susan’s expenditure is $20 * 3 + 3 * 40 + 4 * 5 = 200$. Hence Susan’s allocation is optimal.

(c) Paul’s MRSs are:

\[
MRS_{f,c} = \frac{MU_f}{MU_c} = \frac{3/f}{1/c} = \frac{3c}{f} = \frac{3}{5}
\]

\[
MRS_{c,b} = \frac{MU_c}{MU_b} = \frac{1/c}{2/b} = \frac{b}{2c} = \frac{5}{2}
\]

The price ratios are $p_c/p_b = 8$ and $p_f/p_c = 3/40$, so Paul’s allocation is not optimal (e.g. he should consume more food).

(d) Let’s solve the problem generally and then plug in the numbers. The Lagrangian for Susan is:

\[
\mathcal{L} = 3\ln(f) + 6\ln(c) + \ln(b) + \lambda[M - p_f f - p_c c - p_b b]
\]
The FOCs are
\[
\frac{3}{f} = \lambda p_f \\
\frac{6}{c} = \lambda p_c \\
\frac{1}{b} = \lambda p_b
\]

The equations imply \( cp_c = 2fp_f = 6bp_b \). Using the budget constraint, \( f^* = \frac{3M}{10p_f} \), \( c^* = \frac{3M}{5p_c} \), and \( b^* = \frac{M}{10p_b} \). Given the numbers in the example, we have \( f^* = 20 \), \( c^* = 3 \) and \( b^* = 4 \).

The Lagrangian for Paul is:
\[
\mathcal{L} = 3 \ln(f) + \ln(c) + 2 \ln(b) + \lambda [M - pf - pc - pb]
\]

The FOCs are
\[
\frac{3}{f} = \lambda p_f \\
\frac{1}{c} = \lambda p_c \\
\frac{2}{b} = \lambda p_b
\]

The equations imply \( 6cp_c = 2fp_f = 3bp_b \). Using the budget constraint, \( f^* = \frac{M}{p_f} \), \( c^* = \frac{M}{6p_c} \), and \( b^* = \frac{M}{3p_b} \). Given the numbers in the example, we have \( f^* = 26 \frac{2}{3} \), \( c^* = \frac{2}{3} \) and \( b^* = 10 \frac{2}{3} \).

(e) When Paul’s income rises to $200, the optimal consumption levels become \( f^* = 33 \frac{1}{3} \), \( c^* = \frac{5}{6} \) and \( b^* = 13 \frac{1}{3} \). One can see that Paul’s consumption of all goods increases by the 25%. As we will soon see, this is because the income offer curve is linear with these Cobb–Douglas preferences.

4. Calculating Demand Functions

Find the demand functions for \( x_1 \) and \( x_2 \) in the following utility functions:

a) \( U(x_1, x_2) = x_1^{1/4} x_2^{3/4} \)

b) \( U(x_1, x_2) = \min\{2x_1, 3x_2\} \)

c) \( U(x_1, x_2) = 2x_1 + 3x_2 \)
Solution

(a) The Lagrangian is:
\[ \mathcal{L} = x_1^{1/4} x_2^{3/4} + \lambda [M - p_1 x_1 - p_2 x_2] \]

The FOCs are:
\[ \frac{1}{4} \left( \frac{x_2}{x_1} \right)^{3/4} = \lambda p_1 \]
\[ \frac{3}{4} \left( \frac{x_1}{x_2} \right)^{1/4} = \lambda p_2 \]

These imply that \( 3p_1 x_1 = p_2 x_2 \). Demand are thus \( x_1^* = M/4p_1 \) and \( x_2^* = 3M/4p_2 \).

(b) The utility function is not differentiable at the kink (draw the indifference curves). Hence the Lagrangian method does not work.

From the utility function we know that, at the optimum, \( 2x_1^* = 3x_2^* \). Using the budget constraint,
\[ x_1^* = \frac{3M}{3p_1 + 2p_2} \]
\[ x_2^* = \frac{2M}{3p_1 + 2p_2} \]

(c) The indifference curves are linear, so we will have a boundary problem. [If you use the Lagrangian approach and ignore the boundaries, you will want to consume a negative quantity of one or other good]. We can therefore just compare the two endpoints.

If \( 3p_1 > 2p_2 \), then \( x_1^* = 0 \) and \( x_2^* = M/p_2 \). If \( 3p_1 < 2p_2 \), then \( x_1^* = M/p_1 \) and \( x_2^* = 0 \). If \( 3p_1 = 2p_2 \), then the consumer is indifferent between all points on the budget line.

5. Consumption with Endowments

Robinson is the only person in an island, and has an endowment of 100 units of \( x_1 \) and 120 units of \( x_2 \). His utility function is \( u(x_1, x_2) = \ln x_1 + \ln x_2 \).

a) Find the marginal utility of \( x_1 \) and \( x_2 \).
b) Find the marginal rate of substitution when Robinson consumes all his endowment.

c) Suppose now that Robinson finds that he is not alone on the Island, so now he can trade goods with other people. Find the relative prices $p_1/p_2$ for which Robinson decides to sell $x_1$ and to buy $x_2$.

[Note: In this problem the agent has endowments of goods rather than of income. Later in the course we’ll properly study this model.]

**Solution**

(a) $MU_1 = 1/x_1$, $MU_2 = 1/x_2$.

(b) The MRS is:

$$MRS = \frac{MU_1}{MU_2} = \frac{1}{x_1} \cdot \frac{1}{x_2} = \frac{x_2}{x_1} = \frac{120}{100} = 1.2$$

(c) He will be willing to sell good $x_1$ and buy $x_2$ only if:

$$\frac{p_1}{p_2} \geq MRS = 1.2$$
Economics 11: Practice Problems 3 (Week 4)

October 16, 2008

1. Budget Sets

There are two goods: \( x_1 \) and \( x_2 \) with prices \( p_1 = 2 \) and \( p_2 = 2 \). Suppose the government subsidises the first 10 units of \( x_1 \) by \$1. Draw the budget curves for \( m = 8 \), \( m = 12 \) and \( m = 16 \).

2. Income Effects and Quasilinear Utility

Suppose utility has the form \( u(x_1, x_2) = x_1^{1/2} + x_2 \). Let \( p_1 = 1 \) and \( p_2 = 2 \). Derive the consumer’s income offer curve.

3. Price Changes

Joseph likes roses (R) and tulips (T) equally, and views them as perfect substitutes in proportion 1 to 1.

a) What are Joseph’s demand functions for roses and tulips (as a function of income and prices)?

Suppose the price of a rose is \$3\), the price of a tulip is \$5\), and Joseph has \$30\) to expend in flowers.

c) How much of each flower will Joseph buy?

c) Now, suppose that the price of a rose rises to \$6\). How does the consumption of Joseph change?

d) How much should Joseph’s income increase to compensate for the rise in the price of roses?

4. Taxes with Cobb Douglas Utility

An agent has utility \( u(x_1, x_2) = x_1 x_2^2 \).
a) Find the agent’s Marshallian demand.

b) Find the agent’s indirect utility function.

Now suppose that \( p_1 = 10 \) and \( p_2 = 5 \). The agent’s income is \( m = 300 \).

c) What is the agent’s consumption? What is her utility level?

The government would like to reduce the agent’s consumption of \( x_2 \) by half. It considers two policies to achieve this. Policy A imposes a tax on each unit of \( x_2 \). Policy B imposes a lump sum tax on the agent’s income.

d) Suppose the government uses policy A. How big a tax needs to be imposed on each unit of \( x_2 \)? What is the government’s tax revenue? What is the agent’s utility?

e) Suppose the government uses policy B. By how much would the government have to reduce the agent’s income? What is the government’s tax revenue? What is the agent’s utility?

f) Which policy does the agent prefer? Why?

5. Calculating Elasticities

An agent’s demand for good 1 is given by

\[
x_1^*(p_1, p_2, m) = (m + p_2)/2p_1
\]

Compute the agent’s price elasticity, cross elasticity and income elasticity of \( x_1 \) when \( m = 200 \), \( p_1 = 5 \) and \( p_2 = 10 \).

6. Elasticities

Let \( x \in \mathbb{R}_+ \) and \( y \in \mathbb{R}_+ \) be related by the function: \( y = f(x) \). The elasticity of \( y \) with respect to \( x \) is defined by

\[
\epsilon_{y,x} = \frac{dy}{dx} \frac{x}{y}
\]

Show that

\[
\epsilon_{y,x} = \frac{d\ln y}{d\ln x}
\]
7. Expenditure Minimisation

Suppose that an individual has utility $u(x_1, x_2) = x_1(1 + x_2)$. Throughout assume income $m$ is sufficiently high so there is an internal solution.

a) Find the Marshallian demand for $x_1$ and $x_2$.

b) Are $x_1$ and $x_2$ normal or inferior goods?

c) Find the indirect utility function.

d) Invert the indirect utility function to find the expenditure function. [We will probably cover this in lecture 8]

e) What is the minimum expenditure necessary to achieve a utility level of $u = 72$ with $p_1 = 4$ and $p_2 = 2$?.
Economics 11: Solutions to Practice Problems 3 (Week 4)

October 23, 2008

1. Budget Sets

There are two goods: $x_1$ and $x_2$ with prices $p_1 = 2$ and $p_2 = 2$. Suppose the government subsidises the first 10 units of $x_1$ by $1$. Draw the budget curves for $m = 8$, $m = 12$ and $m = 16$

Solution

When $m = 8$, it is as if $p_1 = 1$ and $p_2 = 2$.

When $m = 12$ or $m = 16$, there is a kink at $x_1 = 10$. The slope is $-1/2$ to the left and $-1$ to the right. The vertical intercept is $x_2 = 6$ when $m = 12$ and $x_2 = 8$ when $m = 16$. The horizontal intercept is $x_1 = 11$ when $m = 12$ and $x_1 = 13$ when $m = 16$.

2. Income Effects and Quasilinear Utility

Suppose utility has the form $u(x_1, x_2) = x_1^{1/2} + x_2$. Let $p_1 = 1$ and $p_2 = 2$. Derive the consumer’s income offer curve.

Solution

Note: Here I use the MRS argument (which is identical to the Lagrange method). We could also use the substitution method.

The consumer’s MRS is

$$\frac{MU_1}{MU_2} = \frac{1}{2} x_1^{-1/2}$$

The price ratio is $p_1/p_2 = \frac{1}{2}$. The solution is $x_1^* = 1$, which is feasible if $m \geq 1$. If $m < 1$ then $MRS > p_1/p_2$, so the consumer will spend all her money on good 1.¹

¹Equivalently, the bang–for–the–back from good 2 is $MU_2/p_2 = 1/2$. This is less than the bang–for–the–back from good 1 if $x_1 < 1$. 

1
We thus find that $x_1^* = m$ for $m < 1$ and $x_1^* = 1$ for $m \geq 1$. More concisely, $x_1^* = \min\{m, 1\}$.

3. Price Changes

Joseph likes roses (R) and tulips (T) equally, and views them as perfect substitutes in proportion 1 to 1.

a) What are Joseph’s demand functions for roses and tulips (as a function of income and prices)?

Suppose the price of a rose is $3, the price of a tulip is $5, and Joseph has $30 to expend in flowers.

c) How much of each flower will Joseph buy?

c) Now, suppose that the price of a rose rises to $6. How does the consumption of Joseph change?

d) How much should Joseph’s income increase to compensate for the rise in the price of roses?

Solution

a) There are three cases:
1) If $p_r < p_t$ then $R = m/p_r$ and $T = 0$
2) If $p_r > p_t$ then $R = 0$ and $T = m/p_t$
3) If $p_r = p_t$ then $R$ and $T$ any non-negative value as long as $p_rR + p_tT = m$.

b) His preferences can be represented by $U = R + T$. If roses are cheaper he will consume only roses. Therefore, $R = 30/3 = 10$ and $T = 0$. Utility is 10.

c) Since tulips are now cheaper he will buy only tulips. $R = 0$ and $T = 30/5 = 6$. Utility is 6.

d) Initial utility is 10. Since he is buying only tulips, he needs $T = m/5 = 10$. So to compensate him to that $m = 50$. Therefore, income has to increase by 20.
4. Taxes with Cobb Douglas Utility

An agent has utility \( u(x_1, x_2) = x_1 x_2^2 \).

a) Find the agent’s Marshallian demand.

b) Find the agent’s indirect utility function.

Now suppose that \( p_1 = 10 \) and \( p_2 = 5 \). The agent’s income is \( m = 300 \).

c) What is the agent’s consumption? What is her utility level?

The government would like to reduce the agent’s consumption of \( x_2 \) by half. It considers two policies to achieve this. Policy A imposes a tax on each unit of \( x_2 \). Policy B imposes a lump sum tax on the agent’s income.

d) Suppose the government uses policy A. How big a tax needs to be imposed on each unit of \( x_2 \)? What is the government’s tax revenue? What is the agent’s utility?

e) Suppose the government uses policy B. By how much would the government have to reduce the agent’s income? What is the government’s tax revenue? What is the agent’s utility?

f) Which policy does the agent prefer? Why?

Solution

a) The Lagrangian is

\[
\mathcal{L} = x_1 x_2^2 + \lambda[m - p_1 x_1 - p_2 x_2]
\]

The FOCs are:

\[
x_2^2 = \lambda p_1
\]

\[
2x_1 x_2 = \lambda p_2
\]

These imply

\[
2p_1 x_1 = p_2 x_2
\]
Using the budget constraint,

\[ x_1^* = \frac{m}{3p_1} \]
\[ x_2^* = \frac{2m}{3p_2} \]

b) The indirect utility is

\[ v = \frac{4m^3}{27p_1p_2^2} \]

c) Plugging in, \( x_1 = 10, \ x_2 = 40 \) and \( v = 16000 \).

d) We need to double the price of \( x_2 \), implying a tax of $5. The agent consumes \( x_2 = 20 \), so the tax revenue is 100. The new utility level is 4000.

e) We need to cut income, reducing it to 150. Tax revenue is 150. The agent’s utility is 2000.

f) The agent prefers the tax on \( x_2 \), because that only reduces her spending on \( x_2 \), whereas the income tax also reduces her consumption of \( x_1 \).

5. Calculating Elasticities

An agent’s demand for good 1 is given by

\[ x_1^*(p_1, p_2, m) = \frac{(m + p_2)}{2p_1} \]

Compute the agent’s price elasticity, cross elasticity and income elasticity of \( x_1 \) when \( m = 200 \), \( p_1 = 5 \) and \( p_2 = 10 \).

Solution

The price elasticity is

\[ \frac{\partial x_1}{\partial p_1} \frac{p_1}{x_1} = -1 \]

The cross elasticity is

\[ \frac{\partial x_1}{\partial p_2} \frac{p_2}{x_1} = 1/21 \]
The income elasticity is
\[ \frac{\partial x_1}{\partial m} \frac{m}{x_1} = \frac{20}{21} \]

Note that the sum of these is zero, as expected.

6. Elasticities

Let \( x \in \mathbb{R}_+ \) and \( y \in \mathbb{R}_+ \) be related by the function: \( y = f(x) \). The elasticity of \( y \) with respect to \( x \) is defined by

\[ \epsilon_{y,x} = \frac{dy}{dx} \frac{x}{y} \]

Show that

\[ \epsilon_{y,x} = \frac{d \ln y}{d \ln x} \]

**Solution**

Observe that

\[ \ln y = \ln f(e^{\ln x}) \]

Using the chain rule,

\[ \frac{d \ln y}{d \ln x} = \frac{f'(e^{\ln x})e^{\ln x}}{f(e^{\ln x})} = \frac{f'(x)}{f(x)} \frac{x}{dy}{dx} \]

as required.

7. Expenditure Minimisation

Suppose that an individual has utility \( u(x_1, x_2) = x_1(1 + x_2) \). Throughout assume income \( m \) is sufficiently high so there is an internal solution.

a) Find the Marshallian demand for \( x_1 \) and \( x_2 \).

b) Are \( x_1 \) and \( x_2 \) normal or inferior goods?

c) Find the indirect utility function.
d) Invert the indirect utility function to find the expenditure function. [We will probably cover this in lecture 8]

e) What is the minimum expenditure necessary to achieve a utility level of \( u = 72 \) with \( p_1 = 4 \) and \( p_2 = 2 \)?

Solution

(a) The Lagrangian is

\[ L = x_1(1 + x_2) + \lambda[m - p_1x_1 - p_2x_2] \]

The FOCs are

\[ 1 + x_2 = \lambda p_1 \]
\[ x_1 = \lambda p_2 \]

We thus get,

\[ \frac{1 + x_2}{x_1} = \frac{p_1}{p_2} \]

Using the budget constraint,

\[ x_1^* = \frac{m + p_2}{2p_1} \]
\[ x_2^* = \frac{m - p_2}{2p_2} \]

(b) Since \( x_1^* \) and \( x_2^* \) are increasing in \( m \), both are normal goods.

(c) The indirect utility is

\[ v = x_1^*(1 + x_2^*) = \frac{(m + p_2)^2}{4p_1p_2} \]

(d) Inverting, we can obtain the expenditure function,

\[ e = 2(p_1p_2u)^{1/2} - p_2 \]

(e) Inserting the values in the expenditure function \( e = 2(4 \times 2 \times 72)^{1/2} - 2 = 46 \).
1. Expenditure Minimisation with CES demand

A consumer has the utility $u(x_1, x_2) = x_1^{1/2} + x_2^{1/2}$.

a) Find the Hicksian demand for $x_1$ and $x_2$.

b) Find the expenditure function.

c) Find the Marshallian demand for $x_1$ and $x_2$.

d) Find the indirect utility function. Depending on how you approach this question, you may find it useful to note that:

$$
\left( \frac{p_2}{p_1} \right)^{1/2} + \left( \frac{p_1}{p_2} \right)^{1/2} = \frac{p_1 + p_2}{(p_1 p_2)^{1/2}}
$$

e) Show that the Slutsky equation holds.

2. Consumption with Endowments: Labour Supply

A particular household consists of two agents who are both potential workers and who pool their budgets. The household’s preferences are represented by a single utility function

$$
u(x_0, x_1, x_2) = \log(x_0 - \alpha) + \log x_1 + \log x_2$$

where $x_1$ is the amount of leisure enjoyed by agent 1, $x_2$ is the amount of leisure enjoyed by agent 2, and $x_0$ is the amount of the single composite consumption good enjoyed by the household. The two agents each have $T$ hours which can either be enjoyed as leisure, or spent in paid work. The hourly wage rates for the two individuals are $w_1$ and $w_2$, respectively, and they jointly have non-wage income of $m$. The price of the composite consumption good is 1.

a) Write down the household budget constraint.
b) Solve for the households optimal choice of $x_1$, $x_2$, and $x_0$. [Throughout this question assume there is an internal solution.]

c) How does the labour supply of the agents, $L_i = T - x_i$, change with the wage rates $(w_1, w_2)$ and the outside income $m$. Explain your results in terms of income and substitution effects.

3. Intertemporal Consumption

An agent allocates consumption across two periods. Let the consumption in period $t$ be $x_t \in \mathbb{R}_+$. The agent’s utility is

$$u(x_1, x_2) = v(x_1) + (1 + \beta)^{-1}v(x_2)$$

where $\beta$ is the agent’s discount rate, and $v(x)$ is an increasing and strictly concave function (this means $v'(x)$ is strictly decreasing).

In period 1 the agent has income $m$. The agent can save at interest rate $r > 0$, so that $\$1$ in period 1 is worth $\$(1+r)$ in period 2. As a result, the agent’s budget constraint is

$$m = x_1 + (1 + r)^{-1}x_2$$

a) Find the agent’s MRS of $x_1$ with respect to $x_2$. [Throughout this question assume there is an internal solution].

b) Find the tangency condition (i.e. first order condition) for the agent’s choice problem.

c) Suppose $r = \beta$. Does the agent consume more in the first or second period? Provide an intuition.

d) Suppose $r > \beta$. Does the agent consume more in the first or second period? Provide an intuition.

e) Suppose $v(x) = \ln(x)$. Solve for the agent’s optimal consumption.

f) Suppose the consumer has no income in period 1, but has income $\tilde{m} = (1 + r)^{-1}m$ in period 2. Also suppose they can borrow at interest rate $r$. How does this affect the agent’s budget constraint? How does this affect the agent’s optimal consumption choice? Provide an intuition.
4. Properties of The Expenditure Function

Prove that $e(p_1, p_2, \pi)$ is (weakly) increasing in $p_1$, $p_2$ and $\pi$. 
1. Expenditure Minimisation with CES demand

A consumer has the utility \( u(x_1, x_2) = x_1^{1/2} + x_2^{1/2} \).

a) Find the Hicksian demand for \( x_1 \) and \( x_2 \).

b) Find the expenditure function.

c) Find the Marshallian demand for \( x_1 \) and \( x_2 \).

d) Find the indirect utility function. Depending on how you approach this question, you may find it useful to note that:

\[
\left( \frac{p_2}{p_1} \right)^{1/2} + \left( \frac{p_1}{p_2} \right)^{1/2} = \frac{p_1 + p_2}{(p_1p_2)^{1/2}}
\]

e) Show that the Slutsky equation holds.

Solution

a) The Lagrangian is

\[
\mathcal{L} = p_1 x_1 + p_2 x_2 + \lambda [u - x_1^{1/2} - x_2^{1/2}]
\]

The FOCs are

\[
p_1 = \frac{1}{2} \lambda x_1^{-1/2}
\]
\[
p_2 = \frac{1}{2} \lambda x_2^{-1/2}
\]

Dividing we get,

\[
\frac{p_1}{p_2} = \frac{x_2^{1/2}}{x_1^{1/2}}
\]
This states that the MRS equals the price ratio. Substituting into the constraint, the Hicksian demands are given by

\[ h_1 = u^2 \left( \frac{p_2}{p_1 + p_2} \right)^2 \]
\[ h_2 = u^2 \left( \frac{p_1}{p_1 + p_2} \right)^2 \]

b) The expenditure function is

\[ e = p_1 x_1^* + p_2 x_2^* = u^2 \frac{p_1 p_2}{p_1 + p_2} \]

c) The Lagrangian is

\[ \mathcal{L} = x_1^{1/2} + x_2^{1/2} + \lambda[m - p_1 x_1 - p_2 x_2] \]

The FOCs are

\[ \frac{1}{2} x_1^{-1/2} = \lambda p_1 \]
\[ \frac{1}{2} x_2^{-1/2} = \lambda p_2 \]

Dividing we get,

\[ \frac{x_2^{1/2}}{x_1^{1/2}} = \frac{p_1}{p_2} \]

This states that the MRS equals the price ratio. Substituting into the constraint,

\[ x_1^* = \frac{m}{p_1} \frac{p_2}{p_1 + p_2} \]
\[ x_2^* = \frac{m}{p_2} \frac{p_1}{p_1 + p_2} \]

e) The indirect utility function is

\[ v = (x_1^*)^{1/2} + (x_2^*)^{1/2} = \left[ m \frac{p_1 + p_2}{p_1 p_2} \right]^{1/2} \]

This can be derived by substituting the Marshallian demands into the utility function, or by inverting the expenditure function.
d) The Slutsky equation states that
\[
\frac{dx_1^i}{dp_1} = \frac{dh_1}{dp_1} - x_i^i \frac{dx_1^i}{dm}
\]
That is,
\[
-m \frac{p_2(2p_1 + p_2)}{p_1^2(p_1 + p_2)^2} = -2u^2 \frac{p_2^2}{(p_1 + p_2)^3} - m \frac{p_2^2}{p_1^2(p_1 + p_2)^2}
\]
Using the indirect utility function,
\[
u^2 = \frac{m(p_1 + p_2)}{p_1 p_2}
\]
Substituting,
\[
-m \frac{p_2(2p_1 + p_2)}{p_1^2(p_1 + p_2)^2} = -2m \frac{p_1 + p_2}{p_1 p_2} \frac{p_2^2}{(p_1 + p_2)^3} - m \frac{p_2^2}{p_1^2(p_1 + p_2)^2}
\]
Simplifying,
\[
-m \frac{p_2(2p_1 + p_2)}{p_1^2(p_1 + p_2)^2} = -m \frac{2p_2 p_1}{p_1^2(p_1 + p_2)^2} - m \frac{p_2^2}{p_1^2(p_1 + p_2)^2}
\]
Which is clearly true.

2. Consumption with Endowments: Labour Supply

A particular household consists of two agents who are both potential workers and who pool their budgets. The households preferences are represented by a single utility function
\[
u(x_0, x_1, x_2) = \log(x_0 - \alpha) + \log x_1 + \log x_2
\]
where \(x_1\) is the amount of leisure enjoyed by agent 1, \(x_2\) is the amount of leisure enjoyed by agent 2, and \(x_0\) is the amount of the single composite consumption good enjoyed by the household. The two agents each have \(T\) hours which can either be enjoyed as leisure, or spent in paid work. The hourly wage rates for the two individuals are \(w_1\) and \(w_2\), respectively, and they jointly have non-wage income of \(m\). The price of the composite consumption good is 1.

a) Write down the household budget constraint

b) Solve for the households optimal choice of \(x_1\), \(x_2\), and \(x_0\). [Throughout this question assume there is an internal solution.]

c) How does the labour supply of the agents, \(L_i = T - x_i\), change with the wage rates \((w_1, w_2)\)
and the outside income $m$. Explain your results in terms of income and substitution effects.

**Solution**

a) The budget constraint then is

$$m + w_1(T - x_1) + w_2(T - x_2) = x_0$$

Equivalently,

$$m + w_1T + w_2T = x_0 + w_1x_1 + w_2x_2$$

b) The Lagrangian is

$$\mathcal{L} = \log(x_0 - \alpha) + \log x_1 + \log x_2 + \lambda[m + w_1T - w_1x_1 + w_2T - w_2x_2 - x_0]$$

The FOCs are:

$$\frac{1}{x_0 - \alpha} = \lambda$$

$$\frac{1}{x_1} = \lambda w_1$$

$$\frac{1}{x_2} = \lambda w_2$$

These imply that $x_0 - \alpha = w_1x_1 = w_2x_2$. Using these three equations and the budget constraint, we find

$$x_0^* = \alpha + \frac{1}{3}[m + w_1T + w_2T - \alpha]$$

$$x_1^* = \frac{1}{3w_1}[m + w_1T + w_2T - \alpha]$$

$$x_2^* = \frac{1}{3w_2}[m + w_1T + w_2T - \alpha]$$

We can think of $\alpha$ as the subsistence level of $x_0$. The agents split their income above the subsistence level,

$$[m + w_1T + w_2T - \alpha],$$

between consumption and leisure.
c) Labour supply is given by:

\[ L_1^* = T - \frac{1}{3w_1}[m + w_1T + w_2T - \alpha] \]
\[ L_2^* = T - \frac{1}{3w_2}[m + w_1T + w_2T - \alpha] \]

The income effect:

\[ \frac{\partial L_1^*}{\partial m} = -\frac{1}{3w_1} < 0 \]

Intuitively, as the household gets richer they can consume more leisure. The price effect:

\[ \frac{\partial L_1^*}{\partial w_1} = \frac{1}{3w_1^2}[m + w_1T + w_2T - \alpha] - \frac{1}{3w_1}T \]
\[ \frac{1}{3w_1^2}[m + w_2T - \alpha] \]

If \( m + w_2T > \alpha \) then 1's wage is not needed to finance subsistence consumption. In this case the substitution effect is dominant: a rise in \( w_1 \) causes agent 1 to work more. If \( m + w_2T < \alpha \) then the income effect is dominant: a rise in \( w_1 \) causes agent 1 to work less.

\[ \frac{\partial L_1^*}{\partial w_2} = -\frac{1}{3w_1}T < 0 \]

As \( w_2 \) rises so the income effect and substitution effect cause agent 1 to work less.

3. Intertemporal Consumption

An agent allocates consumption across two periods. Let the consumption in period \( t \) be \( x_t \in \mathbb{R}_+ \). The agent’s utility is

\[ u(x_1, x_2) = v(x_1) + (1 + \beta)^{-1}v(x_2) \]

where \( \beta \) is the agent’s discount rate, and \( v(x) \) is an increasing and strictly concave function (this means \( v'(x) \) is strictly decreasing).

In period 1 the agent has income \( m \). The agent can save at interest rate \( r > 0 \), so that $1 in period 1 is worth $(1+r) in period 2. As a result, the agent’s budget constraint is

\[ m = x_1 + (1 + r)^{-1}x_2 \]
a) Find the agent’s MRS of $x_1$ with respect to $x_2$. [Throughout this question assume there is an internal solution].

b) Find the tangency condition (i.e. first order condition) for the agent’s choice problem.

c) Suppose $r = \beta$. Does the agent consume more in the first or second period? Provide an intuition.

d) Suppose $r > \beta$. Does the agent consume more in the first or second period? Provide an intuition.

e) Suppose $v(x) = \ln(x)$. Solve for the agent’s optimal consumption.

f) Suppose the consumer has no income in period 1, but has income $\tilde{m} = (1+r)^{-1}m$ in period 2. Also suppose they can borrow at interest rate $r$. How does this affect the agent’s budget constraint? How does this affect the agent’s optimal consumption choice? Provide an intuition.

**Solution**

a) The agent’s MRS is

$$\text{MRS} = \frac{MU_1}{MU_2} = \frac{v'(x_1)}{(1+\beta)^{-1}v'(x_2)}$$

b) The tangency condition sets the MRS equal to the price ratio. That is,

$$\frac{v'(x_1^*)}{(1+\beta)^{-1}v'(x_2^*)} = \frac{1}{(1+r)^{-1}}$$

Rearranging

$$v'(x_1^*) = (1+r)(1+\beta)^{-1}v'(x_2^*)$$

c) If $r = \beta$ then $(1+r) = (1+\beta)$ and $v'(x_1^*) = v'(x_2^*)$. Since $v'(\cdot)$ is strictly decreasing, we have $x_2^* = x_1^*$. Intuitively the agent discounts at the rate of interest, so they wish to perfectly smooth their consumption.

d) If $r > \beta$ then $(1+r) > (1+\beta)$ and $v'(x_1^*) < v'(x_2^*)$. Since $v'(\cdot)$ is decreasing, we have $x_2^* > x_1^*$. Intuitively, the agent is more patient than society since her internal interest rate is lower. As a result she consumes more in period 2.
e) Here $v'(x) = 1/x$. Hence 

$$x_2^* = (1 + r)(1 + \beta)^{-1}x_1^*$$

f) The consumer’s budget constraint is unchanged. As a result, her optimal consumption is unchanged. Intuitively, the agent cares about her lifetime income, not when that income arrives, since she can borrow/save to smooth her consumption.

4. Properties of The Expenditure Function

Prove that $e(p_1, p_2, \bar{u})$ is (weakly) increasing in $p_1$, $p_2$ and $\bar{u}$.

Solution

*Changing the target utility.* Fix prices $(p_1, p_2)$ and suppose $\bar{u} > \bar{u}'$. Let $(h_1, h_2)$ be optimal demands when the target utility is $\bar{u}$, and denote the expenditure by $e(p_1, p_2, \bar{u})$. Now consider the lower target, $\bar{u}'$. Keeping demand the same, $u(h_1, h_2) \geq \bar{u}'$, so the agent can meet the target with expenditure $e(p_1, p_2, \bar{u})$. Hence the minimum expenditure needed to attain $\bar{u}'$ is at most $e(p_1, p_2, \bar{u})$. That is,

$$e(p_1, p_2, \bar{u}') \leq e(p_1, p_2, \bar{u})$$

as required.

*Changing a price.* Fix $(p_2, \bar{u})$ and suppose $p_1 > p_1'$. Let $(h_1, h_2)$ be optimal demands when the price for $x_1$ is $p_1$, and denote the expenditure by $e(p_1, p_2, \bar{u})$. Keeping demand the same, the agent can attain the same utility at a lower level of expenditure since $p_1$ has fallen. That is,

$$e(p_1', p_2, \bar{u}) \leq e(p_1, p_2, \bar{u})$$

as required.
Economics 11: Practice Problems 5 (Week 6)

November 4, 2008

1. Complements and Substitutes

Suppose \( u(x_1, x_2) = x_1x_2 \). Are \( x_1 \) and \( x_2 \) gross complements or substitutes? Are they net complements or substitutes?

2. Choice with Partial Subsidies

Suppose an agent chooses how much bread, \( x_1 \), and wine, \( x_2 \), to consume. The agent views the goods as perfect substitutes and has utility function:

\[
 u(x_1, x_2) = x_1 + x_2
\]

The price of bread is \( p_1 = 4 \), while the price of wine is \( p_2 = 2 \). The agent has income \( m = 4 \).

(a) Solve for the agent’s optimal consumption.

(b) In order to encourage the consumption of bread, the Government subsidises the first unit by $1. Hence the first unit costs \( p_1 = 3 \), while each unit thereafter costs \( p_1 = 4 \). (i) Draw the agent’s budget constraint. (ii) Solve for her optimal consumption. (iii) How much does the subsidy cost the Government?

(c) The Government increases it’s subsidy on the first unit to $3. Hence the first unit costs \( p_1 = 1 \), while each unit thereafter costs \( p_1 = 4 \). (i) Draw the agent’s budget constraint. (ii) Solve for her optimal consumption. (iii) How much does the subsidy cost the Government?

3. Consumer Surplus with Quasilinear Utility

Suppose an agent has utility \( u(x_1, x_2) = 2x_1^{1/2} + x_2 \). The agent has income \( m \) and faces prices \( p_1 \) and \( p_2 \). [Throughout this question assume there is an internal optimum].

(a) Calculate the agent’s Marshallian demand.
(b) Calculate the agent’s indirect utility, \( v(p_1, p_2, m) \).

(c) Fix a target utility \( \bar{u} \). Calculate the agent’s Hicksian demand.

(d) Calculate the agent’s expenditure function.

Suppose the agent starts with income \( m = 10 \) and faces prices \( p_1 = 1 \) and \( p_2 = 1 \).

(e) Calculate the utility of the agent, \( \bar{u} = v(p_1, p_2, m) \).

(f) Suppose \( p_1 = 1 \) rises to \( p_1' = 2 \). How much money must the Government give the agent to compensate for this price rise, so that her utility remains at \( \bar{u} \)?

4. Consumer Surplus with Symmetric Cobb Douglas Utility

Suppose an agent has utility \( u(x_1, x_2) = x_1 x_2 \), income \( m = 4 \) and faces prices \( p_1 = 1 \) and \( p_2 = 1 \). Suppose the price of good 1 rises to \( p_1' = 2 \). Calculate the increase in the agent’s income required to compensate her for this price rise.

5. Production with One Input

Digging clams by hand in Sunset Bay requires only labour input. The total number of clams obtained per hour \( (q) \) is given by:

\[
q = 100L^{1/2}
\]

Where \( L \) is he labour input per hour.

a) Graph the relationship between \( q \) and \( L \)

b) Find the marginal and average product of labour.

6. Production

Suppose output is given by \( q = \left[z_1^\rho + z_2^\rho\right]^{1/\rho} \) for \( 1 > \rho > 0 \).

(a) Does this production function exhibits decreasing, constant or increasing returns to scale?
(b) Derive the MRTS.

(c) Show that the isoquants are convex. That is, show that MRTS is decreasing in $z_1$ along an isoquant.

(d) Suppose that $\rho > 1$. Show the isoquants are concave. Why might this be problematic?
Economics 11: Solutions to Practice Problems 5 (Week 6)

November 4, 2008

1. Complements and Substitutes

Suppose \( u(x_1, x_2) = x_1 x_2 \). Are \( x_1 \) and \( x_2 \) gross complements or substitutes? Are they net complements or substitutes?

Solution

Marshallian demand is given by
\[
x_1^* = \frac{m}{2p_1}
\]
and
\[
x_2^* = \frac{m}{2p_2}.
\]
Since demand for good \( i \) is independent of \( p_j \) these are neither gross complements or gross substitutes.

Hicksian demand is given by
\[
h_1 = \left( \frac{p_2}{p_1} \right)^{1/2}
\]
and
\[
h_2 = \left( \frac{p_1}{p_2} \right)^{1/2}.
\]
Since \( h_i \) is increasing in \( p_j \), these goods are net substitutes.

2. Choice with Partial Subsidies

Suppose an agent chooses how much bread, \( x_1 \), and wine, \( x_2 \), to consume. The agent views the goods as perfect substitutes and has utility function:
\[
u(x_1, x_2) = x_1 + x_2
\]
The price of bread is \( p_1 = 4 \), while the price of wine is \( p_2 = 2 \). The agent has income \( m = 4 \).

(a) Solve for the agent’s optimal consumption.

(b) In order to encourage the consumption of bread, the Government subsidises the first unit by \$1. Hence the first unit costs \( p_1 = 3 \), while each unit thereafter costs \( p_1 = 4 \). (i) Draw the agent’s budget constraint. (ii) Solve for her optimal consumption. (iii) How much does the subsidy cost the Government?

(c) The Government increases it’s subsidy on the first unit to \$3. Hence the first unit costs \( p_1 = 1 \), while each unit thereafter costs \( p_1 = 4 \). (i) Draw the agent’s budget constraint. (ii) Solve for her optimal consumption. (iii) How much does the subsidy cost the Government?
Solution

(a) The agent choose to spend all her money on wine. Hence $x_1^* = 0$ and $x_2^* = 2$.

(b) The agent still chooses to spend all her money on wine. Hence $x_1^* = 0$ and $x_2^* = 2$. The subsidy does not cost the Government anything.

(c) The agent chooses to consume $x_1^* = 1$ and $x_2^* = 3/2$. The subsidy costs the Government $3$.

3. Consumer Surplus with Quasilinear Utility

Suppose an agent has utility $u(x_1, x_2) = 2x_1^{1/2} + x_2$. The agent has income $m$ and faces prices $p_1$ and $p_2$. [Throughout this question assume there is an internal optimum].

(a) Calculate the agent’s Marshallian demand.

(b) Calculate the agent’s indirect utility, $v(p_1, p_2, m)$.

(c) Fix a target utility $\bar{u}$. Calculate the agent’s Hicksian demand.

(d) Calculate the agent’s expenditure function.

Suppose the agent starts with income $m = 10$ and faces prices $p_1 = 1$ and $p_2 = 1$.

(e) Calculate the utility of the agent, $\bar{u} = v(p_1, p_2, m)$.

(f) Suppose $p_1 = 1$ rises to $p'_1 = 2$. How much money must the Government give the agent to compensate for this price rise, so that her utility remains at $\bar{u}$?

Solution

(a) The tangency condition is

$$x_1^{-1/2} = \frac{p_1}{p_2}$$

Rearranging, demand is

$$x_1^*(p_1, p_2, m) = \frac{p_2^2}{p_1^2}$$
Using the budget constraint,
\[ x_2^*(p_1, p_2, m) = \frac{m}{p_2} - \frac{p_2}{p_1}. \]

(b) The agent’s indirect utility is thus:
\[ v(p_1, p_2, m) = 2x_1^{1/2} + x_2 = \frac{m}{p_2} + \frac{p_2}{p_1} \]

(c) The tangency condition is the same as in part (a). Rearranging, Hicksian demand is
\[ h_1(p_1, p_2, \bar{\mu}) = \frac{p_2}{p_1^2} \]

Using the constraint, we have
\[ h_2^*(p_1, p_2, \bar{\mu}) = \bar{\mu} - 2\frac{p_2}{p_1}. \]

(d) The expenditure function is
\[ e(p_1, p_2, \bar{\mu}) = p_1 h_1 + p_2 h_2 = p_2 \left( \bar{\mu} - \frac{p_2}{p_1} \right) \]

(e) Under these parameters, \( v(p_1, p_2, m) = 11 \).

(f) There are three ways to calculate this. First, using the expenditure function,
\[ CV = e(p_1', p_2, \bar{\mu}) - e(p_1, p_2, \bar{\mu}) = p_2^2 - p_2^2 = 1 - \frac{1}{2} = \frac{1}{2} \]

Second, using the Hicksian demand,
\[ CV = \int_1^2 h_1(p_1, p_2, \bar{\mu})dp_1 = p_2^2 \int_1^2 p_1^{-2}dp_1 = p_2^2 [-p_1^{-1}]_1^2 = 1 - \frac{1}{2} = \frac{1}{2} \]

Third, since utility is quasilinear, we can use the Marshallian demand (which is identical to Hicksian demand),
\[ CV = \int_1^2 x_1^*(p_1, p_2, m)dp_1 = p_2^2 \int_1^2 p_1^{-2}dp_1 = p_2^2 [-p_1^{-1}]_1^2 = 1 - \frac{1}{2} = \frac{1}{2} \]

In this case using the Marshallian demand may be easiest since it avoids us having to solve the EMP.
4. Consumer Surplus with Symmetric Cobb Douglas Utility

Suppose an agent has utility $u(x_1, x_2) = x_1 x_2$, income $m = 4$ and faces prices $p_1 = 1$ and $p_2 = 1$. Suppose the price of good 1 rises to $p'_1 = 2$. Calculate the increase in the agent’s income required to compensate her for this price rise.

Solution

We derived the Marshallian demand and indirect utility function in class. The indirect utility function is

$$v(p_1, p_2, m) = \frac{m^2}{4p_1 p_2}$$

Plugging in the parameters, we see $v(p_1, p_2, m) = 4$. Inverting the indirect utility function we obtain the expenditure function,

$$e(p_1, p_2, \pi) = 2(\pi p_1 p_2)^{1/2}$$

Letting, $\pi = 4$, the compensating variation is

$$CV = e(p'_1, p_2, \pi) - e(p_1, p_2, \pi) = 2(\pi p'_1 p_2)^{1/2} - 2(\pi p_1 p_2)^{1/2} = 4(\sqrt{2} - 1)$$

5. Production with One Input

Digging clams by hand in Sunset Bay requires only labour input. The total number of clams obtained per hour ($q$) is given by:

$$q = 100L^{1/2}$$

Where $L$ is the labour input per hour.

a) Graph the relationship between $q$ and $L$

b) Find the marginal and average product of labour.

Solution

(a) This is a concave function starting at the origin.
(b) The marginal product of labour is

\[ MP_L = \frac{dq}{dL} = 50L^{-1/2} \]

The average product is

\[ AP_L = q/L = 100L^{-1/2} \]

6. Production

Suppose output is given by \( q = [z_1^\rho + z_2^\rho]^{1/\rho} \) for \( 1 > \rho > 0 \).

(a) Does this production function exhibits decreasing, constant or increasing returns to scale?

(b) Derive the MRTS.

(c) Show that the isoquants are convex. That is, show that MRTS is decreasing in \( z_1 \) along an isoquant.

(d) Suppose that \( \rho > 1 \). Show the isoquants are concave. Why might this be problematic?

Solution

(a) For \( \alpha > 0 \),

\[ f(\alpha z_1, \alpha z_2) = [\alpha^\rho z_1^\rho + \alpha^\rho z_2^\rho]^{1/\rho} = \alpha [z_1^\rho + z_2^\rho]^{1/\rho} = \alpha f(z_1, z_2) \]

(b) The MRTS is

\[ \frac{MP_1}{MP_2} = \frac{z_2^{\rho-1} [z_1^\rho + z_2^\rho]^{(1/\rho)-1}}{z_1^{\rho-1} [z_1^\rho + z_2^\rho]^{(1/\rho)-1}} = \left( \frac{z_2}{z_1} \right)^{1-\rho} \]

(c) The equation of an isoquant is

\[ [z_1^\rho + z_2^\rho]^{1/\rho} = k \]

Rearranging,

\[ z_2 = [k^\rho - z_1^\rho]^{1/\rho} \]
Substituting into the MRTS,

\[ MRTS = \left( \frac{z_2}{z_1} \right)^{1-\rho} = \left( \frac{[k^\rho - z_1^{\rho 1/\rho}]}{z_1} \right)^{1-\rho} \]

which is decreasing in \( z_1 \).

(d) As in part (c), we have

\[ MRTS = \left( \frac{[k^\rho - z_1^{\rho 1/\rho}]}{z_1} \right)^{1-\rho} \]

The difference is that \( 1 - \rho < 0 \). Hence this is increasing in \( z_1 \) and the isoquants are concave. This is problematic since there cost minimisation problem will not have an internal optimum: the firm will wish to use one input rather than a mix.
1. Cost Functions

A firm has cost function $c(q) = 100 - 10q + 5q^2$.

a) Find the fixed cost.\(^1\)

b) Find the variable cost.\(^2\)

c) Find the average cost.

d) Find the marginal cost.

e) Draw the relationship between MC and AC. Prove that they always intersect at the minimum AC.

2. Cost Functions

A firm has production function

$$f(z_1, z_2) = z_1^{1/2} (z_2 - 1)^{1/2}$$

The prices of the inputs are $r_1$ and $r_2$.

(a) Find $MP_1$, $MP_2$, and $MRTS$.

(b) If $z_2$ is fixed at 5, what is the short-run cost function? Find the short-run marginal cost and average cost.

(c) What are the long-run input demand functions? What is the long-run cost function? Find the long-run marginal cost and average cost.

(d) Does the production function exhibit increasing, constant or decreasing returns to scale.

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\(^1\) Definition: The fixed cost is the cost that is independent of output.

\(^2\) Definition: The variable cost is the cost that varies with the level of output.
3. Cost Minimisation: Cobb Douglas

Suppose that a firm production function is given by the Cobb-Douglas function: \( f(z_1, z_2) = z_1^\alpha z_2^\beta \). The cost of the inputs is \( z_1 \) and \( z_2 \).

a) Find marginal and average productivity of the two factors.

b) Does this production function have increasing, constant or decreasing returns to scale?

c) Show that cost minimisation requires \( \beta r_1 z_1 = \alpha r_2 z_2 \).

d) Suppose \( \alpha = \beta = 1/4 \). Find the cost function.

4. Returns to Scale

A firm has production function \( f(z) = z^\alpha \). The price of the input is \( r \).

(a) For which values of \( \alpha \) does the technology have increasing, decreasing and constant returns?

(b) Show that the cost function is convex when technology has constant or decreasing returns. Provide an intuition.

5. Average Cost and Marginal Cost

(a) Show that \( AC(q) \) is increasing when \( MC(q) \geq AC(q) \), and \( AC(q) \) is decreasing when \( MC(q) \leq AC(q) \). [Hint: Differentiate \( AC(q) \).]

(b) Provide an intuition for the result in part (a).

(c) Suppose \( AC(q) \) is U-shaped. Argue that \( AC(q) = MC(q) \) when \( AC(q) \) is at its lowest point.

6. Profit Maximisation

A firm has production function \( f(z) = 2z^{1/2} \). The output price is \( p \); the input price is \( r \).
What is the firm’s optimal output? What is the optimal input? What is the firm’s profit?

7. Revenue Maximisation

A firm has cost function $c(q) = 3750 + \frac{1}{2}q^2$. The price of output is $p = 100$.

(a) What is the profit maximising quantity?

(b) What are the maximal profits?

(c) Which quantity maximises revenue,\(^3\) subject to the constraint that profits are positive?

Note: For this question, you may find the quadratic formula useful. If $ax^2 + bx + c = 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

\(^3\)Definition: revenue equals the money the firm gets from selling it’s goods, ignoring the cost.
1. Cost Functions

A firm has cost function \( c(q) = 100 - 10q + 5q^2 \).

a) Find the fixed cost. \(^1\)

b) Find the variable cost. \(^2\)

c) Find the average cost.

d) Find the marginal cost.

e) Draw the relationship between MC and AC. Prove that they always intersect at the minimum AC.

Solution

(a) The fixed cost is 100.

(b) The variable cost is \(-10q + 5q^2\).

(c) The average cost is

\[
AC(q) = \frac{100}{q} - 10 + 5q
\]

(d) The marginal cost is

\[
MC(q) = -10 + 10q
\]

(e) \(AC\) is a convex function. Differentiating, it is minimised when

\[
\frac{d}{dq} AC(q) = -100q^{-2} + 5q = 0
\]

\(^1\) Definition: The fixed cost is the cost that is independent of output.

\(^2\) Definition: The variable cost is the cost that varies with the level of output.
Rearranging, we find that \( q = \sqrt{20} \).

We have \( AC = MC \) when
\[
\frac{100}{q} - 10 + 5q = -10 + 10q
\]
Rearranging, we find that \( q = \sqrt{20} \).

2. Cost Functions

A firm has production function
\[
f(z_1, z_2) = z_1^{1/2}(z_2 - 1)^{1/2}
\]
The prices of the inputs are \( r_1 \) and \( r_2 \).

(a) Find \( MP_1 \), \( MP_2 \), and \( MRTS \).

(b) If \( z_2 \) is fixed at 5, what is the short-run cost function? Find the short-run marginal cost and average cost.

(c) What are the long-run input demand functions? What is the long-run cost function? Find the long-run marginal cost and average cost.

(d) Does the production function exhibit increasing, constant or decreasing returns to scale.

Solution

(a) The marginal products are
\[
MP_1 = \frac{1}{2}z_1^{-1/2}(z_2 - 1)^{1/2}
\]
\[
MP_2 = \frac{1}{2}z_1^{1/2}(z_2 - 1)^{-1/2}
\]

Hence we have
\[
MRTS = \frac{MP_1}{MP_2} = \frac{z_2 - 1}{z_1}
\]
(b) The cost minimisation problem is to choose $z_1$ to minimise $r_1 z_1 + 5r_2$ subject to $2z_1^{1/2} \geq q$. At the optimum, the constraint binds and we have

$$z_1^* = \frac{q^2}{4}$$

and the short-run total cost is

$$SRTC(q) = r_1 \frac{q^2}{4} + 5r_2$$

The average cost is

$$SRAC = r_1 \frac{q}{4} + \frac{5r_2}{q}$$

The marginal cost is

$$SRMC = r_1 \frac{q}{2}$$

(c) In the long run, both factors are flexible. The tangency condition is

$$\frac{z_2 - 1}{z_1} = \frac{r_1}{r_2}$$

Rearranging, we have $(z_2 - 1)r_2 = z_1 r_1$. The constraint says that

$$q = z_1^{1/2}(z_2 - 1)^{1/2}$$

Substituting, we see that

$$z_1^* = \left( \frac{r_2}{r_1} \right)^{1/2} q \quad \text{and} \quad z_2^* = 1 + \left( \frac{r_1}{r_2} \right)^{1/2} q$$

The cost is therefore

$$c(q) = 2q \sqrt{r_1 r_2} + r_2$$

Intuitively, after the firm pays a fixed cost of 1 unit of $z_2$, they have Cobb-Douglas technology. The average cost is

$$AC(q) = 2\sqrt{r_1 r_2} + \frac{r_2}{q}$$

The marginal cost is

$$MC(q) = 2\sqrt{r_1 r_2}$$

(d) It exhibits increasing returns. You can verify this directly from the production function:

$$f(\alpha z_1, \alpha z_2) = (\alpha z_1)^{1/2}(\alpha z_2 - 1)^{1/2} \geq (\alpha z_1)^{1/2}(\alpha z_2 - \alpha)^{1/2} = \alpha z_1^{1/2}(z_2 - 1)^{1/2} = \alpha f(z_1, z_2)$$
You can also verify this from the cost function, which is concave in $q$.

3. Cost Minimisation: Cobb Douglas

Suppose that a firm production function is given by the Cobb-Douglas function: $f(z_1, z_2) = z_1^\alpha z_2^\beta$. The cost of the inputs is $z_1$ and $z_2$.

a) Find marginal and average productivity of the two factors.

b) Does this production function have increasing, constant or decreasing returns to scale?

c) Show that cost minimisation requires $\beta r_1 z_1 = \alpha r_2 z_2$.

d) Suppose $\alpha = \beta = 1/4$. Find the cost function.

Solution

(a) The average productivity of 1 is

$$AP_1 = z_1^{\alpha-1} z_2^\beta$$

The marginal productivity of 1 is

$$MP_1 = \alpha z_1^{\alpha-1} z_2^\beta$$

Factor 2 is similar.

(b) It has constant returns if $\alpha + \beta = 1$. It has increasing returns if $\alpha + \beta > 1$. It has decreasing returns if $\alpha + \beta < 1$. To see this, just use the definition of returns to scale.

(c) We have

$$MRTS = \frac{MP_1}{MP_2} = \frac{\alpha z_1^{\alpha-1} z_2^\beta}{\beta z_1^{\alpha} z_2^{\beta-1}} = \frac{\alpha z_2}{\beta z_1}$$

The tangency condition is therefore

$$\frac{\alpha z_2}{\beta z_1} = \frac{r_1}{r_2}$$

Rearranging, yields the result.
(d) The tangency condition becomes $r_1 z_1 = r_2 z_2$. The constraint is

$$q = z_1^{1/4} z_2^{1/4}$$

Solving yields,

$$z_1^* = \left( \frac{r_2}{r_1} \right)^{1/2} q^2 \quad \text{and} \quad z_2^* = \left( \frac{r_1}{r_2} \right)^{1/2} q^2$$

The cost is

$$c(q) = r_1 z_1^* + r_2 z_2^* = 2 \left( r_1 r_2 \right)^{1/2} q^2$$

4. Returns to Scale

A firm has production function $f(z) = z^\alpha$. The price of the input is $r$.

(a) For which values of $\alpha$ does the technology have increasing, decreasing and constant returns?

(b) Show that the cost function is convex when technology has constant or decreasing returns. Provide an intuition.

Solution

(a) For $t > 1$, we have

$$f(tz_1) = (tz)^\alpha = t^\alpha f(z)$$

Hence this has IRS if $\alpha > 1$, CRS if $\alpha = 1$ and DRS if $\alpha < 1$.

(b) In order to produce $q$ the firm needs inputs $z = q^{1/\alpha}$. Hence the cost function is $c(q) = rq^{1/\alpha}$. If $\alpha \leq 1$, then $1/\alpha \geq 1$ and the cost function is convex.

Intuitively, when the production function exhibits DRS each extra unit of input yields less output. Hence the cost of producing an extra unit of output increases.

5. Average Cost and Marginal Cost

(a) Show that $AC(q)$ is increasing when $MC(q) \geq AC(q)$, and $AC(q)$ is decreasing when $MC(q) \leq AC(q)$. [Hint: Differentiate $AC(q)$.]
(b) Provide an intuition for the result in part (a).

(c) Suppose $AC(q)$ is U-shaped. Argue that $AC(q) = MC(q)$ when $AC(q)$ is at its lowest point.

Solution

(a) Differentiating,
\[
\frac{d}{dq} AC(q) = \frac{d}{dq} c(q) = \frac{c'(q)q - c(q)}{q^2}
\]
Hence $AC(q)$ is increasing if and only if $c'(q)q \geq c(q)$. Rearranging, this condition is just $MC(q) \geq AC(q)$.

(b) Suppose the firm currently produces $n$ units of output, and that the cost of the $(n + 1)^{st}$ unit is higher than the average cost of the first $n$. Then the average cost of producing $n + 1$ units is higher than producing $n$ units since the costs is being dragged up by the final unit.

(c) If $AC(q)$ is U-shaped it is downward sloping at first, implying that $MC(q) \leq AC(q)$, and is then upward sloping, implying that $MC(q) \geq AC(q)$. At the minimum we must therefore have $MC(q) = AC(q)$.

6. Profit Maximisation

A firm has production function $f(z) = 2z^{1/2}$. The output price is $p$; the input price is $r$.

What is the firm’s optimal output? What is the optimal input? What is the firm’s profit?

Solution

The firm maximises
\[
2pz^{1/2} - rz
\]
The FOC is
\[
pz^{-1/2} = r
\]
Rearranging,

\[ z^* = \frac{p^2}{r^2} \]

Output is

\[ y^* = 2(z^*)^{1/2} = \frac{2p}{r} \]

Profit is

\[ \pi^* = p y^* - rz^* = 2\frac{p^2}{r} - \frac{p^2}{r} = \frac{p^2}{r} \]

### 7. Revenue Maximisation

A firm has cost function \( c(q) = 3750 + \frac{1}{2}q^2 \). The price of output is \( p = 100 \).

(a) What is the profit maximising quantity?

(b) What are the maximal profits?

(c) Which quantity maximises revenue,\(^3\) subject to the constraint that profits are positive?

Note: For this question, you may find the quadratic formula useful. If \( ax^2 + bx + c = 0 \) then

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

### Solution

(a) The profit of the firm is

\[ \pi = 100q - 3750 - \frac{1}{2}q^2 \]

The FOC yields

\[ q^* = 100 \]

(b) The maximal profit level is

\[ \pi^* = (100)^2 - 3750 - \frac{1}{2}(100)^2 = 1250 \]

\(^3\)Definition: revenue equals the money the firm gets from selling its goods, ignoring the cost.
(c) Profit is inverse-U-shaped, so the revenue maximising firm will make zero profits. This implies

\[ 100q - 3750 - \frac{1}{2}q^2 = 0 \]

Rearranging, this yields the quadratic

\[ q^2 - 200q - 7500 = 0 \]

Using the quadratic formula, this has the solution

\[ q = \frac{200 \pm (200^2 - 4 \times 7500)^{1/2}}{2} = \frac{200 \pm 100}{2} = 100 \pm 50 \]

Profit is therefore positive if \( 150 \geq q \geq 50 \), and revenue is maximised at \( q = 150 \).
Economics 11: Practice Problems 7 (Week 8)

November 19, 2008

1. Properties of the Profit Function

A firm has cost function \( c(q) = q^2 \).

(a) Calculate the optimal supply function, \( q^*(p) \).

(b) Calculate the optimal profit function, \( \pi^*(p) \).

(c) Show that \( \frac{d}{dp} \pi^*(p) = q^*(p) \).

(d) Show that \( \pi^*(p) \) is convex.

(e) Fix a level of output, \( q \), and define the profit the firm makes when the price is \( p \) by

\[
\pi(p; q) = pq - c(q)
\]

On a single picture, draw \( \pi(p; q) \) for each \( q \in \{0, 1, 2, 3, 4\} \). Also draw \( \pi^*(p) \). Discuss your findings and the relationship to (c) and (d). [You may want to use a computer program such as Excel or Mathematica to draw the picture.]

2. Profit Maximisation

(a) A firm has cost function \( c(q) = 20q - 10q^2 + q^3 \). The output price is \( p = 8 \). Solve for the optimal output.

(b) A firm has cost function \( c(q) = 40q - 10q^2 + q^3 \). The output price is \( p = 8 \). Solve for the optimal output.
3. Market Demand

Suppose three consumers have the following demand curves:

\[ x_1^*(p) = 8 - p \]
\[ x_2^*(p) = 6 - 2p \]
\[ x_3^*(p) = 12 - 3p \]

where the subscript identifies the agent. Find the market demand (either mathematically or graphically).

4. Perfect Competition

A perfectly competitive industry has a large number of potential entrants. Each firm has an identical cost function given by \( c(q) = 100 + \frac{q^2}{4} \). Total market demand is \( X = 1500 - 50p \).

a) What is the long-run equilibrium price, the output of each firm (q), the number of firms in the industry, the total industry output (Q), and the profits of each firm?

b) What is the long-run industry supply curve?

Suppose demand falls to \( X = 1200 - 50p \).

c) In the very short-run, firms cannot change their output. What is the new price?

d) In the short-run, firms cannot exit or enter. Their fixed cost of 100 is also sunk, so the firms have cost curve \( c(q) = \frac{q^2}{4} \). Calculate the short-run supply curve. Using this, find the short-run equilibrium price, the total industry output, and the output of each firm. Show that, taking into account the sunk cost, firms would like to exit.

e) In the long-run, firms can enter and exit. What is the long-run equilibrium price, the output of each firm, the number of firms in the industry, the total industry output (Q), and the profits of each firm?
5. Equilibrium and Inputs Markets

A firm has a Cobb–Douglas production function \( f(z_1) = 10z_1^{1/2} \). The price of the input is \( r_1 \).

(a) Find the supply function of the firm and its input demand (as a function of \( p \) and \( r_1 \)).

Suppose there are 4,000 identical firms in the market, and the supply of \( z_1 \) is given by \( z_1^S = 40r_1^2 \).

b) Assume that the output price is \( p = 1 \). Calculate the equilibrium input price \( r_1 \), the amount of \( z_1 \) each firm demands, the amount of \( z_1 \) the entire market demands and the output of each firm.

c) Suppose the output price rises to \( p = 2 \). Repeat part (b). What is the effect of an increase in the output price?
Economics 11: Solutions to Practice Problems 7 (Week 8)

November 19, 2008

1. Properties of the Profit Function

A firm has cost function $c(q) = q^2$.

(a) Calculate the optimal supply function, $q^s(p)$.

(b) Calculate the optimal profit function, $\pi^*(p)$.

(c) Show that $\frac{d}{dp} \pi^*(p) = q^s(p)$.

(d) Show that $\pi^*(p)$ is convex.

(e) Fix a level of output, $q$, and define the profit the firm makes when the price is $p$ by

$$\pi(p; q) = pq - c(q)$$

On a single picture, draw $\pi(p; q)$ for each $q \in \{0, 1, 2, 3, 4\}$. Also draw $\pi^*(p)$. Discuss your findings and the relationship to (c) and (d). [You may want to use a computer program such as Excel or Mathematica to draw the picture.]

Solution

(a) The FOC of the profit maximisation problem is

$$p = 2q$$

Rearranging, $q^*(p) = p/2$.

(b) The profit function is given by

$$\pi^*(p) = pq^* - c(q^*) = \frac{p^2}{4}$$
(c) Differentiating,
\[ \frac{d}{dp} \pi^*(p) = \frac{p}{2} = y^*(p) \]

(d) Differentiating again,
\[ \frac{d^2}{dp^2} \pi^*(p) = \frac{1}{2} \geq 0 \]

Hence the profit function is convex.

(c) See figure 1. The profit function is the maximum of the \( \pi(p; q) \) functions. This implies that the profit function is convex. It also implies that the slope of the profit function at any point equals the slope of \( \pi(p; q^*(p)) \), which equals \( q^*(p) \).

2. Profit Maximisation

(a) A firm has cost function \( c(q) = 20q - 10q^2 + q^3 \). The output price is \( p = 8 \). Solve for the optimal output.

(b) A firm has cost function \( c(q) = 40q - 10q^2 + q^3 \). The output price is \( p = 8 \). Solve for the optimal output.
Solution

(a) The firm’s profit is

$$\pi(q) = 8q - 20q + 10q^2 - q^3 = -12q + 10q^2 - q^3$$

The FOC is

$$\frac{d}{dq} \pi(q) = -12 + 20q - 3q^2 = 0$$

The SOC is

$$\frac{d^2}{dq^2} \pi(q) = 20 - 6q$$

Solving the quadratic, the FOC yields

$$q^* = \frac{20 \pm \sqrt{400 - 4 \times 3 \times 12}}{6} = \frac{20 \pm 16}{6}$$

The solutions are $q^* = 2/3$ and $q^* = 6$. Of these, only $q^* = 6$ satisfies the SOC.

We also need to check that there is not a boundary solution. If $q = 0$ then $\pi = 0$. If $q = 6$ then $\pi = 72$, which is better than 0. Hence the optimal solution is $q^* = 6$.

(b) The firm’s profit is

$$\pi(q) = 8q - 40q + 10q^2 - q^3 = -32q + 10q^2 - q^3$$

The FOC is

$$\frac{d}{dq} \pi(q) = -32 + 20q - 3q^2 = 0$$

The SOC is

$$\frac{d^2}{dq^2} \pi(q) = 20 - 6q$$

Solving the quadratic, the FOC yields

$$q^* = \frac{20 \pm \sqrt{400 - 4 \times 3 \times 32}}{6} = \frac{20 \pm 4}{6}$$

The solutions are $q^* = 8/3$ and $q^* = 4$. Of these, only $q^* = 4$ satisfies the SOC.

We also need to check that there is not a boundary solution. If $q = 0$ then $\pi = 0$. If $q = 4$ then
\[ \pi = -32, \text{ which is worse than } 0. \text{ Hence the optimal solution is } q^* = 0. \]

3. Market Demand

Suppose three consumers have the following demand curves:

\[ x_1^*(p) = 8 - p \]
\[ x_2^*(p) = 6 - 2p \]
\[ x_3^*(p) = 12 - 3p \]

where the subscript identifies the agent. Find the market demand (either mathematically or graphically).

Solution

The demand curves are shown in figure 2. Agent 1 buys when \( p \in [0, 8] \); agent 2 buys when \( p \in [0, 3] \); agent 3 buys when \( p \in [0, 4] \). Summing up, market demand is given by

\[ X(p) = \begin{cases} 8 - p & \text{for } p \in [4, 8] \\ 20 - 4p & \text{for } p \in [3, 4] \\ 26 - 6p & \text{for } p \in [0, 3] \end{cases} \]

4. Perfect Competition

A perfectly competitive industry has a large number of potential entrants. Each firm has an identical cost function given by \( c(q) = 100 + \frac{q^2}{4} \). Total market demand is \( X = 1500 - 50p \).

a) What is the long-run equilibrium price, the output of each firm \( q \), the number of firms in the industry, the total industry output \( Q \), and the profits of each firm?

b) What is the long-run industry supply curve?

Suppose demand falls to \( X = 1200 - 50p \).
c) In the very short–run, firms cannot change their output. What is the new price?

d) In the short–run, firms cannot exit or enter. Their fixed cost of 100 is also sunk, so the firms have cost curve \( c(q) = \frac{q^2}{4} \). Calculate the short–run supply curve. Using this, find the short–run equilibrium price, the total industry output, and the output of each firm. Show that, taking into account the sunk cost, firms would like to exit.

e) In the long–run, firms can enter and exit. What is the long–run equilibrium price, the output of each firm, the number of firms in the industry, the total industry output (Q), and the profits of each firm?

**Solution**

(a) In the long run \( AC(q) \) is minimised. Using the cost function

\[
AC(q) = 100q^{-1} + \frac{1}{4}q
\]

This is minimised at \( q = 20 \), yielding \( p = AC(q) = 10 \). By construction, the profits are zero. Industry output is \( 1500 - 50 \times 10 = 1000 \). The number of firms is \( N = 1000/20 = 50 \).
(b) The supply function is a horizontal line at \( p = 10 \).

(c) In the very short–run, industry supply is stuck at \( Q = 1000 \). Using the demand curve, the new price is \( p = 4 \).

(d) In the short–run, firms can change their output. The firm’s profit is

\[
pq - \frac{1}{4}q^2
\]

The FOC is \( p = q/2 \). Rearranging, the short–run supply curve is \( q = 2p \). The industry supply is \( Q = 50q = 100p \). The new equilibrium price solves

\[
100p = 1200 - 50p
\]

Rearranging, \( p = 8 \). Each firm produces \( q = 16 \) and industry output is \( Q = 800 \). Taking into account the sunk cost, profit is \( 128 - 100 - 64 = -36 \), so that each firm would like to exit.

(e) In the long–run, firms exit until the profits are zero. Hence we find that \( p = 10 \) and \( Q = 700 \). The number of firms in the industry is \( N = 700/20 = 35 \).

5. Equilibrium and Inputs Markets

A firm has a Cobb–Douglas production function \( f(z_1) = 10z_1^{1/2} \). The price of the input is \( r_1 \).

(a) Find the supply function of the firm and its input demand (as a function of \( p \) and \( r_1 \)).

Suppose there are 4,000 identical firms in the market, and the supply of \( z_1 \) is given by \( z_1^S = 40r_1^2 \).

b) Assume that the output price is \( p = 1 \). Calculate the equilibrium input price \( r_1 \), the amount of \( z_1 \) each firm demands, the amount of \( z_1 \) the entire market demands and the output of each firm.

c) Suppose the output price rises to \( p = 2 \). Repeat part (b). What is the effect of an increase in the output price?
Solution

(a) We use the two-step method. Inverting the production function, we find that

\[ z_1 = \frac{q^2}{100} \]  

(1)

The cost function is therefore

\[ c(q) = r_1 \frac{q^2}{100} \]

The profit maximisation problem is

\[ \max_q pq - r_1 \frac{q^2}{100} \]

This is concave. The FOC is

\[ p = r_1 \frac{q}{50} \]

Rearranging,

\[ q^* = 50 \frac{p}{r_1} \]

Using (1),

\[ z_1^* = 25 \frac{p^2}{r_1^2} \]

(b) Suppose \( p = 1 \). The equilibrium price equates supply and demand:

\[ 4000 \times 25 \frac{1}{r_1^2} = 40r_1^2 \]

Rearranging, \( r_1^4 = 2500 \) or \( r_1 = (50)^{1/2} \). Each firm demands \( z_1 = 25/50 = 1/2 \). The market demand \( 4000 \times 1/2 = 2000 \). Firm output is \( 50/\sqrt{50} = \sqrt{50} \).

(c) Suppose \( p = 2 \). The equilibrium price equates supply and demand:

\[ 4000 \times 25 \frac{4}{r_1^2} = 40r_1^2 \]

Rearranging, \( r_1^4 = 10000 \) or \( r_1 = 10 \). Each firm demands \( z_1 = 100/100 = 1 \). The market demand \( 4000 \times 1 = 4000 \). Firm output is \( 100/10 = 10 \). Hence the higher output price raises the demand for \( z_1 \) and raises the equilibrium price.
Economics 11: Practice Problems 8 (Weeks 9–10)

December 4, 2008

1. First Welfare Theorem

There are two goods ($x$ and $y$) and two agents (A and B). The agents’ utility functions are

$$u_A = v_A(x) + y$$
$$u_B = v_B(x) + y$$

where $v_A = 2 \ln(x)$ and $v_B(x) = 4x^{1/2}$, The agents have incomes $m_A = 10$ and $m_B = 10$. The price of good $y$ is $p_y = 1$; the price of good $x$ is to be determined.

A single firm produces good $x$. It has cost function $c(q) = q^2/2$.

(a) Show that $p = 2$ is an equilibrium price. Find the equilibrium allocations $(x_A, x_B, q)$.

(b) Suppose a social planner chooses $(x_A, x_B, q)$ to maximise the total surplus,

$$v_A(x_A) + v_B(x_B) - c(q)$$

subject to $q = x_A + x_B$. Verify your allocations from part (a) satisfy the FOCs from this optimisation problem.

2. Shifts in Supply Functions

Suppose the supply function is $q = p$, and the demand function is $q = 10 - p$.

(a) Find the equilibrium price and quantity.

(b) Suppose the supply curve shifts up to $q = p - 2$, so each unit costs $2 more to produce. Derive the new price and quantity.

Suppose the supply function is $q = p$, and the demand function is $q = 6 - p/5$.

(c) Find the equilibrium price and quantity.
(d) Suppose the supply curve shifts up to $q = p - 2$. Derive the new price and quantity.

Suppose the supply function is $q = p$, and the demand function is $q = 25 - 4p$.

(e) Find the equilibrium price and quantity.

(f) Suppose the supply curve shifts up to $q = p - 2$. Derive the new price and quantity.

(g) Given these results, explain how the elasticity of the demand curve affects the impact of a shift in the supply function.

3. Taxation

Suppose utility is quasilinear in good $x$ and the demand function is $q = 10 - p$. The supply function is $q = p$.

(a) Solve for the equilibrium price. Solve for consumer and producer surplus.

(b) Suppose there is a $2 producer tax. What is the new equilibrium price? What price does the firm receive? What is the change in producer and consumer surplus? What is government revenue? What is the deadweight loss?

(c) Suppose there is a $2 consumer tax. What is the new equilibrium price? What price does the firm receive? What is the change in producer and consumer surplus? What is government revenue? What is the deadweight loss?

4. Imports

The demand for portable radios is given by: $Q = 5000 - 100p$. The local supply curve is given by $Q = 150p$.

a) Find the market equilibrium.

b) Suppose that radios can be imported at a price of $10 per unit. Find the market equilibrium and the amount of radios imported.
c) Suppose that the local producers convince the government to impose a tariff of $5 per radio. Find the market equilibrium, the total revenue of the tariff and the effect on the consumer and producer surplus, and the deadweight loss.

5. General Equilibrium

Consider an economy with two goods \((x, y)\) and three agents \((A, B, C)\). The agents have preferences:

\[
\begin{align*}
u_A &= xy \\
u_A &= x^2 y \\
u_A &= xy^2
\end{align*}
\]

The agents have endowments \(\omega^A = (10, 20)\), \(\omega^B = (30, 15)\) and \(\omega^C = (15, 45)\).

Find the competitive equilibrium prices and allocations.
1. First Welfare Theorem

There are two goods \( (x \text{ and } y) \) and two agents (A and B). The agents' utility functions are

\[
\begin{align*}
    u_A &= v_A(x) + y \\
    u_B &= v_B(x) + y
\end{align*}
\]

where \( v_A = 2 \ln(x) \) and \( v_B(x) = 4x^{1/2} \), The agents have incomes \( m_A = 10 \) and \( m_B = 10 \). The price of good \( y \) is \( p_y = 1 \); the price of good \( x \) is to be determined.

A single firm produces good \( x \). It has cost function \( c(q) = q^2/2 \).

(a) Show that \( p = 2 \) is an equilibrium price. Find the equilibrium allocations \( (x_A, x_B, q) \).

(b) Suppose a social planner chooses \( (x_A, x_B, q) \) to maximise the total surplus,

\[
v_A(x_A) + v_B(x_B) - c(q)
\]

subject to \( q = x_A + x_B \). Verify your allocations from part (a) satisfy the FOCs from this optimisation problem.

Solution

(a) Substituting her budget into her utility, A maximises

\[
2 \ln(x) + (m - px)
\]

The FOC implies that demand is given by \( x = 2/p \). Similarly, B maximises

\[
4x^{1/2} + (m - px)
\]

The FOC implies that demand is given by \( x = 4/p^2 \).
The firm’s marginal cost is $MC = q$. Hence the supply curve is $q = p$. The equilibrium price thus solves
\[
\frac{4}{p^2} + \frac{2}{p} = p
\]
Substituting in, $p = 2$ solves this equation. At this price the allocations are
\[
(x_A, x_B, q) = (1, 1, 2)
\]

(b) Using the fact that $q = x_A + x_B$, the social planner wishes to maximise
\[
S = 2 \ln(x_A) + 4x_B^{1/2} - \frac{1}{2}(x_A + x_B)^2
\]
(where the $S$ stands for ‘surplus’). The FOCs are
\[
\frac{dS}{dx_A} = 2x_A^{-1} - (x_A + x_B) = 0
\]
\[
\frac{dS}{dx_B} = 2x_B^{1/2} - (x_A + x_B) = 0
\]
Substituting in $x_A = 1$ and $x_B = 1$ we see that both are satisfied.

2. Shifts in Supply Functions

Suppose the supply function is $q = p$, and the demand function is $q = 10 - p$.

(a) Find the equilibrium price and quantity.

(b) Suppose the supply curve shifts up to $q = p - 2$, so each unit costs $\$2$ more to produce. Derive the new price and quantity.

Suppose the supply function is $q = p$, and the demand function is $q = 6 - p/5$.

(c) Find the equilibrium price and quantity.

(d) Suppose the supply curve shifts up to $q = p - 2$. Derive the new price and quantity.

Suppose the supply function is $q = p$, and the demand function is $q = 25 - 4p$.

(e) Find the equilibrium price and quantity.
(f) Suppose the supply curve shifts up to \( q = p - 2 \). Derive the new price and quantity.

(g) Given these results, explain how the elasticity of the demand curve affects the impact of a shift in the supply function.

**Solution**

(a) The equilibrium price and quantity are \( p = 5 \) and \( q = 5 \).

(b) The equilibrium price and quantity are \( p = 6 \) and \( q = 4 \).

(c) The equilibrium price and quantity are \( p = 5 \) and \( q = 5 \).

(d) The equilibrium price and quantity are \( p = \frac{62}{15} \) and \( q = \frac{42}{5} \).

(e) The equilibrium price and quantity are \( p = 5 \) and \( q = 5 \).

(f) The equilibrium price and quantity are \( p = \frac{52}{5} \) and \( q = \frac{32}{5} \).

(g) When demand is inelastic (part (d)) then the shift in the supply curve has little effect on quantity but increases the price a lot. When demand is elastic (part (f)) then the shift in the supply curve has a large effect on quantity but a small effect on price.

3. **Taxation**

Suppose utility is quasilinear in good \( x \) and the demand function is \( q = 10 - p \). The supply function is \( q = p \).

(a) Solve for the equilibrium price. Solve for consumer and producer surplus.

(b) Suppose there is a $2 producer tax. What is the new equilibrium price? What price does the firm receive? What is the change in producer and consumer surplus? What is government revenue? What is the deadweight loss?

(c) Suppose there is a $2 consumer tax. What is the new equilibrium price? What price does the firm receive? What is the change in producer and consumer surplus? What is government revenue? What is the deadweight loss?
Solution

(a) Setting supply equal to demand

\[
10 - p = p
\]

implies that \( p = 5 \). The quantity traded is \( q' = 5 \). Consumer surplus and producer surplus are both \( 5 \times 5/2 = 12\frac{1}{2} \).

(b) Suppose there is a $2 producer tax. The new supply curve is \( q = p - 2 \). The equilibrium price is \( p' = 6 \), although the firm receives \( p' - t = 4 \). The quantity traded is \( q' = 4 \).

With the tax, the consumer and producer surplus are both \( 4 \times 4/2 = 8 \), a change of \( 4\frac{1}{2} \). Government revenue is \( 4 \times 2 = 8 \). The deadweight loss therefore equals 1.

(c) Suppose there is a $2 consumer tax. The new demand curve is \( q = 8 - p \). The equilibrium price is \( p' = 4 \), although the consumer pays \( p' + t = 6 \). The quantity traded is \( q' = 4 \).

With the tax, the consumer and producer surplus are both \( 4 \times 4/2 = 8 \), a change of \( 4\frac{1}{2} \). Government revenue is \( 4 \times 2 = 8 \). The deadweight loss therefore equals 1.

4. Imports

The demand for portable radios is given by: \( Q = 5000 - 100p \). The local supply curve is given by \( Q = 150p \).

a) Find the market equilibrium.

b) Suppose that radios can be imported at a price of $10 per unit. Find the market equilibrium and the amount of radios imported.

c) Suppose that the local producers convince the government to impose a tariff of $5 per radio. Find the market equilibrium, the total revenue of the tariff and the effect on the consumer and producer surplus, and the deadweight loss.

Solution

a) Equating supply and demand, \( 150p = 5000 - 100p \). Hence \( p = 20 \) and \( Q = 3000 \).
b) The price drops to 10. As a result the quantity demanded is \( Q = 4000 \). Domestic production is \( 150 \times 10 = 1500 \), while imports \( 4000 - 1500 = 2500 \).

c) The price rises to 15, and demand falls to \( Q = 3500 \). Domestic production is \( 150 \times 15 = 2250 \) and imports are \( 3500 - 2250 = 1250 \).

Tariff revenues are \( 1250 \times 5 = 6250 \). The consumer surplus without the tariff is \( (50 - 10) \times 4000/2 = 80,000 \). The CS with tariff is \( (50 - 15) \times 3500/2 = 61,250 \), resulting in a loss of 18,750. The transfer to producers is \( 5 \times 1500 + (15 - 10) \times (2250 - 1500)/2 = 9375 \). The deadweight loss is \( 18750 - 9375 - 6250 = 3125 \).

5. General Equilibrium

Consider an economy with two goods \( (x \) and \( y) \) and three agents \((A, B \) and \( C)\). The agents have preferences:

\[
\begin{align*}
  u_A &= xy \\
  u_A' &= x^2 y \\
  u_A'' &= xy^2
\end{align*}
\]

The agents have endowments \( \omega^A = (10, 20) \), \( \omega^B = (30, 15) \) and \( \omega^C = (15, 45) \).

Find the competitive equilibrium prices and allocations.

Solution

Utilities are Cobb–Douglas. Demands are given by

\[
\begin{align*}
  x^A &= \frac{1}{2} \frac{m^A}{p_x} \\
  y^A &= \frac{1}{2} \frac{m^A}{p_y} \\
  x^B &= \frac{2}{3} \frac{m^B}{p_x} \\
  y^B &= \frac{1}{3} \frac{m^B}{p_y} \\
  x^C &= \frac{1}{3} \frac{m^C}{p_x} \\
  y^C &= \frac{2}{3} \frac{m^C}{p_y}
\end{align*}
\]
The agent’s incomes are

\[ m^A = 10p_x + 20p_y \]
\[ m^B = 30p_x + 15p_y \]
\[ m^C = 15p_x + 45p_y \]

Market clearing for good \( x \) means

\[ \frac{1}{2} \left( \frac{10p_x + 20p_y}{p_x} \right) + \frac{2}{3} \left( \frac{30p_x + 15p_y}{p_x} \right) + \frac{1}{3} \left( \frac{15p_x + 45p_y}{p_x} \right) = 55 \]

Solving yields the price ratio:

\[ p_y = \frac{5}{7} p_x \]

Intuitively, there is more of good \( y \) so it is cheaper.

Normalise \( p_x = 7 \), so that \( p_y = 5 \). The incomes are \( m^A = 170 \), \( m^B = 285 \) and \( m^C = 330 \). Using the demand functions, allocation are

\[
(x_A, y_A, x_B, y_B, x_C, y_C) = \left( \frac{85}{7}, \frac{85}{5}, \frac{190}{7}, \frac{95}{5}, \frac{110}{7}, \frac{220}{5} \right) \]
1. Optimisation (4 points)

We wish to maximise \( f(x) = 5 - 2x - x^2 \).

(a) Find the optimal value of \( x \) and the resulting value of the objective. Verify the SOC.

(b) Suppose we introduce the constraint that \( x \geq 0 \). Find the optimal value of \( x \) and the resulting value of the objective. [Hint: Plot the function]

2. Optimisation with 2 variables (4 points)

Find the values of \((x_1, x_2)\) that maximise \( f(x) = 20x_1^{1/4}x_2^{1/4} - x_1 - x_2 \).

3. Constrained Optimisation (4 points)

Suppose that \( f(x_1, x_2) = x_1x_2 \). Suppose the constraint is \( x_1 + x_2 = 1 \).

(a) Draw the constraint in \((x_1, x_2)\) space.

(b) Draw the level curves of the objective in \((x_1, x_2)\) space. A level curve are the values of \((x_1, x_2)\) such that \( f(x_1, x_2) \) is constant.\(^1\)

We now wish to find the maximum value of \( f \) given the constraint.

(c) Solve the problem by substituting the constraint into the objective.

(d) Solve the problem by using the Lagrange method.

4. Constrained Optimisation (4 points)

The dual problem to the last question is to minimise \( g(x_1, x_2) = x_1 + x_2 \) subject to \( x_1x_2 = 1/4 \).

\(^1\)We will later think of these as indifference curves.
(a) Solve this problem using the Lagrangian technique.

(b) How does the solution compare to part (d) of the last question?

5. Partial Derivatives and Total Differential (4 points)

Suppose \( u = f(x_1, x_2) = 4x_1^2 + 3x_2^2 \).

(a) Calculate \( \partial u/\partial x_1 \) and \( \partial u/\partial x_2 \).

(b) Write the total differential for \( u \).

(c) Calculate \( dx_2/dx_1 \) when \( du = 0 \). That is, what is the implied trade-off between \( x_1 \) and \( x_2 \) holding \( u \) constant.
Economics 11: Solutions to Homework 1

1. Optimisation (4 points)

We wish to maximise \( f(x) = 5 - 2x - x^2 \).

(a) Find the optimal value of \( x \) and the resulting value of the objective. Verify the SOC.

(b) Suppose we introduce the constraint that \( x \geq 0 \). Find the optimal value of \( x \) and the resulting value of the objective. [Hint: Plot the function]

Solution

(a) Differentiating, the FOC is

\[
\frac{df(x)}{dx} = -2 - 2x = 0
\]

This yields \( x^* = -1 \) and \( f(x^*) = 6 \). The SOC is \(-2\), and is thus satisfied.

(b) The function is concave, so when we introduce the constraint, the function is maximised at the boundary, \( x^* = 0 \) where \( f(x^*) = 5 \).

2. Optimisation with 2 variables (4 points)

Find the values of \((x_1, x_2)\) that maximise \( f(x) = 20x_1^{1/4} x_2^{1/4} - x_1 - x_2 \).

Solution

THE FOCs are

\[
\begin{align*}
\frac{\partial f(x_1, x_2)}{\partial x_1} &= 5x_1^{-3/4} x_2^{1/4} - 1 = 0 \\
\frac{\partial f(x_1, x_2)}{\partial x_2} &= 5x_1^{1/4} x_2^{-3/4} - 1 = 0
\end{align*}
\]

Solving this system yields \( x_1^* = 25 \) and \( x_2^* = 25 \).
3. Constrained Optimisation (4 points)

Suppose that \( f(x_1, x_2) = x_1 x_2 \). Suppose the constraint is \( x_1 + x_2 = 1 \).

(a) Draw the constraint in \((x_1, x_2)\) space.

(b) Draw the level curves of the objective in \((x_1, x_2)\) space. A level curve are the values of \((x_1, x_2)\) such that \(f(x_1, x_2)\) is constant.\(^1\)

We now wish to find the maximum value of \(f\) given the constraint.

(c) Solve the problem by substituting the constraint into the objective.

(d) Solve the problem by using the Lagrange method.

Solution

(a) The constraint in a straight line with slope \(-1\).

(b) The level curves are parabolas.

(c) Substituting,
\[
f(x_1, x_2) = x_1 x_2 = x_1 (1 - x_1)
\]
This is maximised at \(x_1^* = 1/2\), where \(x_2^* = 1/2\) and \(f(x_1^*, x_2^*) = 1/4\).

(d) The Lagrangian is
\[
\mathcal{L} = x_1 x_2 - \lambda [1 - x_1 - x_2]
\]
The FOCs are
\[
x_2 = \lambda \\
x_1 = \lambda
\]
Hence \(x_1 = x_2\). The constraint thus implies that \(x_1^* = 1/2, x_2^* = 1/2\) and \(f(x_1^*, x_2^*) = 1/4\). The multiplier is \(\lambda = 1/2\).

\(^1\)We will later think of these as indifference curves.
4. Constrained Optimisation (4 points)

The dual problem to the last question is to *minimise* \( g(x_1, x_2) = x_1 + x_2 \) subject to \( x_1 x_2 = 1/4 \) and \( x_1, x_2 \geq 0 \).

(a) Solve this problem using the Lagrangian technique.

(b) How does the solution compare to part (d) of the last question?

**Solution**

(a) The Lagrangian is

\[
L = x_1 + x_2 + \mu [0.25 - x_1 x_2]
\]

where we subtract the penalty because we now wish to minimise \( L \). The FOCs are

\[
1 = \mu x_2 \\
1 = \mu x_2
\]

Hence \( x_1 = x_2 \). The constraint thus implies that \( x_1^* = 1/2, x_2^* = 1/2 \) and \( g(x_1^*, x_2^*) = 1 \). The Lagrange multiplier is \( \mu = 2 \).

(b) First, observe that the optimal choices of \((x_1, x_2)\) are the same under the two problems. Second, observe that the value of the objective in the second problem is the value of the constraint in the first. Finally, observe that \( \mu = 1/\lambda \). Intuitively, \( \lambda \) measures the extra utility from a $1 increase in income, while \( \mu \) measures the extra expenditure required to obtain 1 util, which are just the inverse of each other.[Note: We will talk more about duality as the class progresses.]

5. Partial Derivatives and Total Differential (4 points)

Suppose \( u = f(x_1, x_2) = 4x_1^2 + 3x_2^2 \).

(a) Calculate \( \partial u/\partial x_1 \) and \( \partial u/\partial x_2 \).

(b) Write the total differential for \( u \).
(c) Calculate \(dx_2/dx_1\) when \(du = 0\). That is, what is the implied trade-off between \(x_1\) and \(x_2\) holding \(u\) constant.

**Solution**

(a) \(\partial u/\partial x_1 = 8x_1\) and \(\partial u/\partial x_2 = 6x_2\).

(b) Totally differentiating,

\[
du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2
\]

\[
= (8x_1)dx_1 + (6x_2)dx_2
\]

(c) If \(du = 0\) then

\[
\frac{dx_2}{dx_1} = -\frac{8x_1}{6x_2} = -\frac{4x_1}{3x_2}
\]
1. Utility Functions (4 points)

Define the utility functions and draw the indifference curves for each of the following cases:

a) “Every time I consume one unit of $x_1$, I want to consume 2 units of $x_2$.”

b) “If the price of $x_1$ is bigger than the price of $x_2$, I will only consume $x_2$. If the price of $x_2$ is bigger than the price of $x_1$, I will only consume $x_1$.”

c) “I don’t care about the prices of $x_1$ and $x_2$, I only want to consume $x_2$.”

2. Consumption with Perfect Substitutes (4 points)

Suppose John views butter and margarine as perfectly substitutable in the ratio of one to one for each other.

a) Draw a set of indifference curves that describes John's preferences for butter and margarine.

b) If butter costs $2 per package, while margarine costs $1, and John has a $20 budget to spend for the month, which butter-margarine combination will he choose?

(c) Can you show this graphically? What happens when the price of margarine increases to $3?

3. Convexity and Monotonicity (6 points)

For the following utility functions, sketch the indifference curves and explain whether the underlying preferences satisfy convexity and monotonicity. Throughout, assume $x_1, x_2 \geq 0$.

(a) $u(x_1, x_2) = x_1 + x_2$

(b) $u(x_1, x_2) = x_1^2 + x_2^2$

(c) $u(x_1, x_2) = \min\{x_1 + x_2, 1\}$
(d) \( u(x_1, x_2) = I(x_1) + I(x_2) \), where \( I(x) \) is the integer component of \( x \). For example, \( I(3.64) = 3 \) and \( I(5.2) = 5 \).

(e) \( u(x_1, x_2) = -(x_1 - 1)^2 - (x_2 - 1)^2 \).

(f) \( u(x_1, x_2) = \min\{x_1, x_2\} \).

4. Monotone Transformations of Utility Functions (6 points)

The agent consumes \( x \) units of a pizza, where we assume that \( 0 \leq x \leq 4 \). The agent receives utility

\[
u(x) = \begin{cases} 
  x & \text{if } 0 < x \leq 2 \\
  4 - x & \text{if } 2 < x \leq 4
\end{cases}
\]

Which of the following utility functions represent the same preferences as above:

(a) \( v(x) = 2x \) if \( 0 \leq x \leq 2 \), and \( v(x) = 4 - x \) if \( 2 < x \leq 4 \).

(b) \( v(x) = 2x \) if \( 0 \leq x \leq 2 \), and \( v(x) = 8 - 2x \) if \( 2 < x \leq 4 \).

(c) \( v(x) = 4 - (2 - x)^2 \) if \( 0 \leq x \leq 4 \).

(d) \( v(x) = 4 - (2 - x)^3 \) if \( 0 \leq x \leq 4 \).

(e) \( v(x) = 4 - (2 - x)^4 \) if \( 0 \leq x \leq 4 \).

(f) \( v(x) = -(2 - x)^2 \) if \( 0 \leq x \leq 4 \).
Economics 11: Solutions to Homework 2

1. Utility Functions (4 points)

Define the utility functions and draw the indifference curves for each of the following cases:

a) “Every time I consume one unit of $x_1$, I want to consume 2 units of $x_2$”.

b) “If the price of $x_1$ is bigger than the price of $x_2$, I will only consume $x_2$. If the price of $x_2$ is bigger than the price of $x_1$, I will only consume $x_1$”.

c) “I don’t care about the prices of $x_1$ and $x_2$, I only want to consume $x_2$”.

Solution

I won’t draw the pictures. I refer you to the lectures/TA sessions.

(a) Preferences are represented by $u(x_1, x_2) = \min\{2x_1, x_2\}$. That is, preferences are perfect complements.

(b) Preferences are represented by $u(x_1, x_2) = x_1 + x_2$. That is, preferences are perfect substitutes.

(c) Preferences are represented by $u(x_1, x_2) = x_2$.

2. Consumption with Perfect Substitutes (4 points)

Suppose John views butter and margarine as perfectly substitutable in the ratio of one to one for each other.

a) Draw a set of indifference curves that describes Johns preferences for butter and margarine.

b) If butter costs $2 per package, while margarine cost only $1, and John has a $20 budget to spend for the month, which butter-margarine combination will he choose?

(c) Can you show this graphically? What happens when the price of margarine increases to $3$?
Solution

(a) Indifference curves are straight lines. See the lecture.

(b) The MRS between butter and margarine is equal to 1 at every consumption level. As the cost of margarine is less, John will choose to buy margarine with all his income and will buy no butter. So he will buy 20 units of margarine and 0 units of butter.

(c) If the price of margarine increases to 3 he will consume only butter (10 units).

3. Convexity and Monotonicity (6 points)

For the following utility functions, sketch the indifference curves and explain whether the underlying preferences satisfy convexity and monotonicity. Throughout, assume $x_1, x_2 \geq 0$.

(a) $u(x_1, x_2) = x_1 + x_2$

(b) $u(x_1, x_2) = x_1^2 + x_2^2$

(c) $u(x_1, x_2) = \min\{x_1 + x_2, 1\}$

(d) $u(x_1, x_2) = I(x_1) + I(x_2)$, where $I(x)$ is the integer component of $x$. For example, $I(3.64) = 3$ and $I(5.2) = 5$.

(e) $u(x_1, x_2) = -(x_1 - 1)^2 - (x_2 - 1)^2$.

(f) $u(x_1, x_2) = \min\{x_1, x_2\}$.

Solution

(a) The goods here are perfect substitutes. These preferences satisfy monotonicity and convexity.

(b) These preferences satisfy monotonicity but not convexity.

(c) The consumer has a maximum utility of 1. Hence these preferences satisfy convexity but not monotonicity.
(d) The goods here are indivisible. These preferences satisfy neither monotonicity or convexity.

(e) This has a bliss point at (1, 1). These preferences satisfy convexity but not monotonicity.

(f) Preferences are perfect complements. These preferences satisfy convexity but not monotonicity.

4. Monotone Transformations of Utility Functions (6 points)

The agent consumes $x$ units of a pizza, where we assume that $0 \leq x \leq 4$. The agent receives utility

\[
u(x) = \begin{cases} 
  x & \text{if } 0 \leq x \leq 2 \\
  4 - x & \text{if } 2 < x \leq 4
\end{cases}
\]

Which of the following utility functions represent the same preferences as above:

(a) $v(x) = 2x$ if $0 \leq x \leq 2$, and $v(x) = 4 - x$ if $2 < x \leq 4$.

(b) $v(x) = 2x$ if $0 \leq x \leq 2$, and $v(x) = 8 - 2x$ if $2 < x \leq 4$.

(c) $v(x) = 4 - (2 - x)^2$ if $0 \leq x \leq 4$.

(d) $v(x) = 4 - (2 - x)^3$ if $0 \leq x \leq 4$.

(e) $v(x) = 4 - (2 - x)^4$ if $0 \leq x \leq 4$.

(f) $v(x) = -(2 - x)^2$ if $0 \leq x \leq 4$.

Solution

(a) This is different since $v(1) = 2 > 1 = v(3)$.

(b) This is the same since $v(x) = 2u(x)$. 
(c) This is the same since $v(x)$ is increasing for $x < 2$, decreasing for $x > 2$, and is symmetric around 2.

(d) This is different since $v(x)$ is increasing for all $x$.

(e) This is the same since $v(x)$ is the square of part (c).

(f) This is the same since $v(x)$ is a vertical shift of part (c).
1. Consumer choice problem (4 points)

Sam eats breakfast at Luvalle Commons everyday. She likes bagels and coffee and they provide her a utility of $U(b, c) = b^{2/3}c^{1/3}$. Bagels cost $2 and coffee costs $1 per cup. Her daily budget for breakfast is $12.

a) Suppose that she consumes 2 bagels and 8 coffees. Find the marginal rate of substitution. Is this an optimal choice? Why?

b) Suppose that she consumes 3 bagels and 3 coffees. Find the marginal rate of substitution. Is this an optimal choice? Why?

c) Given prices $(p_b, p_c)$, find the optimal choice for Sam.

2. Consumer choice problem (4 points)

Joe is a teenager who likes video games $G$ and music $M$. Each week, he receives $60 to buy games and music CDs. His utility function for these two goods is $U(G, M) = G^{1/2} + M^{1/2}$. The price of a game is $20 and the price of a CD is $10.

a) Solve for Joe’s demand using the Lagrangian Method.

b) Solve the same problem as in (a) using the Substitution Method.

c) Joe is a bit disappointed with your calculations. He thinks he can buy more games and CDs if his utility function is logarithmic instead. Take the (natural) log of $U(G, M)$ and solve the optimisation problem again using the Lagrangian method. Compare the optimal levels with those found in part (a).

3. Consumer choice with three goods (4 points)

Susan and Paul are college students whose parents give them monthly allowances to be spent on food ($f$), clothing ($c$) and books ($b$). Prices are as follows: one meal costs $3, one piece of
clothing costs $40, and one book costs $5. Assume the goods are perfectly divisible. Susan is a senior and she receives $200 per month; Paul is a freshman and he receives $160 per month. Susan’s preferences are given by the following utility function:

$$U(f, c, b) = 3 \ln(f) + 6 \ln(c) + \ln(b)$$

Paul’s preferences are given by the following utility function:

$$U(f, c, b) = 3 \ln(f) + \ln(c) + 2 \ln(b)$$

a) Write down the budget constraint for each individual.

b) Suppose Susan buys 20 meals, 3 pieces of clothing and 4 books. What is her marginal rate of substitution between food and clothing. What is her marginal rate of substitution between clothing and books? Is this allocation optimal for her?

c) Suppose Paul buys 10 meals, 2 pieces of clothing and 10 books. What is his marginal rate of substitution between food and clothing. What is his marginal rate of substitution between clothing and books? Is this allocation optimal for him?

d) Using the Lagrangian method, find the optimal allocations for food, clothing, and books for Susan and Paul.

e) Suppose that the students parents decide that even though Susan is older, both children are facing the same prices and thus should have the same allowance. They increase Paul’s allowance to $200. Find his new optimal allocations. Which good has the largest percentage increase in consumption, if any?

4. Calculating Demand Functions (4 points)

Find the demand functions for $x_1$ and $x_2$ in the following utility functions:

a) $$U(x_1, x_2) = x_1^{1/4} x_2^{3/4}$$

b) $$U(x_1, x_2) = \min\{2x_1, 3x_2\}$$

c) $$U(x_1, x_2) = 2x_1 + 3x_2$$
5. Budget Sets (4 points)

There are two goods: $x_1$ and $x_2$ with prices $p_1 = 2$ and $p_2 = 2$. Suppose the government subsidises the first 10 units of $x_1$ by $1$. Draw the budget curves for $m = 8$, $m = 12$ and $m = 16$.
1. Consumer choice problem (4 points)

Sam eats breakfast at Luvalle Commons everyday. She likes bagels and coffee and they provide her a utility of $U(b, c) = b^{2/3}c^{1/3}$. Bagels cost $2 and coffee costs $1 per cup. Her daily budget for breakfast is $12.

a) Suppose that she consumes 2 bagels and 8 coffees. Find the marginal rate of substitution. Is this an optimal choice? Why?

b) Suppose that she consumes 3 bagels and 3 coffees. Find the marginal rate of substitution. Is this an optimal choice? Why?

c) Given prices $(p_b, p_c)$, find the optimal choice for Sam.

**Solution**

(a) The MRS is

$$MRS = \frac{MU_b}{MU_c} = \frac{2c^{1/3}}{\frac{1}{3}b^{2/3}} = \frac{2c}{b^{1/3}} = \frac{16}{2} = 8$$

The price ratio is $p_b/p_c = 2$, so Sam would prefer to eat more bagels.

(b) The MRS equals 2. Sam’s total expenditure is $9. Since her preferences are monotone, this cannot be optimal.

(c) The Lagrangian is:

$$\mathcal{L} = b^{2/3}c^{1/3} + \lambda[M - p_bb - p_cc]$$

The FOCs are:

$$\frac{2}{3} \left( \frac{c}{b} \right)^{1/3} = \lambda p_b$$

$$\frac{1}{3} \left( \frac{b}{c} \right)^{2/3} = \lambda p_c$$

These imply that $2p_c = p_b$. Using the budget equation, $b^* = 4$ and $c^* = 4$. 

---

1
2. Consumer choice problem (4 points)

Joe is a teenager who likes video games G and music M. Each week, he receives $60 to buy games and music CDs. His utility function for these two goods is $U(G, M) = G^{1/2} + M^{1/2}$. The price of a game is $20 and the price of a CD is $10.

a) Solve for Joe’s demand using the Lagrangian Method.

b) Solve the same problem as in (a) using the Substitution Method.

c) Joe is a bit disappointed with your calculations. He thinks he can buy more games and CDs if his utility function is logarithmic instead. Take the (natural) log of $U(G, M)$ and solve the optimisation problem again using the Lagrangian method. Compare the optimal levels with those found in part (a).

Solution

(a) Joe’s budget constraint is $20G + 10M = 60$. The Lagrangian is:

$$\mathcal{L} = G^{1/2} + M^{1/2} + \lambda[60 - 20G - 10M]$$

The FOCs are

$$\frac{1}{2}G^{-1/2} = 20\lambda$$
$$\frac{1}{2}M^{-1/2} = 10\lambda$$

Rearranging, $M = 4G$. Using the budget constraint, $G^* = 1$ and $M^* = 4$.

(b) Using the budget constraint, $M = 6 - 2G$. The utility function is thus:

$$U = G^{1/2} + (6 - 2G)^{1/2}$$

The FOC is

$$\frac{1}{2}G^{-1/2} - (6 - 2G)^{-1/2} = 0$$

Rearranging, $G^* = 1$. Using the budget constraint, $M^* = 4$. 

2
(c) The Lagrangian is:
\[
\mathcal{L} = \ln\left(\frac{G^{1/2}}{2} + \frac{M^{1/2}}{2}\right) + \lambda[60 - 20G - 10M]
\]
The FOCs are
\[
\frac{1}{2 \frac{G^{-1/2}}{G^{1/2} + M^{1/2}}} = 20\lambda
\]
\[
\frac{1}{2 \frac{M^{-1/2}}{G^{1/2} + M^{1/2}}} = 10\lambda
\]
Rearranging, \( M = 4G \). Using the budget constraint, \( G^* = 1 \) and \( M^* = 4 \). The solution is the same since we have just applied an increasing function to the utility function.

3. Consumer choice with three goods (4 points)

Susan and Paul are college students whose parents give them monthly allowances to be spent on food (\( f \)), clothing (\( c \)) and books (\( b \)). Prices are as follows: one meal costs $3, one piece of clothing costs $40, and one book costs $5. Assume the goods are perfectly divisible. Susan is a senior and she receives $200 per month; Paul is a freshman and he receives $160 per month. Susan’s preferences are given by the following utility function:
\[
U(f, c, b) = 3 \ln(f) + 6 \ln(c) + \ln(b)
\]
Paul’s preferences are given by the following utility function:
\[
U(f, c, b) = 3 \ln(f) + \ln(c) + 2 \ln(b)
\]
a) Write down the budget constraint for each individual.

b) Suppose Susan buys 20 meals, 3 pieces of clothing and 4 books. What is her marginal rate of substitution between food and clothing. What is her marginal rate of substitution between clothing and books? Is this allocation optimal for her?

c) Suppose Paul buys 10 meals, 2 pieces of clothing and 10 books. What is his marginal rate of substitution between food and clothing. What is his marginal rate of substitution between clothing and books? Is this allocation optimal for him?
d) Using the Lagrangian method, find the optimal allocations for food, clothing, and books for Susan and Paul.

e) Suppose that the students parents decide that even though Susan is older, both children are facing the same prices and thus should have the same allowance. They increase Paul’s allowance to $200. Find his new optimal allocations. Which good has the largest percentage increase in consumption, if any?

Solution

(a) Susan’s budget is $3f + 40c + 5b = 200$. Paul’s is $3f + 40c + 5b = 160$.

(b) Susan’s MRSs are:

\[ MRS_{f,c} = \frac{MU_f}{MU_c} = \frac{3/f}{6/c} = \frac{c}{2f} = \frac{3}{40} \]
\[ MRS_{c,b} = \frac{MU_c}{MU_b} = \frac{6/c}{1/b} = \frac{6b}{c} = 8 \]

For the allocation to be optimal we need to verify (a) Susan’s marginal rates of substitution equal the price ratios and (b) Susan spends all her budget. The price ratios are $p_c/p_b = 8$ and $p_f/p_c = 3/40$. Moreover, Susan’s expenditure is $20 \cdot 3 + 3 \cdot 40 + 4 \cdot 5 = 200$. Hence Susan’s allocation is optimal.

(c) Paul’s MRSs are:

\[ MRS_{f,c} = \frac{MU_f}{MU_c} = \frac{3/f}{1/c} = \frac{3c}{f} = \frac{3}{5} \]
\[ MRS_{c,b} = \frac{MU_c}{MU_b} = \frac{1/c}{2/b} = \frac{b}{2c} = \frac{5}{2} \]

The price ratios are $p_c/p_b = 8$ and $p_f/p_c = 3/40$, so Paul’s allocation is not optimal (e.g. he should consume more food).

(d) Let’s solve the problem generally and then plug in the numbers. The Lagrangian for Susan is:

\[ \mathcal{L} = 3 \ln(f) + 6 \ln(c) + \ln(b) + \lambda[M - p_f f - p_c c - p_b b] \]
The FOCs are

\[
\frac{3}{f} = \lambda p_f \\
\frac{6}{c} = \lambda p_c \\
\frac{1}{b} = \lambda p_b
\]

The equations imply \( cp_c = 2fp_f = 6bp_b \). Using the budget constraint, \( f^* = \frac{3M}{lp_f} \), \( c^* = \frac{3M}{6pc} \), and \( b^* = \frac{M}{5pb} \). Given the numbers in the example, we have \( f^* = 20 \), \( c^* = 3 \) and \( b^* = 4 \).

The Lagrangian for Paul is:

\[
\mathcal{L} = 3\ln(f) + \ln(c) + 2\ln(b) + \lambda[M - p_f f - p_c c - p_b b]
\]

The FOCs are

\[
\frac{3}{f} = \lambda p_f \\
\frac{1}{c} = \lambda p_c \\
\frac{2}{b} = \lambda p_b
\]

The equations imply \( 6cp_c = 2fp_f = 3bp_b \). Using the budget constraint, \( f^* = \frac{M}{p_f} \), \( c^* = \frac{M}{3pc} \), and \( b^* = \frac{M}{5pb} \). Given the numbers in the example, we have \( f^* = 26\frac{2}{3} \), \( c^* = 2\frac{2}{3} \) and \( b^* = 10\frac{2}{3} \).

(e) When Paul’s income rises to $200, the optimal consumption levels become \( f^* = 33\frac{1}{3} \), \( c^* = \frac{5}{6} \) and \( b^* = 13\frac{1}{3} \). One can see that Paul’s consumption of all goods increases by the 25%. This is because the income offer curve is linear with these Cobb–Douglas preferences.

4. Calculating Demand Functions (4 points)

Find the demand functions for \( x_1 \) and \( x_2 \) in the following utility functions:

a) \( U(x_1, x_2) = x_1^{1/4} x_2^{3/4} \)

b) \( U(x_1, x_2) = \min\{2x_1, 3x_2\} \)

c) \( U(x_1, x_2) = 2x_1 + 3x_2 \)
(a) The Lagrangian is:
$$\mathcal{L} = x_1^{1/4} x_2^{3/4} + \lambda [m - p_1 x_1 - p_2 x_2]$$

The FOCs are:
$$\frac{1}{4} \left( \frac{x_2}{x_1} \right)^{3/4} = \lambda p_1$$
$$\frac{3}{4} \left( \frac{x_1}{x_2} \right)^{1/4} = \lambda p_2$$

These imply that $3p_1 x_1 = p_2 x_2$. Demand are thus $x_1^* = m/4p_1$ and $x_2^* = 3m/4p_2$.

(b) The utility function is not differentiable at the kink (draw the indifference curves). Hence the Lagrangian method does not work.

From the utility function we know that, at the optimum, $2x_1^* = 3x_2^*$. Using the budget constraint,
$$x_1^* = \frac{3m}{3p_1 + 2p_2}$$
$$x_2^* = \frac{2m}{3p_1 + 2p_2}$$

(c) The indifference curves are linear, so we will have a boundary problem. [If you use the Lagrangian approach and ignore the boundaries, you will want to consume a negative quantity of one or other good]. We can therefore just compare the two endpoints.

If $3p_1 > 2p_2$, then $x_1^* = 0$ and $x_2^* = m/p_2$. If $3p_1 < 2p_2$, then $x_1^* = m/p_1$ and $x_2^* = 0$. If $3p_1 = 2p_2$, then the consumer is indifferent between all points on the budget line.

5. Budget Sets (4 points)

There are two goods: $x_1$ and $x_2$ with prices $p_1 = 2$ and $p_2 = 2$. Suppose the government subsidises the first 10 units of $x_1$ by $1$. Draw the budget curves for $m = 8$, $m = 12$ and $m = 16$.
Solution

When $m = 8$, it is as if $p_1 = 1$ and $p_2 = 2$.

When $m = 12$ or $m = 16$, there is a kink at $x_1 = 10$. The slope is $-1/2$ to the left and $-1$ to the right. The vertical intercept is $x_2 = 6$ when $m = 12$ and $x_2 = 8$ when $m = 16$. The horizontal intercept is $x_1 = 11$ when $m = 12$ and $x_1 = 13$ when $m = 16$. 
1. Consumption with Three Goods

Suppose an agent consumes three goods, \( \{x_1, x_2, x_3\} \). Her utility is

\[
\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}
\]

The agent has income \( m = 1 \).

(a) Suppose the prices are \( p_1 = 1 \), \( p_2 = 1 \) and \( p_3 = 1 \). Solve for the agent’s optimal consumption.

(b) Suppose the prices are \( p_1 = 3 \), \( p_2 = 1 \) and \( p_3 = 1 \). Solve for the agent’s optimal consumption.

2. Income Effects and Quasilinear Utility

Suppose utility has the form \( u(x_1, x_2) = x_1^{1/2} + x_2 \). Let \( p_1 = 1 \) and \( p_2 = 2 \).

(a) Suppose \( m \geq 1 \). Solve for the agent’s optimal consumption of \( (x_1, x_2) \).

(b) Suppose \( m < 1 \). Solve for the agent’s optimal consumption of \( (x_1, x_2) \). [Hint: Beware of boundary constraints].

3. Consumer Choice with Price Discounts

There are two goods: \( x_1 \) and \( x_2 \). The seller of \( x_2 \) charges \( p_2 = 2 \). The seller of \( x_1 \) offers price discounts. If the agent buys \( x_1 < 10 \) she pays \( p_1 = 2 \) for every unit. If the agent buys \( x_1 \geq 10 \) she pays \( p_1 = 1 \) for every unit (including the first 10).

(a) Suppose \( m = 30 \). Draw the agent’s budget set.
(b) Assume the agent has monotone preferences. Do preferences exist such that the agent chooses (i) \( x_1 = 3 \), (ii) \( x_1 = 7 \) and (iii) \( x_1 = 12 \)? Explain your answers.

4. Taxes with Cobb Douglas Utility

An agent has utility \( u(x_1, x_2) = x_1 x_2^2 \).

a) Find the agent’s Marshallian demand.

b) Find the agent’s indirect utility function.

c) Verify Roy’s identity for good 1. That is, show the following holds:

\[
\frac{\partial v(p_1, p_2, m)}{\partial p_1} = -x_1^*(p_1, p_2, m) \frac{\partial v(p_1, p_2, m)}{\partial m}
\]

Now suppose that \( p_1 = 10 \) and \( p_2 = 5 \). The agent’s income is \( m = 300 \).

d) What is the agent’s consumption? What is her utility level?

The government would like to reduce the agent’s consumption of \( x_2 \) by half. It considers two policies to achieve this. Policy A imposes a tax on each unit of \( x_2 \). Policy B imposes a lump sum tax on the agent’s income.

e) Suppose the government uses policy A. How big a tax needs to be imposed on each unit of \( x_2 \)? What is the governments tax revenue? What is the agent’s utility?

f) Suppose the government uses policy B. By how much would the government have to reduce the agent’s income? What is the governments tax revenue? What is the agent’s utility?

g) Which policy does the agent prefer? Why?

5. Expenditure Minimisation with CES demand

A consumer has the utility \( u(x_1, x_2) = x_1^{1/2} + x_2^{1/2} \). She faces prices \( p_1 = 1 \) and \( p_2 = 2 \).

(a) Fix income \( m \). Find the Marshallian demand for \( x_1 \) and \( x_2 \).
(b) Substituting the Marshallian demands into the utility function, show the indirect utility is

\[ v = \left( \frac{3}{m-2} \right)^{1/2} \]

(c) Fix target utility, \( u \). Find the Hicksian demand for \( x_1 \) and \( x_2 \).

(d) Find the expenditure function directly, by calculating \( e = p_1 h_1 + p_2 h_2 \).

(e) Verify the expenditure function is the inverse of the indirect utility function.
1. Consumption with Three Goods

Suppose an agent consumes three goods, \( \{x_1, x_2, x_3\} \). Her utility is

\[
\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}
\]

The agent has income \( m = 1 \).

(a) Suppose the prices are \( p_1 = 1, p_2 = 1 \) and \( p_3 = 1 \). Solve for the agent’s optimal consumption.

(b) Suppose the prices are \( p_1 = 3, p_2 = 1 \) and \( p_3 = 1 \). Solve for the agent’s optimal consumption.

Solution

(a) The Lagrangian is

\[
L = \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \lambda (m - x_1 - x_2 - x_3)
\]

The FOCs are

\[
\frac{1}{2} x_1^{-1/2} = \lambda
\]

\[
\frac{1}{2} x_2^{-1/2} = \lambda
\]

\[
\frac{1}{2} x_3^{-1/2} = \lambda
\]

The equations imply \( x_1 = x_2 = x_3 \). Using the budget constraint, \( x_1^* = 1/3, x_2^* = 1/3, x_3^* = 1/3 \).

(b) The Lagrangian is

\[
L = \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \lambda (m - 3x_1 - x_2 - x_3)
\]
The FOCs are

\[
\frac{1}{2} x_1^{-1/2} = \lambda \\
\frac{1}{2} x_2^{-1/2} = \lambda \\
\frac{1}{2} x_3^{-1/2} = 3\lambda
\]

The equations imply \(9x_1 = x_2 = x_3\). Using the budget constraint, \(x_1^* = 1/21\), \(x_2^* = 9/21\), \(x_3^* = 9/12\).

2. Income Effects and Quasilinear Utility

Suppose utility has the form \(u(x_1, x_2) = x_1^{1/2} + x_2\). Let \(p_1 = 1\) and \(p_2 = 2\).

(a) Suppose \(m \geq 1\). Solve for the agent’s optimal consumption of \((x_1, x_2)\).

(b) Suppose \(m < 1\). Solve for the agent’s optimal consumption of \((x_1, x_2)\). [Hint: Beware of boundary constraints].

Solution

(a) The consumer’s MRS is

\[
\frac{MU_1}{MU_2} = \frac{1}{2} x_1^{-1/2}
\]

The price ratio is \(p_1/p_2 = \frac{1}{2}\). Equating these, the solution is \(x_1^* = 1\), which is feasible if \(m \geq 1\).

(b) If \(m < 1\) then \(MRS > p_1/p_2\), so the consumer will spend all her money on good 1.\(^1\)

We thus find that \(x_1^* = m\) for \(m < 1\) and \(x_1^* = 1\) for \(m \geq 1\). More concisely, \(x_1^* = \min\{m, 1\}\).

3. Consumer Choice with Price Discounts

There are two goods: \(x_1\) and \(x_2\). The seller of \(x_2\) charges \(p_2 = 2\). The seller of \(x_1\) offers price discounts. If the agent buys \(x_1 < 10\) she pays \(p_1 = 2\) for every unit. If the agent buys \(x_1 \geq 10\)

\(^1\)Equivalently, the bang-for-the-buck from good 2 is \(MU_2/p_2 = 1/2\). This is less than the bang-for-the-buck from good 1 if \(x_1 < 1\).
she pays \( p_1 = 1 \) for every unit (including the first 10).

(a) Suppose \( m = 30 \). Draw the agent’s budget set.

(b) Assume the agent has monotone preferences. Do preferences exist such that the agent chooses (i) \( x_1 = 3 \), (ii) \( x_1 = 7 \) and (iii) \( x_1 = 12 \)? Explain your answers.

**Solution**

(a) The budget line is given by

\[
x_2 = 15 - x_1 \quad \text{for } x_1 < 10
\]

\[
x_2 = 15 - \frac{1}{2} x_1 \quad \text{for } x_1 \geq 10
\]

As a result, there is a discontinuity at \( x_1 = 10 \).

(b) (i) Yes.
(ii) No. Because of the discontinuity, the agent can obtain more of both goods.
(iii) Yes.

4. Taxes with Cobb Douglas Utility

An agent has utility \( u(x_1, x_2) = x_1 x_2^2 \).

a) Find the agent’s Marshallian demand.

b) Find the agent’s indirect utility function.

c) Verify Roy’s identity for good 1. That is, show the following holds:

\[
\frac{\partial v(p_1, p_2, m)}{\partial p_1} = -x_1^*(p_1, p_2, m) \frac{\partial v(p_1, p_2, m)}{\partial m}
\]

Now suppose that \( p_1 = 10 \) and \( p_2 = 5 \). The agent’s income is \( m = 300 \).

d) What is the agent’s consumption? What is her utility level?
The government would like to reduce the agent’s consumption of $x_2$ by half. It considers two policies to achieve this. Policy A imposes a tax on each unit of $x_2$. Policy B imposes a lump sum tax on the agent’s income.

e) Suppose the government uses policy A. How big a tax needs to be imposed on each unit of $x_2$? What is the government’s tax revenue? What is the agent’s utility?

f) Suppose the government uses policy B. By how much would the government have to reduce the agent’s income? What is the government’s tax revenue? What is the agent’s utility?

g) Which policy does the agent prefer? Why?

**Solution**

a) The Lagrangian is

$$\mathcal{L} = x_1 x_2^2 + \lambda [m - p_1 x_1 - p_2 x_2]$$

The FOCs are:

$$x_2^2 = \lambda p_1$$

$$2 x_1 x_2 = \lambda p_2$$

These imply

$$2 p_1 x_1 = p_2 x_2$$

Using the budget constraint,

$$x_1^* = \frac{m}{3 p_1}$$

$$x_2^* = \frac{2m}{3 p_2}$$

b) The indirect utility is

$$\nu = \frac{4 m^3}{27 p_1 p_2^2}$$

c) The LHS of Roy’s identity is

$$\frac{\partial \nu}{\partial p_1} = - \frac{4 m^3}{27 p_1^2 p_2^2}$$
The RHS of Roy’s identity is

\[ -x_1^* \frac{\partial v}{\partial m} = - \frac{m}{3p_1} \frac{4m^3}{27p_1^2p_2^2} = - \frac{4m^3}{27p_1^2p_2^2} \]

as required.

d) Plugging in, \( x_1 = 10, x_2 = 40 \) and \( v = 16000 \).

e) We need to double the price of \( x_2 \), implying a tax of $5. The agent consumes \( x_2 = 20 \), so the tax revenue is 100. The new utility level is 4000.

f) We need to cut income, reducing it to 150. Tax revenue is 150. The agent’s utility is 2000.

g) The agent prefers the tax on \( x_2 \), because that only reduces her spending on \( x_2 \), whereas the income tax also reduces her consumption of \( x_1 \).

5. Expenditure Minimisation with CES demand

A consumer has the utility \( u(x_1, x_2) = x_1^{1/2} + x_2^{1/2} \). She faces prices \( p_1 = 1 \) and \( p_2 = 2 \).

(a) Fix income \( m \). Find the Marshallian demand for \( x_1 \) and \( x_2 \).

(b) Substituting the Marshallian demands into the utility function, show the indirect utility is

\[ v = \left( \frac{3}{2} \right)^{1/2} \]

(c) Fix target utility, \( u \). Find the Hicksian demand for \( x_1 \) and \( x_2 \).

(d) Find the expenditure function directly, by calculating \( e = p_1 h_1 + p_2 h_2 \).

(e) Verify the expenditure function is the inverse of the indirect utility function.

Solution

a) The Lagrangian is

\[ \mathcal{L} = x_1^{1/2} + x_2^{1/2} + \lambda [m - x_1 - 2x_2] \]
The FOCs are
\[
\frac{1}{2} x_1^{-1/2} = \lambda \\
\frac{1}{2} x_2^{-1/2} = 2\lambda
\]
Dividing we get,
\[
\frac{x_2^{1/2}}{x_1^{1/2}} = \frac{1}{2}
\]
This states that the MRS equals the price ratio. Simplifying, \(x_1 = 4x_2\). Substituting into the constraint,
\[
x_1^* = \frac{4m}{6} \quad \text{and} \quad x_2^* = \frac{m}{6}
\]

b) The indirect utility is
\[
v = (x_1^*)^{1/2} + (x_2^*)^{1/2} = m\frac{\sqrt{4} + \sqrt{1}}{\sqrt{6}} = m\frac{3}{\sqrt{6}} = m\left(\frac{9}{6}\right)^{1/2} = \left(\frac{3}{2}\right)^{1/2}
\]
c) The Lagrangian is
\[
\mathcal{L} = x_1 + 2x_2 + \lambda[u - x_1^{1/2} - x_2^{1/2}]
\]
The FOCs are
\[
1 = \frac{1}{2} \lambda x_1^{-1/2} \\
2 = \frac{1}{2} \lambda x_2^{-1/2}
\]
Dividing we get,
\[
\frac{1}{2} = \frac{x_2^{1/2}}{x_1^{1/2}}
\]
This states that the MRS equals the price ratio. Simplifying, \(x_1 = 4x_2\). Substituting into the constraint, the Hicksian demands are given by
\[
h_1 = \frac{4}{9} u^2 \quad \text{and} \quad h_2 = \frac{1}{9} u^2
\]
d) The expenditure function is
\[
e = p_1 h_1 + p_2 h_2 = \frac{2}{3} u^2
\]
e) Letting $m = e$ and $v = u$ the expenditure function becomes

$$m = \frac{2}{3} v^2$$

Rearranging yields the indirect utility function, as required.
1. Expenditure Minimisation (5 points)

Suppose that an individual has utility \( u(x_1, x_2) = x_1(1 + x_2) \). Throughout assume income \( m \) is sufficiently high so there is an internal solution.

a) Find the Marshallian demand for \( x_1 \) and \( x_2 \).

b) Are \( x_1 \) and \( x_2 \) normal or inferior goods?

c) Find the indirect utility function.

d) Invert the indirect utility function to find the expenditure function.

e) What is the minimum expenditure necessary to achieve a utility level of \( u = 72 \) with \( p_1 = 4 \) and \( p_2 = 2 \)?

2. Calculating Elasticities (4 points)

An agent’s demand for good 1 is given by

\[
x_1^*(p_1, p_2, m) = (m + p_2)/2p_1
\]

Compute the agent’s price elasticity, cross elasticity and income elasticity of \( x_1 \) when \( m = 200 \), \( p_1 = 5 \) and \( p_2 = 10 \).

3. Complements and Substitutes (4 points)

Suppose \( u(x_1, x_2) = x_1x_2 \). Are \( x_1 \) and \( x_2 \) gross complements or substitutes? Are they net complements or substitutes?
4. Slutsky Equation (7 points)

An agent has utility $u(x_1, x_2) = (x_1^{-1} + x_2^{-1})^{-1}$ for goods $x_1$ and $x_2$. The prices of the goods are $p_1$ and $p_2$. The agent has income $m$.

(a) Show preference are convex. You can do this graphically or by showing that MRS is decreasing in $x_1$.

(b) Solve for the agent’s optimal choice of $(x_1, x_2)$.

(c) Show the agent’s indirect utility function is given by

$$v = \frac{m}{(p_1^{1/2} + p_2^{1/2})^2}$$

(d) Solve for the agent’s Hicksian demand.

(e) Solve for the expenditure function.

(f) Verify the Slutsky equation for good $x_1$. In particular, show that both sides are as follows:

$$\frac{\partial x_1^*}{\partial p_1} = -\left[p_1^{-1} + \frac{1}{2}p_1^{-3/2} p_2^{1/2}\right] \frac{m}{(p_1^{1/2} + p_2^{1/2})^2} = \frac{\partial h_1}{\partial p_1} - x_1^* \frac{\partial x_1^*}{\partial m}$$
1. Expenditure Minimisation (5 points)

Suppose that an individual has utility \( u(x_1, x_2) = x_1(1 + x_2) \). Throughout assume income \( m \) is sufficiently high so there is an internal solution.

a) Find the Marshallian demand for \( x_1 \) and \( x_2 \).

b) Are \( x_1 \) and \( x_2 \) normal or inferior goods?

c) Find the indirect utility function.

d) Invert the indirect utility function to find the expenditure function.

e) What is the minimum expenditure necessary to achieve a utility level of \( u = 72 \) with \( p_1 = 4 \) and \( p_2 = 2 \)?

Solution

(a) The Lagrangian is

\[
\mathcal{L} = x_1(1 + x_2) + \lambda[m - p_1 x_1 - p_2 x_2]
\]

The FOCs are

\[
1 + x_2 = \lambda p_1 \\
x_1 = \lambda p_2
\]

We thus get,

\[
\frac{1 + x_2}{x_1} = \frac{p_1}{p_2}
\]

Using the budget constraint,

\[
x_1^* = \frac{m + p_2}{2p_1} \\
x_2^* = \frac{m - p_2}{2p_2}
\]
(b) Since \( x^*_1 \) and \( x^*_2 \) are increasing in \( m \), both are normal goods.

(c) The indirect utility is

\[
v = x^*_1(1 + x^*_2) = \frac{(m + p_2)^2}{4p_1p_2}
\]

(d) Inverting, we can obtain the expenditure function,

\[
e = 2(p_1p_2u)^{1/2} - p_2
\]

(e) Inserting the values in the expenditure function \( e = 2(4 \times 2 \times 72)^{1/2} - 2 = 46 \).

2. Calculating Elasticities (4 points)

An agent’s demand for good 1 is given by

\[
x^*_1(p_1, p_2, m) = \frac{(m + p_2)}{2p_1}
\]

Compute the agent’s price elasticity, cross elasticity and income elasticity of \( x_1 \) when \( m = 200 \), \( p_1 = 5 \) and \( p_2 = 10 \).

Solution

The price elasticity is

\[
\frac{\partial x_1}{\partial p_1} \frac{p_1}{x_1} = -\frac{1}{2} \frac{(m + p_2)}{p_1} \frac{p_1}{(m + p_2)p_1} = -1
\]

The cross elasticity is

\[
\frac{\partial x_1}{\partial p_2} \frac{p_2}{x_1} = 1/21
\]

The incomes elasticity is

\[
\frac{\partial x_1}{\partial m} \frac{m}{x_1} = 20/21
\]

Note that the sum of these is zero, as expected.
3. Complements and Substitutes (4 points)

Suppose \( u(x_1, x_2) = x_1x_2 \). Are \( x_1 \) and \( x_2 \) gross complements or substitutes? Are they net complements or substitutes?

Solution

Marshallian demand is given by \( x_1^* = m/2p_1 \) and \( x_2^* = m/2p_2 \). Since demand for good \( i \) is independent of \( p_j \) these are neither gross complements or gross substitutes.

Hicksian demand is given by \( h_1 = (\bar{u}p_2/p_1)^{1/2} \) and \( h_2 = (\bar{u}p_1/p_2)^{1/2} \). Since \( h_i \) is increasing in \( p_j \), these goods are net substitutes.

4. Slutsky Equation (7 points)

An agent has utility \( u(x_1, x_2) = (x_1^{-1} + x_2^{-1})^{-1} \) for goods \( x_1 \) and \( x_2 \). The prices of the goods are \( p_1 \) and \( p_2 \). The agent has income \( m \).

(a) Show preference are convex. You can do this graphically or by showing that MRS is decreasing in \( x_1 \).

(b) Solve for the agent’s optimal choice of \((x_1, x_2)\).

(c) Show the agent’s indirect utility function is given by

\[
v = \frac{m}{(p_1^{1/2} + p_2^{1/2})^2}
\]

(d) Solve for the agent’s Hicksian demand.

(e) Solve for the expenditure function.

(f) Verify the Slutsky equation for good \( x_1 \). In particular, show that both sides are as follows:

\[
\frac{\partial x_1^*}{\partial p_1} = -\left[ p_1^{-1} + \frac{1}{2} p_1^{-3/2} p_2^{1/2} \right] \frac{m}{(p_1^{1/2} + p_2^{1/2})^2} = \frac{\partial h_1}{\partial p_1} - x_1^* \frac{\partial x_1^*}{\partial m}
\]
Solution

(a) To verify convexity mathematically, you can use two methods. First, observe that the indifference curve is given by

\[(x_1^{-1} + x_2^{-1})^{-1} = \bar{u}\]

Solving for \(x_2\),

\[x_2 = \frac{x_1 \bar{u}}{x_1 - \bar{u}} \quad (1)\]

Differentiating,

\[\text{MRS} = -\frac{dx_2}{dx_1} = \frac{\bar{u}^2}{(x_1 - \bar{u})^2}\]

Differentiating again,

\[\frac{d}{dx_1} \text{MRS} = \frac{-2\bar{u}^2}{(x_1 - \bar{u})^3}\]

Hence MRS is decreasing in \(x_1\) and preferences are convex. Economically, this means that when \(x_1\) is higher, the consumer is willing to give up fewer units of \(x_2\) for a unit of \(x_1\).

The second method is to observe the MRS is

\[\text{MRS} = \frac{MU_1}{MU_2} = \frac{(x_1^{-1} + x_2^{-1})^{-2}x_1^{-2}}{(x_1^{-1} + x_2^{-1})^{-2}x_2^{-2}} = \frac{x_2^2}{x_1^2}\]

Using (1) we obtain

\[\text{MRS} = \frac{\bar{u}^2}{(x_1 - \bar{u})^2}\]

As in part (a).

(b) The tangency condition says that \(MRS = p_1/p_2\). That is,\(^1\)

\[\frac{x_2^2}{x_1^2} = \frac{p_1}{p_2}\]

Rearranging, \(x_1p_1^{1/2} = x_2p_2^{1/2}\). Using the budget constraint,

\[x_1^* = \frac{m}{p_1^{1/2}(p_1^{1/2} + p_2^{1/2})} \quad \text{and} \quad x_2^* = \frac{m}{p_2^{1/2}(p_1^{1/2} + p_2^{1/2})}\]

\(^1\)One can derive the same equation using a Lagrangian or the substitution method.
(c) Substituting,
\[ v = \frac{m}{(p_1^{1/2} + p_2^{1/2})^2} \]
For future reference, it is useful to invert this to find the expenditure function:
\[ e = \pi(p_1^{1/2} + p_2^{1/2})^2 \]

(d) For Hicksian demand, we attain the same tangency condition: \( p_1^{1/2} x_1 = p_2^{1/2} x_2 \). Using the constraint,
\[ (x_1^{-1} + x_2^{-1})^{-1} = \pi \]
we obtain
\[ h_1 = \pi(1 + p_1^{-1/2} p_2^{1/2}) \quad \text{and} \quad h_2 = \pi(1 + p_1^{1/2} p_2^{-1/2}) \]

(e) Plugging in, the expenditure function is
\[ e = p_1 h_1 + p_2 h_2 = \pi(p_1^{1/2} + p_2^{1/2})^2 \]
Just to check this is correct, one can verify this is the same as the answer in part (f).

(f) The LHS of the Slutsky equation is
\[ \frac{\partial x_i^*}{\partial p_1} - x_1 \frac{\partial x_i^*}{\partial m} = -\frac{1}{2} \pi p_1^{-3/2} p_2^{1/2} - \frac{m p_1^{-1/2}}{(p_1^{1/2} + p_2^{1/2})^2} - \frac{1}{2} p_1^{-1} \frac{m}{(p_1^{1/2} + p_2^{1/2})^2} \]
\[ = - \left[ p_1^{-1} + \frac{1}{2} p_1^{-3/2} p_2^{1/2} \right] \frac{m}{(p_1^{1/2} + p_2^{1/2})^2} \]
The RHS of the Slutsky equation is
\[ \frac{\partial h_1}{\partial p_1} - x_1 \frac{\partial h_1}{\partial m} = -\frac{1}{2} \pi p_1^{-3/2} p_2^{1/2} - \frac{m p_1^{-1/2}}{(p_1^{1/2} + p_2^{1/2})^2} - p_1^{-1} \frac{m}{(p_1^{1/2} + p_2^{1/2})^2} \]
\[ = - \left[ p_1^{-1} + \frac{1}{2} p_1^{-3/2} p_2^{1/2} \right] \frac{m}{(p_1^{1/2} + p_2^{1/2})^2} \]
which is the same as the LHS.
Economics 11: Homework 6

1. Complements or Substitutes? (4 points)

A consumer has utility function

\[ u(x_1, x_2) = \frac{x_1^\delta}{\delta} + \frac{x_2^\delta}{\delta} \]

She has income \( m \) and faces prices \( p_1 \) and \( p_2 \). [Throughout this question, assume there is internal solution to the agent’s problem.]

(a) Suppose \( \delta = 1/2 \). Calculate the Marshallian demand for good 1 and 2. Are the goods gross substitutes or complements?

(b) Suppose \( \delta = -1/2 \). Calculate the Marshallian demand for good 1 and 2. Are the goods gross substitutes or complements?

2. Consumer Surplus with Quasilinear Utility (4 points)

Suppose an agent has utility \( u(x_1, x_2) = 2x_1^{1/2} + x_2 \). The agent has income \( m \) and faces prices \( p_1 \) and \( p_2 \). [Throughout this question assume there is an internal optimum].

(a) Calculate the agent’s Marshallian demand.

(b) Calculate the agent’s indirect utility, \( v(p_1, p_2, m) \).

(c) Fix a target utility \( \bar{u} \). Calculate the agent’s Hicksian demand.

(d) Calculate the agent’s expenditure function.

Suppose the agent starts with income \( m = 10 \) and faces prices \( p_1 = 1 \) and \( p_2 = 1 \).

(e) Calculate the utility of the agent, \( \bar{u} = v(p_1, p_2, m) \).

(f) Suppose \( p_1 = 1 \) rises to \( p'_1 = 2 \). How much money must the Government give the agent to compensate for this price rise, so that her utility remains at \( \bar{u} \)?
3. Consumer Surplus with CES utility (4 points)

An agent has utility \( u(x_1, x_2) = x_1^{1/2} + x_2^{1/2} \), giving rise to indirect utility

\[
v(p_1, p_2, m) = \left[ m \frac{p_1 + p_2}{p_1 p_2} \right]^{1/2}.
\]

She has income \( m = 4 \) and faces prices \( p_1 = 2 \) and \( p_2 = 2 \). Suppose the price of good 1 rises to \( p'_1 = 3 \). Calculate the increase in the agent’s income required to compensate her for this price rise.

4. Labour Supply (4 points)

Suppose an agent has the same utility function as in question 4 on HW5,

\[
u(x_1, x_2) = (x_1^{-1} + x_2^{-1})^{-1}
\]

Suppose that \( x_1 \) is hours of leisure and \( x_2 \) is quantity of food. The agent is endowed with \( T \) hours to divide between work and leisure. Her wage rate is \( w \), while the price of \( x_2 \) is \( p_2 = 1 \). The agent has no outside income, \( m = 0 \).

(a) Derive the budget constraint of the agent.

(b) Solve for the agent’s optimal choice of \((x_1, x_2)\). [Hint: the answer is very similar to 4(a) from HW5.]

(c) How does the agent’s leisure consumption change as a function of her wage? Explain this intuitively in terms of income and substitution effects.

5. Intertemporal Choice with Differential Interest Rates (4 points)

[Note: Parts (b) and (c) of this question is harder than normal. Nevertheless, it’s an important economic application.]

An agent allocates consumption across two periods. Let the consumption in period \( t \) be \( x_t \), and
the income in period $t$ be $m_t$. The agent’s utility is

$$u(x_1, x_2) = \ln(x_1) + \frac{3}{4} \ln(x_2)$$

(using the notation from class, this corresponds to $\beta = 1/3$). The agent is poor in period 1 but wealthy in period 2. In particular, she has income $m_1 = 3$ in period 1 and $m_2 = 4$ in period 2.

(a) Suppose the agent can borrow and save at interest rate $r = 1/3$, so $\$1$ in period 1 is worth $\$(1 + 1/3)$ in period 2.

(i) Sketch the agent’s budget constraint.

(ii) Solve for her optimal consumption $(x_1^*, x_2^*)$.

(iii) Given her optimal consumption, does the agent borrow or save in period 1? [Note: The agent saves if $m_1 > x_1$ and borrows if $m_1 < x_1$].

(b) Suppose the agent can still save at $r = 1/3$, but can only borrow at $r = 1/2$.

(i) Sketch the agent’s budget constraint [Hint: There’s a kink in it at the agent’s endowment].

(ii) Solve for her optimal consumption. [Hint: Give your answer in part (a) you should know whether the agent is interested in saving or borrowing, and therefore which interest rate is relevant.]

(c) Suppose the agent can still save at $r = 1/3$, but can only borrow at $r = 1$.

(i) Sketch the agent’s budget constraint.

(ii) Solve for her optimal consumption. [Hint: Be careful of the kink.]
1. Complements or Substitutes? (4 points)

A consumer has utility function

\[ u(x_1, x_2) = \frac{x_1^\delta}{\delta} + \frac{x_2^\delta}{\delta} \]

She has income \( m \) and faces prices \( p_1 \) and \( p_2 \). [Throughout this question, assume there is internal solution to the agent’s problem.]

(a) Suppose \( \delta = 1/2 \). Calculate the Marshallian demand for good 1 and 2. Are the goods gross substitutes or complements?

(b) Suppose \( \delta = -1/2 \). Calculate the Marshallian demand for good 1 and 2. Are the goods gross substitutes or complements?

Solution

I’m going to derive demand for an arbitrary CES utility function. Setting the MRS equal to the price ratio,

\[ \frac{x_1^\delta}{x_2^\delta} = \frac{p_1}{p_2} \]

Rearranging,

\[ x_2 = x_1 \left( \frac{p_1}{p_2} \right)^{1/(1-\delta)} \]

Substituting into the budget constraint,

\[ m = x_1 \left[ p_1 + x_2 \left( \frac{p_1}{p_2} \right) \right]^{1/(1-\delta)} = x_1 p_1 \left[ 1 + \frac{\delta/(1-\delta)}{\delta/(1-\delta)} \right] \]

Hence demand is

\[ x_1^* = \frac{m}{p_1} \left[ 1 + \frac{\delta/(1-\delta)}{\delta/(1-\delta)} \right]^{-1} \quad \text{and} \quad x_2^* = \frac{m}{p_2} \left[ 1 + \frac{\delta/(1-\delta)}{\delta/(1-\delta)} \right]^{-1} \]
Rearranging, the demand for good 1 can be written as

\[
x_1^* = \frac{m}{p_1} \left[ 1 + \frac{p_2^{-\delta/(1-\delta)}}{p_1^{-\delta/(1-\delta)}} \right]^{-1} = \frac{m}{p_1^{1/(1-\delta)}} \left( \frac{1}{p_1^{-\delta/(1-\delta)}} + p_2^{-\delta/(1-\delta)} \right) = \frac{m}{p_1^{1/(1-\delta)}} \frac{1}{p_1^{-\sigma} + p_2^{-\sigma}}
\]

where \(\sigma = 1/(1-\delta)\) is the elasticity of substitution. The symmetric expression holds for \(x_2^*\).

When \(\delta = 1/2\), we have

\[
x_1^* = \frac{m}{p_1} \left[ 1 + \frac{p_1}{p_2} \right]^{-1} \quad \text{and} \quad x_2^* = \frac{m}{p_2} \left[ 1 + \frac{p_2}{p_1} \right]^{-1}
\]

When \(\delta = -1/2\), we have

\[
x_1^* = \frac{m}{p_1} \left[ 1 + \frac{p_1^{-1/3}}{p_2^{-1/3}} \right]^{-1} \quad \text{and} \quad x_2^* = \frac{m}{p_2} \left[ 1 + \frac{p_2^{-1/3}}{p_1^{-1/3}} \right]^{-1}
\]

Differentiating,

\[
\frac{\partial x_1^*}{\partial p_2} = \frac{m}{p_1^2} (\sigma - 1) \frac{p_2^{-\sigma}}{(p_1^{-\sigma} + p_2^{-\sigma})^2}
\]

Differentiating \(x_2^*\) with respect to \(p_1\) is symmetric. If \(\delta > 0\) then \(\sigma > 0\) and \(\partial x_1^*/\partial p_2 > 0\), implying that the goods are substitutes. Conversely, if \(\delta < 0\) then \(\partial x_1^*/\partial p_2 < 0\) and the goods are complements.

Note that perfect complements are equivalent to \(\delta = -\infty\), symmetric Cobb–Douglas is equivalent to \(\delta = 0\) and perfect substitutes are equivalent to \(\delta = 1\), so this result should not be a surprise.

2. Consumer Surplus with Quasilinear Utility (4 points)

Suppose an agent has utility \(u(x_1, x_2) = 2x_1^{1/2} + x_2\). The agent has income \(m\) and faces prices \(p_1\) and \(p_2\). [Throughout this question assume there is an internal optimum].

(a) Calculate the agent’s Marshallian demand.

(b) Calculate the agent’s indirect utility, \(v(p_1, p_2, m)\).

(c) Fix a target utility \(\bar{u}\). Calculate the agent’s Hicksian demand.
(d) Calculate the agent’s expenditure function.

Suppose the agent starts with income \( m = 10 \) and faces prices \( p_1 = 1 \) and \( p_2 = 1 \).

(e) Calculate the utility of the agent, \( \bar{u} = v(p_1, p_2, m) \).

(f) Suppose \( p_1 = 1 \) rises to \( p_1' = 2 \). How much money must the Government give the agent to compensate for this price rise, so that her utility remains at \( \bar{u} \)?

Solution

(a) The tangency condition is

\[
x_1^{-1/2} = \frac{p_1}{p_2}
\]

Rearranging, demand is

\[
x_1^*(p_1, p_2, m) = \frac{p_2^2}{p_1^2}
\]

Using the budget constraint,

\[
x_2^*(p_1, p_2, m) = \frac{m}{p_2} - \frac{p_2}{p_1}.
\]

(b) The agent’s indirect utility is thus:

\[
v(p_1, p_2, m) = 2x_1^{1/2} + x_2 = \frac{m}{p_2} + \frac{p_2}{p_1}
\]

(c) The tangency condition is the same as in part (a). Rearranging, Hicksian demand is

\[
h_1(p_1, p_2, \bar{u}) = \frac{p_2^2}{p_1^2}
\]

Using the constraint, we have

\[
h_2^*(p_1, p_2, \bar{u}) = \bar{u} - 2\frac{p_2}{p_1}.
\]

(d) The expenditure function is

\[
e(p_1, p_2, \bar{u}) = p_1 h_1 + p_2 h_2 = p_2 \left( \bar{u} - \frac{p_2}{p_1} \right)
\]

(e) Under these parameters, \( v(p_1, p_2, m) = 11 \).
(f) There are three ways to calculate this. First, using the expenditure function,

\[ CV = e(p'_1, p_2, \bar{u}) - e(p_1, p_2, \bar{u}) = \frac{p_2^2}{p_1} - \frac{p_2^2}{p'_1} = 1 - \frac{1}{2} = \frac{1}{2} \]

Second, using the Hicksian demand,

\[ CV = \int_1^2 h_1(p_1, p_2, \bar{u}) dp_1 = p_2^2 \int_1^2 p_1^{-2} dp_1 = p_2^2[-p_1^{-1}]_1^2 = 1 - \frac{1}{2} = \frac{1}{2}. \]

Third, since utility is quasilinear, we can use the Marshallian demand (which is identical to Hicksian demand),

\[ CV = \int_1^2 x_1^*(p_1, p_2, m) dp_1 = p_2^2 \int_1^2 p_1^{-2} dp_1 = p_2^2[-p_1^{-1}]_1^2 = 1 - \frac{1}{2} = \frac{1}{2}. \]

In this case using the Marshallian demand may be easiest since it avoids us having to solve the EMP.

3. Consumer Surplus with CES utility (4 points)

An agent has utility \( u(x_1, x_2) = x_1^{1/2} + x_2^{1/2} \), giving rise to indirect utility

\[ v(p_1, p_2, m) = \left[ m \frac{p_1 + p_2}{p_1 p_2} \right]^{1/2}. \]

She has income \( m = 4 \) and faces prices \( p_1 = 2 \) and \( p_2 = 2 \). Suppose the price of good 1 rises to \( p'_1 = 3 \). Calculate the increase in the agent’s income required to compensate her for this price rise.

Solution

Plugging in the parameters, we see \( v(p_1, p_2, m) = 2 \). Inverting the indirect utility function we obtain the expenditure function,

\[ e(p_1, p_2, \bar{u}) = u^2 \frac{p_1 p_2}{p_1 + p_2} \]
Letting, \( \bar{u} = 2 \), the compensating variation is
\[
CV = e(p_1', p_2, \bar{u}) - e(p_1, p_2, \bar{u}) = u^2 \frac{p_1' p_2}{p_1' + p_2} - u^2 \frac{p_1 p_2}{p_1 + p_2} = \frac{4}{5}
\]

4. Labour Supply (4 points)

Suppose an agent has the same utility function as in question 4 on HW5,
\[
u(x_1, x_2) = (x_1^{-1} + x_2^{-1})^{-1}
\]
Suppose that \( x_1 \) is hours of leisure and \( x_2 \) is quantity of food. The agent is endowed with \( T \) hours to divide between work and leisure. Her wage rate is \( w \), while the price of \( x_2 \) is \( p_2 = 1 \). The agent has no outside income, \( m = 0 \).

(a) Derive the budget constraint of the agent.

(b) Solve for the agent’s optimal choice of \((x_1, x_2)\). [Hint: the answer is very similar to 4(a) from HW5.]

(c) How does the agent’s leisure consumption change as a function of her wage? Explain this intuitively in terms of income and substitution effects.

Solution

(a) The budget is
\[(T - x_2)w = x_1\]
Equivalently,
\[x_1 + wx_2 = wT\]
We can thus think of \( wT \) as the income of the agent.

(b) The tangency condition says that \( x_1 p_1^{1/2} = x_2 p_2^{1/2} \). Using the budget constraint,
\[x_1^* = \frac{wT}{w^{1/2}(1 + w^{1/2})} \quad \text{and} \quad x_2^* = \frac{wT}{(1 + w^{1/2})}\]
Note: This is the same as last week, where we have replaced \( m \) with \( wT \).
(c) Leisure is increasing in $w$. Intuitively, as $w$ increase the value of the agent’s endowment $wT$ increases, leading to an increase in the demand for leisure. This income effect dominates the substitution effect.

5. Intertemporal Choice with Differential Interest Rates (4 points)

[Note: Parts (b) and (c) of this question is harder than normal. Nevertheless, it’s an important economic application.]

An agent allocates consumption across two periods. Let the consumption in period $t$ be $x_t$, and the income in period $t$ be $m_t$. The agent’s utility is

$$u(x_1, x_2) = \ln(x_1) + \frac{3}{4} \ln(x_2)$$

(using the notation from class, this corresponds to $\beta = 1/3$). The agent is poor in period 1 but wealthy in period 2. In particular, she has income $m_1 = 3$ in period 1 and $m_2 = 4$ in period 2.

(a) Suppose the agent can borrow and save at interest rate $r = 1/3$, so $\$1$ in period 1 is worth $\$(1 + 1/3) in period 2.
  (i) Sketch the agent’s budget constraint.
  (ii) Solve for her optimal consumption $(x_1^*, x_2^*)$.
  (iii) Given her optimal consumption, does the agent borrow or save in period 1? [Note: The agent saves if $m_1 > x_1$ and borrows if $m_1 < x_1$].

(b) Suppose the agent can still save at $r = 1/3$, but can only borrow at $r = 1/2$.
  (i) Sketch the agent’s budget constraint [Hint: There’s a kink in it at the agent’s endowment].
  (ii) Solve for her optimal consumption. [Hint: Give your answer in part (a) you should know whether the agent is interested in saving or borrowing, and therefore which interest rate is relevant.]

(c) Suppose the agent can still save at $r = 1/3$, but can only borrow at $r = 1$.
  (i) Sketch the agent’s budget constraint.
  (ii) Solve for her optimal consumption. [Hint: Be careful of the kink.]
Solution

(a) Writing everything in terms of period 1 money, the agent’s budget constraint is

\[ m_1 + \frac{3}{4}m_2 = x_1 + \frac{3}{4}x_2 \]

The tangency condition states that \( MRS = \frac{p_1}{p_2} \). That is,

\[ \frac{4x_2}{3x_1} = \frac{4}{3} \]

which implies \( x_1 = x_2 \). The LHS of the budget equation is 6. Using this, we obtain \( x_1^* = x_2^* = 24/7 \). Since \( x_1^* > 3 \), the agent borrows in period 1.

(b) To the left of \((x_1, x_2) = (3, 4)\), the agent’s budget constraint has a slope of \(-4/3\). To the right it has a slope of \(-3/2\). From part (a) we know the agent does not wish to save at \( r = 1/3 \). When the agent borrows, the equation for the budget line is

\[ m_1 + \frac{2}{3}m_2 = x_1 + \frac{2}{3}x_2 \]

Suppose the agent wishes to borrow a positive amount of money. The tangency condition states that

\[ \frac{4x_2}{3x_1} = \frac{3}{2} \]

which implies \( x_2 = \frac{9}{5}x_1 \), so the agent now wishes to consume more in the second period due to the high interest rate of borrowing. The LHS of the budget equation is 17/3. Using this, we obtain \( x_1^* = 68/21 \) and \( x_2^* = 51/14 \). Observe \( x_1^* > 3 \), so the agent still borrows.

(c) To the left of \((x_1, x_2) = (3, 4)\), the agent’s budget constraint has a slope of \(-4/3\). To the right it has a slope of \(-2\). As before, the agent will never wish to save. When she borrows, the equation for the budget line is

\[ m_1 + \frac{1}{2}m_2 = x_1 + \frac{1}{2}x_2 \]

Suppose the agent wishes to borrow a positive amount of money. The tangency condition states that

\[ \frac{4x_2}{3x_1} = 2 \]

which implies \( x_2 = \frac{3}{2}x_1 \). The LHS of the budget equation is 5. Using this, we obtain \( x_1^* = 20/7 \) and \( x_2^* = 30/7 \). Observe \( x_1^* < 3 \) so the agent is not willing to borrow at this interest rate. Since they are not willing to save at \( r = 1/3 \), she must choose to consume her endowment:
\((x_1^*, x_2^*) = (3, 4)\).

Another way of seeing the same result is to note that the agent prefers to consume at the kink if

\[
\frac{p_1}{p_2} \mid x_1 < 3 \geq \frac{MU_1}{MU_2} \mid x_1 = 3 \geq \frac{p_1}{p_2} \mid x_1 > 3
\]

Plugging in the numbers,

\[
\frac{4}{3} \geq \frac{16}{9} \geq 2
\]

we see that the agent neither wishes to save or borrow.
Economics 11: Homework 7

November 6, 2009

1. Production with One Input (4 points)

Digging clams by hand in Sunset Bay requires only labour input. The total number of clams obtained per hour \((q)\) is given by: \(q = 100L^{1/2}\) Where \(L\) is he labour input per hour.

a) Graph the relationship between \(q\) and \(L\)

b) Find the marginal and average product of labour.

2. Production Function (4 points)

A firm has one input, \(z\), and the following production function

\[
f(z) = \begin{cases} 2x & \text{if } 1 \geq x \\ 2 & \text{if } 2 \geq x \geq 1 \\ x & \text{if } x \geq 2 \end{cases}
\]

(a) Is this production function concave?

(b) Does it exhibit decreasing, constant or increasing returns to scale (or neither)?

3. Cost Functions (4 points)

A firms has cost function \(c(q) = 100 - 10q + 5q^2\).

a) Find the fixed cost.\(^1\)

b) Find the variable cost.\(^2\)

\(^1\)Definition: The fixed cost is the cost that is independent of output.

\(^2\)Definition: The variable cost is the cost that varies with the level of output.
c) Find the average cost.

d) Find the marginal cost.

e) Draw the relationship between MC and AC. Prove that they always intersect at the minimum AC.

4. Cost Minimisation: Cobb Douglas (4 points)

Suppose that a firm production function is given by the Cobb-Douglas function: \( f(z_1, z_2) = z_1^\alpha z_2^\beta \). The cost of the inputs is \( z_1 \) and \( z_2 \).

a) Find marginal and average productivity of the two factors.

b) Does this production function have increasing, constant or decreasing returns to scale?

c) Show that cost minimisation requires \( \beta r_1 z_1 = \alpha r_2 z_2 \).

d) Suppose \( \alpha = \beta = 1/4 \). Find the cost function.

5. Cost Minimisation Problem (4 points)

A firm has production function

\[
f(z_1, z_2) = z_1^{1/2}(z_2 - 1)^{1/2}
\]

The prices of the inputs are \( r_1 \) and \( r_2 \).

(a) Find \( MP_1 \), \( MP_2 \), and \( MRTS \).

(b) If \( z_2 \) is fixed at 5, what is the short-run cost function? Find the short-run marginal cost and average cost.

(c) What are the long-run input demand functions? What is the long-run cost function? Find the long-run marginal cost and average cost.

(d) Does the production function exhibit increasing, constant or decreasing returns to scale.
1. Production with One Input (4 points)

Digging clams by hand in Sunset Bay requires only labour input. The total number of clams obtained per hour \(q\) is given by: \(q = 100L^{1/2}\) Where \(L\) is he labour input per hour.

a) Graph the relationship between \(q\) and \(L\)

b) Find the marginal and average product of labour.

Solution

(a) This is a concave function starting at the origin.

(b) The marginal product of labour is

\[
MP_L = \frac{dq}{dL} = 50L^{-1/2}
\]

The average product is

\[
AP_L = \frac{q}{L} = 100L^{-1/2}
\]

2. Production Function (4 points)

A firm has one input, \(z\), and the following production function

\[
f(z) = \begin{cases} 
2z & \text{if } 1 \geq z \\
2 & \text{if } 2 \geq z \geq 1 \\
z & \text{if } z \geq 2 
\end{cases}
\]

(a) Is this production function concave?

(b) Does it exhibit decreasing, constant or increasing returns to scale (or neither)?
Solution

[This example shows that a function that exhibits DRS may not be concave. In contrast, a function that is concave exhibits DRS, as mentioned in class.]

(a) It is not concave. In particular, there is an upward kink at $x = 2$.

(b) It exhibits decreasing returns to scale. This can be seen with a picture. Pick any point on the production function, and dray a ray through that point and the origin. Then every point with higher $x$ is below the ray, implying decreasing returns.

3. Cost Functions (4 points)

A firm has cost function $c(q) = 100 - 10q + 5q^2$.

a) Find the fixed cost.\(^1\)

b) Find the variable cost.\(^2\)

c) Find the average cost.

d) Find the marginal cost.

e) Draw the relationship between MC and AC. Prove that they always intersect at the minimum AC.

Solution

(a) The fixed cost is 100.

(b) The variable cost is $-10q + 5q^2$.

(c) The average cost is

$$AC(q) = \frac{100}{q} - 10 + 5q$$

\(^1\)Definition: The fixed cost is the cost that is independent of output.

\(^2\)Definition: The variable cost is the cost that varies with the level of output.
(d) The marginal cost is
\[ MC(q) = -10 + 10q \]

(e) \( AC \) is a convex function. Differentiating, it is minimised when
\[ \frac{d}{dq} AC(q) = -100q^{-2} + 5q = 0 \]
Rearranging, we find that \( q = \sqrt{20} \).

We have \( AC = MC \) when
\[ \frac{100}{q} - 10 + 5q = -10 + 10q \]
Rearranging, we find that \( q = \sqrt{20} \).

4. Cost Minimisation: Cobb Douglas (4 points)

Suppose that a firm production function is given by the Cobb-Douglas function: \( f(z_1, z_2) = z_1^\alpha z_2^\beta \). The cost of the inputs is \( z_1 \) and \( z_2 \).

a) Find marginal and average productivity of the two factors.

b) Does this production function have increasing, constant or decreasing returns to scale?

c) Show that cost minimisation requires \( \beta r_1 z_1 = \alpha r_2 z_2 \).

d) Suppose \( \alpha = \beta = 1/4 \). Find the cost function.

Solution

(a) The average productivity of 1 is
\[ AP_1 = z_1^{\alpha-1} z_2^\beta \]
The marginal productivity of 1 is
\[ MP_1 = \alpha z_1^{\alpha-1} z_2^\beta \]
Factor 2 is similar.
(b) It has constant returns if \( \alpha + \beta = 1 \). It has increasing returns if \( \alpha + \beta > 1 \). It has decreasing returns if \( \alpha + \beta < 1 \). To see this, just use the definition of returns to scale.

(c) We have

\[
MRTS = \frac{MP_1}{MP_2} = \frac{\alpha z_1^{\alpha-1} z_2^\beta}{\beta z_1^\alpha z_2^{\beta-1}} = \frac{\alpha z_2}{\beta z_1}
\]

The tangency condition is therefore

\[
\frac{\alpha z_2}{\beta z_1} = \frac{r_1}{r_2}
\]

Rearranging, yields the result.

(d) The tangency condition becomes \( r_1 z_1 = r_2 z_2 \). The constraint is

\[
q = z_1^{1/4} z_2^{1/4}
\]

Solving yields,

\[
z_1^* = \left( \frac{r_2}{r_1} \right)^{1/2} q^2 \quad \text{and} \quad z_2^* = \left( \frac{r_1}{r_2} \right)^{1/2} q^2
\]

The cost is

\[
c(q) = r_1 z_1^* + r_2 z_2^* = 2 \left( r_1 r_2 \right)^{1/2} q^2
\]
1. Cost Minimisation Problem (5 points)

A firm has production function

\[ f(z_1, z_2) = z_1^{1/2}(z_2 - 1)^{1/2} \]

The prices of the inputs are \( r_1 \) and \( r_2 \).

(a) Find \( MP_1 \), \( MP_2 \), and \( MRTS \).

(b) If \( z_2 \) is fixed at 5, what is the short-run cost function? Find the short-run marginal cost and average cost.

(c) What are the long-run input demand functions? What is the long-run cost function? Find the long-run marginal cost and average cost.

(d) Does the production function exhibit increasing, constant or decreasing returns to scale.

2. Cost Minimisation with Quasilinear Production (5 points)

A firm has production function

\[ f(z_1, z_2) = 2z_1^{1/2} + z_2 \]

The firm’s inputs cost \( r_1 \) and \( r_2 \).

(a) Does this production function exhibit decreasing, constant or increasing returns to scale?

(b) Derive the MRTS and show that the isoquants are convex.

(c) Solve the firm’s cost minimisation problem. You may find it easy to do this in two parts. First, assume

\[ q \geq 2 \frac{r_2}{r_1} \]
so there is an internal solution and solve for the resulting cost function. Second, assume

\[ q \leq \frac{r_2}{r_1} \]

so there is a boundary solution and solve for the resulting cost function.

(d) Is the cost function concave or convex in \( q \)? Explain this finding.

3. Basic Profit Maximisation (5 points)

A firm has production function \( f(z) = 2z^{1/2} \). The output price is \( p \); the input price is \( r \).

What is the firm’s optimal output? What is the optimal input? What is the firm’s profit?

4. Firm’s Problem with CES Production (5 points)

A firm has production function

\[ q = f(z_1, z_2) = z_1^{1/2} + z_2^{1/2} \]

The firm’s inputs cost \( r_1 \) and \( r_2 \). The output price is \( p \).

(a) Suppose \( z_2 = 1 \), and assume \( q \geq 1 \) to avoid boundary problems. Derive the firm’s short–run cost and short–run marginal cost.

(b) Calculate the long-run cost function, when both factors are flexible.

(c) Using the cost function in (b), solve for the firm’s profit maximising output, and their maximal profit.

(d) Solve the firm’s profit maximisation problem directly, whereby the firm chooses \( (z_1, z_2) \) to maximise

\[ \pi = pf(z_1, z_2) - r_1 z_1 - r_2 z_2 \]

What are the optimal inputs? What is the firm’s output and profit?
1. Cost Minimisation Problem (5 points)

A firm has production function

\[ f(z_1, z_2) = z_1^{1/2}(z_2 - 1)^{1/2} \]

The prices of the inputs are \( r_1 \) and \( r_2 \).

(a) Find \( MP_1 \), \( MP_2 \), and \( MRTS \).

(b) If \( z_2 \) is fixed at 5, what is the short-run cost function? Find the short-run marginal cost and average cost.

(c) What are the long-run input demand functions? What is the long-run cost function? Find the long-run marginal cost and average cost.

(d) Does the production function exhibit increasing, constant or decreasing returns to scale.

Solution

(a) The marginal products are

\[ MP_1 = \frac{1}{2} z_1^{-1/2}(z_2 - 1)^{1/2} \]
\[ MP_2 = \frac{1}{2} z_1^{1/2}(z_2 - 1)^{-1/2} \]

Hence we have

\[ MRTS = \frac{MP_1}{MP_2} = \frac{z_2 - 1}{z_1} \]

(b) The cost minimisation problem is to choose \( z_1 \) to minimise \( r_1 z_1 + 5r_2 \) subject to \( 2z_1^{1/2} \geq q \).
At the optimum, the constraint binds and we have
\[ z_1^* = \frac{q^2}{4} \]
and the short-run total cost is
\[ SRTC(q) = r_1 \frac{q^2}{4} + 5r_2 \]
The average cost is
\[ SRAC = r_1 \frac{q}{4} + \frac{5r_2}{q} \]
The marginal cost is
\[ SRMC = r_1 \frac{q}{2} \]

(c) In the long run, both factors are flexible. The tangency condition is
\[ \frac{z_2 - 1}{z_1} = \frac{r_1}{r_2} \]
Rearranging, we have \((z_2 - 1)r_2 = z_1r_1\). The constraint says that
\[ q = z_1^{1/2}(z_2 - 1)^{1/2} \]
Substituting, we see that
\[ z_1^* = \left( \frac{r_2}{r_1} \right)^{1/2} q \quad \text{and} \quad z_2^* = 1 + \left( r_1 \frac{r_2}{r_1} \right)^{1/2} \frac{q}{r_2} \quad q \]
The cost is therefore
\[ c(q) = 2q \sqrt{r_1 r_2} + r_2 \]
Intuitively, after the firm pays a fixed cost of 1 unit of \(z_2\), they have Cobb-Douglas technology.
The average cost is
\[ AC(q) = 2 \sqrt{r_1 r_2} + \frac{r_2}{q} \]
The marginal cost is
\[ MC(q) = 2 \sqrt{r_1 r_2} \]

(d) It exhibits increasing returns. You can verify this directly from the production function:
\[ f(\alpha z_1, \alpha z_2) = (\alpha z_1)^{1/2} (\alpha z_2 - 1)^{1/2} \geq (\alpha z_1)^{1/2} (z_2 - 1)^{1/2} = \alpha f(z_1, z_2) \]
You can also verify this from the cost function, which is concave in $q$.

2. Cost Minimisation with Quasilinear Production (5 points)

A firm has production function

$$f(z_1, z_2) = 2z_1^{1/2} + z_2$$

The firm’s inputs cost $r_1$ and $r_2$.

(a) Does this production function exhibit decreasing, constant or increasing returns to scale?

(b) Derive the MRTS.

(c) Show that the isoquants are convex.

(d) Solve the firm’s cost minimisation problem. You may find it easy to do this in two parts. First, assume

$$q \geq \frac{r_2}{r_1}$$

so there is an internal solution and solve for the resulting cost function. Second, assume

$$q \leq \frac{r_2}{r_1}$$

so there is a boundary solution and solve for the resulting cost function.

(e) Is the cost function concave or convex in $q$? Explain this finding.

Solution

(a) Decreasing returns. For $t > 1$,

$$f(tz_1, tz_2) = 2t^{1/2}z_1^{1/2} + tz_2 \leq 2tz_1^{1/2} + tz_2 = tf(z_1, z_2)$$

(b) We have

$$MRTS = \frac{MP_1}{MP_2} = \frac{z_1^{-1/2}}{1} = z_1^{-1/2}$$
(c) From part (b), MRTS is clearly decreasing in $z_1$.

(d) The firm minimises the Lagrangian

$$\mathcal{L} = r_1 z_1 + r_2 z_2 + \lambda [q - 2z_1^{1/2} - z_2]$$

Assuming an internal optimum, the FOCs are

$$r_1 = \lambda z_1^{-1/2}$$
$$r_2 = \lambda$$

These yield

$$z_1^* = \frac{r_2^2}{r_1^2} \quad \text{and} \quad z_2^* = q - 2\frac{r_2}{r_1}$$

The cost function is

$$c(q) = r_1 z_1^* + r_2 z_2^* = qr_2 - \frac{r_2^2}{r_1}$$

For $q \leq 2\frac{r_2}{r_1}$, we have $z_2^* = 0$. The production function is therefore $q = 2z_1^{1/2}$. Inverting,

$$z_1^* = \frac{1}{4} q^2 \quad \text{and} \quad z_2^* = 0$$

The cost function is

$$c(q) = r_1 z_1^* + r_2 z_2^* = \frac{r_1}{4} q^2$$

(e) This cost function is convex in $q$. This follows from the concavity of the production function. Intuitively, $z_1$ has decreasing marginal product so the marginal cost increases in $q$.

3. Basic Profit Maximisation (5 points)

A firm has production function $f(z) = 2z^{1/2}$. The output price is $p$; the input price is $r$.

What is the firm’s optimal output? What is the optimal input? What is the firm’s profit?
Solution

The firm maximises

\[ 2pz^{1/2} - rz \]

The FOC is

\[ pz^{-1/2} = r \]

Rearranging,

\[ z^* = \frac{p^2}{r^2} \]

Output is

\[ y^* = 2(z^*)^{1/2} = \frac{2p}{r} \]

Profit is

\[ \pi^* = py^* - rz^* = 2\frac{p^2}{r} - \frac{p^2}{r} = \frac{p^2}{r} \]

4. Firm’s Problem with CES Production (5 points)

A firm has production function

\[ f(z_1, z_2) = z_1^{1/2} + z_2^{1/2} \]

The firm’s inputs cost \( r_1 \) and \( r_2 \). The output price is \( p \).

(a) Suppose \( z_2 = 1 \), and assume \( q \geq 1 \) to avoid boundary problems. Derive the firm’s short-run cost and short-run marginal cost.

(b) In the long-run both factors are flexible. Show that the cost function is given by

\[ c(q, r_1, r_2) = \frac{r_1r_2}{r_1 + r_2 q^2} \]

(c) Using the cost function in (b), solve for the firm’s profit maximising output, and their maximal profit.

(d) Solve the firm’s profit maximisation problem using the one-step method, whereby the firm chooses \((z_1, z_2)\) to maximise

\[ \pi = pf(z_1, z_2) - r_1z_1 - r_2z_2 \]

What are the optimal inputs? What is the firm’s output and profit?
Solution

(a) The firm’s short run production function is

\[ f(z_1, z_2) = z_1^{1/2} + 1 \]

Hence the inputs needed to produce \( q \) are given by

\[ z_1^* = (q - 1)^2 \]

where we assumed that \( q \geq 1 \). The cost function is given by

\[ SRTC(q) = r_1 z_1 + r_2 z_2 = r_1 (q - 1)^2 + r_2 \]

The marginal cost is

\[ SRMC(q) = 2r_1(q - 1) \]

(b) The firm minimises the Lagrangian

\[ \mathcal{L} = r_1 z_1 + r_2 z_2 + \lambda [q - z_1^{1/2} - z_2^{1/2}] \]

The FOCs are

\[ r_1 = \lambda \frac{1}{2} z_1^{-1/2} \]
\[ r_2 = \lambda \frac{1}{2} z_2^{-1/2} \]

Rearranging, we have

\[ \frac{r_1^2}{r_2^2} = \frac{z_2}{z_1} \]

Substituting into the production function, we obtain

\[ z_1^* = \left( q \frac{r_2}{r_1 + r_2} \right)^2 \quad \text{and} \quad z_2^* = \left( q \frac{r_1}{r_1 + r_2} \right)^2 \]

The cost function is given by

\[ c(q) = r_1 z_1^* + r_2 z_2^* = \frac{r_1 r_2}{r_1 + r_2} q^2 \]

[Note the similarity between this and the expenditure function used in the first question.]
(c) The firm chooses $q$ to maximise

$$\pi = pq - \frac{r_1 r_2}{r_1 + r_2} q^2$$

This is a concave problem. The FOC is

$$p = 2 \frac{r_1 r_2}{r_1 + r_2} q$$

Rearranging,

$$q^* = \frac{p r_1 + r_2}{2} \frac{2}{r_1 r_2}$$

Profit is

$$\pi^* = pq^* - c(q^*) = \frac{p^2 r_1 + r_2}{2} \frac{2}{r_1 r_2} - \frac{p^2 r_1 + r_2}{4} \frac{2}{r_1 r_2} = \frac{p^2 r_1 + r_2}{4} \frac{2}{r_1 r_2}$$

(d) The firm maximises

$$\pi = p\left[\frac{1}{2} z_1^{1/2} + \frac{1}{2} z_2^{1/2}\right] - r_1 z_1 - r_2 z_2$$

The FOCs are

$$\frac{1}{p z_1} z_1^{-1/2} = r_1$$

$$\frac{1}{p z_2} z_2^{-1/2} = r_2$$

Rearranging,

$$z_1^* = \frac{p^2}{4 r_1^2}$$

and

$$z_2^* = \frac{p^2}{4 r_2^2}$$

Output is

$$q^* = (z_1^*)^{1/2} + (z_2^*)^{1/2} = \frac{p r_1 + r_2}{2} \frac{2}{r_1 r_2}$$

Profit is

$$\pi^* = p\left[\frac{1}{2} (z_1^*)^{1/2} + \frac{1}{2} (z_2^*)^{1/2}\right] - r_1 z_1^* - r_2 z_2^* = \frac{p^2 r_1 + r_2}{2} \frac{2}{r_1 r_2} - \frac{p^2 r_1 + r_2}{4} \frac{2}{r_1 r_2} = \frac{p^2 r_1 + r_2}{4} \frac{2}{r_1 r_2}$$
1. Profit Maximisation (3 points)

A firm has cost curve
\[ c(q) = 100 + q^2 \]

The firm can also shut down and make profits \( \pi = 0 \).

(a) Suppose the firm faces price \( p = 30 \). What is the profit maximising quantity? What is the maximal profit? Will the firm shut down?

(b) Suppose \( p = 10 \). What is the profit maximising quantity? What is the maximal profit? Will the firm shut down?

(c) For which price levels will the firm shut down?

(d) Draw the supply function \( q^*(p) \) as a function of the output price \( p \).

2. Deriving Supply Functions (3 points)

(a) A firm has cost function
\[
   c(q) =\begin{cases} 
   q^2 & \text{for } q < 2 \\
   q^2 + q - 2 & \text{for } q \geq 2
   \end{cases}
\]

Derive the firm’s supply curve, \( q^*(p) \).

(b) A firm has cost function
\[
   c(q) =\begin{cases} 
   q^2 & \text{for } q < 2 \\
   q^2 - q + 2 & \text{for } q \geq 2
   \end{cases}
\]

Derive the firm’s supply curve, \( q^*(p) \).
3. Properties of the Profit Function (3 points)

A firm has cost function \( c(q) = q^2 \).

(a) Calculate the optimal supply function, \( q^*(p) \).

(b) Calculate the optimal profit function, \( \pi^*(p) \).

(c) Show that \( \frac{d}{dp} \pi^*(p) = q^*(p) \).

(d) Show that \( \pi^*(p) \) is convex in \( p \).

(e) Fix a level of output, \( q \), and define the profit the firm makes when the price is \( p \) by

\[
\pi(p; q) = pq - c(q)
\]

On a single picture, draw \( \pi(p; q) \) for each \( q \in \{0, 1, 2, 3, 4\} \). Also draw \( \pi^*(p) \). Discuss your findings and the relationship to (c) and (d). [You may want to use a computer program such as Excel to draw the picture.]

4. Market Demand (3 points)

Suppose three consumers have the following demand curves:

\[
\begin{align*}
x_1^*(p) &= 8 - p \\
x_2^*(p) &= 6 - 2p \\
x_3^*(p) &= 12 - 3p
\end{align*}
\]

where the subscript identifies the agent. Find the market demand (either mathematically or graphically).

5. Equilibrium (4 points)

There is an economy with 50 agents. Of these agents, ten have income \( m = 10 \), ten have \( m = 20 \), ten have \( m = 30 \), ten have \( m = 40 \) and ten have \( m = 50 \). Each agent has utility
function

\[ u(x_1, x_2) = x_1^{1/2} + x_2^{1/2} \]

over goods \( x_1 \) and \( x_2 \). The price of good 2 equals 1. The price of good 1 is to be determined.

(a) Derive each agent’s demand curve for good 1.

(b) Derive the market demand for good 1.

There are \( J = 40 \) firms who produce good 1. Each has production function

\[ q = (z_1 - 1)^{1/4}(z_2 - 1)^{1/4} \]

The cost of the inputs is \( r_1 = 1 \) and \( r_2 = 1 \).

(c) Derive each firm’s supply curve.

(d) Derive the market supply curve.

(e) Verify the equilibrium price for good 1 is \( p = 5 \). Show that, at this price, new entrants will wish to enter.

(f) Find the long-run free-entry equilibrium price for good 1, assuming all potential entrants have the same production technology. In addition, find the output of each firm (\( q \)), the number of firms in the industry (\( J \)) and the total industry output (\( Q \)).

6. Equilibrium and Changes in Input Prices (4 points)

A perfectly competitive industry has a large number of potential entrants. Each firm has an identical production function given by \( f(z) = (z - 100)^{1/2} \). Total market demand is \( X = 1800 - 10p \). The input price is \( r = 4 \).

(a) Derive the long-run equilibrium price, the output of each firm (\( q \)), the number of firms in the industry (\( J \)), the total industry output (\( Q \)), and the profits of each firm (\( \pi \)).

Suppose the input price falls to \( r = 1 \).

(b) In the very short-run, firms cannot change their output. What is the new price?
(c) In the short–run, firms cannot exit or enter. Their fixed cost is also sunk, so the firms cannot produce 0 for $c(0) = 0$. Calculate the short–run equilibrium price, the total industry output, and the output of each firm. Show that, taking into account the sunk cost, firms would like to enter.

(d) In the long–run, firms can enter and exit. Derive the the long–run equilibrium price, the output of each firm ($q$), the number of firms in the industry ($J$), the total industry output ($Q$), and the profits of each firm ($\pi$).
1. Profit Maximisation (4 points)

A firm has cost curve
\[ c(q) = 100 + q^2 \]
The firm can also shut down and make profits \( \pi = 0 \).

(a) Suppose the firm faces price \( p = 30 \). What is the profit maximising quantity? What is the maximal profit? Will the firm shut down?

(b) Suppose \( p = 10 \). What is the profit maximising quantity? What is the maximal profit? Will the firm shut down?

(c) For which price levels will the firm shut down?

(d) Draw the supply function \( q^*(p) \) as a function of the output price \( p \).

Solution

The firm maximises
\[ \pi = pq - 100 - q^2 \]
Assuming an internal optimum, the FOC implies
\[ q^* = \frac{p}{2} \]
Maximal profits are
\[ \pi^* = pq^* - 100 - (q^*)^2 = \frac{p^2}{4} - 100 \]
Recall, the firm can also shut down and make \( \pi = 0 \).

(a) If \( p = 30 \), then \( q^* = 15 \) and \( \pi^* = 125 \). This is positive, so the firm should not shut down.
(b) If $p = 10$, then the internal optimum yields $q^* = 5$ and $\pi^* = -75$. Hence the firm is better off by shutting down.

(c) Setting $\pi^* = 0$, we find that $p = 20$. Hence the firm shuts down if $p < 20$, operates if $p > 20$, and is indifferent at $p = 20$.

(d) The supply function is $q^*(p) = 0$ for $p \leq 20$ and $q^*(p) = p/2$ for $p \geq 20$.

2. Deriving Supply Functions (4 points)

(a) A firm has cost function

\[
c(q) = \begin{cases} 
q^2 & \text{for } q < 2 \\
q^2 + q - 2 & \text{for } q \geq 2
\end{cases}
\]

Derive the firm’s supply curve, $q^*(p)$.

(b) A firm has cost function

\[
c(q) = \begin{cases} 
q^2 & \text{for } q < 2 \\
q^2 - q + 2 & \text{for } q \geq 2
\end{cases}
\]

Derive the firm’s supply curve, $q^*(p)$.

Solution

(a) First suppose $q < 2$. The FOC is

\[p = 2q\]

Hence $q^*(p) = p/2$. This assumes that $q < 2$, and hence $p < 4$.

Next suppose $q \geq 2$. The FOC is

\[p = 2q + 1\]

Hence $q^*(p) = (p - 1)/2$. This assumes that $q \geq 2$ and hence $p \geq 5$.

For $p \in [4, 5]$, the firm does not want to produce more than $q = 2$ at $MC = 2q + 1$. It also
Figure 1: Convex Cost Function. The black line shows the MC curve. The red line shows the supply curve.

does not want to produce less that \( q = 2 \) at \( MC = 2q \). Hence the firm will produce \( q = 2 \). See figure 3.

(a) First suppose \( q < 2 \). The FOC is

\[
p = 2q
\]

Hence \( q^*(p) = p/2 \). This assumes that \( q < 2 \), and hence \( p < 4 \).

Next suppose \( q \geq 2 \). The FOC is

\[
p = 2q - 1
\]

Hence \( q^*(p) = (p + 1)/2 \). This assumes that \( q \geq 2 \) and hence \( p \geq 3 \).

For \( p \in [3, 4] \) we therefore have two locally optimal solutions. To see which is the global optimum, we can calculate the profits. If \( q^*(p) = p/2 \) then profit equals

\[
\pi = pq^* - c(q^*) = \frac{p}{2} \left( \frac{p^2}{2} - \frac{p^2}{4} \right) = \frac{p^2}{4}
\]

(1)

If \( q^*(p) = (p + 1)/2 \) then profit equals

\[
\pi = pq^* - c(q^*) = p \left( \frac{p+1}{2} - \frac{(p+1)^2}{4} + \frac{(p+1)}{2} - 2 \right) = \frac{p^2}{4} + \frac{p}{2} - \frac{7}{4}
\]

(2)
Comparing (1) and (2), we see that the optimal supply is $q^*(p) = p/2$ if $p < 7/2$ and $q^*(p) = (p + 1)/2$ if $p \geq 7/2$. See figure 4.

3. Properties of the Profit Function (4 points)

A firm has cost function $c(q) = q^2$.

(a) Calculate the optimal supply function, $q^*(p)$.

(b) Calculate the optimal profit function, $\pi^*(p)$.

(c) Show that $\frac{d}{dp} \pi^*(p) = q^*(p)$.

(d) Show that $\pi^*(p)$ is convex in $p$.

(e) Fix a level of output, $q$, and define the profit the firm makes when the price is $p$ by

$$\pi(p; q) = pq - c(q)$$

On a single picture, draw $\pi(p; q)$ for each $q \in \{0, 1, 2, 3, 4\}$. Also draw $\pi^*(p)$. Discuss your findings and the relationship to (c) and (d). [You may want to use a computer program such
as Excel to draw the picture.]

Solution

(a) The FOC of the profit maximisation problem is

\[ p = 2q \]

Rearranging, \( q^*(p) = p/2 \).

(b) The profit function is given by

\[ \pi^*(p) = pq^* - c(q^*) = \frac{p^2}{4} \]

(c) Differentiating,

\[ \frac{d}{dp} \pi^*(p) = \frac{p}{2} = y^*(p) \]

(d) Differentiating again,

\[ \frac{d^2}{dp^2} \pi^*(p) = \frac{1}{2} \geq 0 \]

Hence the profit function is convex.

(c) See figure 3. The profit function is the maximum of the \( \pi(p; q) \) functions. This implies that the profit function is convex. It also implies that the slope of the profit function at any point equals the slope of \( \pi(p; q^*(p)) \), which equals \( q^*(p) \).

4. Market Demand (4 points)

Suppose three consumers have the following demand curves:

\[ x_1^*(p) = 8 - p \]
\[ x_2^*(p) = 6 - 2p \]
\[ x_3^*(p) = 12 - 3p \]
where the subscript identifies the agent. Find the market demand (either mathematically or graphically).

Solution

The demand curves are shown in figure 4. Agent 1 buys when \( p \in [0, 8] \); agent 2 buys when \( p \in [0, 3] \); agent 3 buys when \( p \in [0, 4] \). Summing up, market demand is given by

\[
X(p) = 8 - p \quad \text{for } p \in [4, 8] \\
= 20 - 4p \quad \text{for } p \in [3, 4] \\
= 26 - 6p \quad \text{for } p \in [0, 3]
\]

5. Equilibrium (4 points)

There is an economy with 50 agents. Of these agents, ten have income \( m = 10 \), ten have \( m = 20 \), ten have \( m = 30 \), ten have \( m = 40 \) and ten have \( m = 50 \). Each agent has utility function

\[
u(x_1, x_2) = x_1^{1/2} + x_2^{1/2}
\]
over goods $x_1$ and $x_2$. The price of good 2 equals 1. The price of good 1 is to be determined.

(a) Derive each agent’s demand curve for good 1.

(b) Derive the market demand for good 1.

There are $J = 40$ firms who produce good 1. Each has production function

$$q = (z_1 - 1)^{1/4}(z_2 - 1)^{1/4}$$

The cost of the inputs is $r_1 = 1$ and $r_2 = 1$.

(c) Derive each firm’s supply curve.

(d) Derive the market supply curve.

(e) Verify the equilibrium price for good 1 is $p = 5$. Show that, at this price, new entrants will wish to enter.

(f) Find the long-run free-entry equilibrium price for good 1, assuming all potential entrants have the same production technology. In addition, find the output of each firm ($q$), the number
of firms in the industry (J) and the total industry output (Q).

Solution

(a) As in Practice Problem Set 4, the demand for each agent is

\[ x_{i,1}^* = \frac{m_i}{p_1} \frac{p_2}{p_1 + p_2} = \frac{m_i}{p_1(p_1 + 1)} \]

(b) Summing up, market demand is

\[ X_1 = \sum_i \frac{m_i}{p_1(p_1 + 1)} = \frac{1500}{p_1(p_1 + 1)} \]

(b) The cost minimisation problem is

\[ \min_{z_1, z_2} \mathcal{L} = r_1 z_1 + r_2 z_2 + \lambda[q - (z_1 - 1)^{1/4}(z_2 - 1)^{1/4}] \]

The FOCs imply that \( r_1(z_1 - 1) = r_2(z_2 - 1) \). The optimal input demands are

\[ z_1^* = \left( \frac{r_2}{r_1} \right)^{1/2} q^2 + 1 \quad \text{and} \quad z_2^* = \left( \frac{r_1}{r_2} \right)^{1/2} q^2 + 1 \]

The resulting cost function is

\[ c(q; r_1, r_2) = 2q^2(r_1 r_2)^{1/2} + (r_1 + r_2) \]

The firm’s profit maximisation problem implies \( p = MC \). That is,

\[ p = 4q(r_1 r_2)^{1/2} \]

Hence the firm’s supply curve is

\[ q^*(p) = \frac{p}{4(r_1 r_2)^{1/2}} \]

(d) There are 40 firms, so the market supply is

\[ Q^*(p) = 10 \frac{p}{(r_1 r_2)^{1/2}} \]
Using $r_1 = 1$ and $r_2 = 1$, we have $Q^*(p) = 10p$.

(e) Equilibrium is characterised by

$$\frac{1500}{p(p + 1)} = 10p$$

Rearranging, $p^2(p + 1) = 150$. This is solved by $p = 5$. The total supply is $Q = 50$. Each firm produces $q = 5/4$. A firm’s profit is

$$\pi = pq - c(q) = \frac{25}{4} - \frac{25}{16} - 2 = \frac{9}{8} > 0$$

Hence entry is profitable.

(f) The average cost is

$$AC = 2q + 2q^{-1}$$

This is minimised at $q = 1$. The average cost is $AC = 4$. Hence the long run price is $p = 4$. Demand is $X = 1500/20 = 75$. Hence there are $J = 75$ firms.

6. Equilibrium and Changes in Input Prices (4 points)

A perfectly competitive industry has a large number of potential entrants. Each firm has an identical production function given by $f(z) = (z - 100)^{1/2}$. Total market demand is $X = 1800 - 10p$. The input price is $r = 4$.

(a) Derive the long–run equilibrium price, the output of each firm ($q$), the number of firms in the industry ($J$), the total industry output ($Q$), and the profits of each firm ($\pi$).

Suppose the input price falls to $r = 1$.

(b) In the very short–run, firms cannot change their output. What is the new price?

(c) In the short–run, firms cannot exit or enter. Their fixed cost is also sunk, so the firms cannot produce 0 for $c(0) = 0$. Calculate the short–run equilibrium price, the total industry output, and the output of each firm. Show that, taking into account the sunk cost, firms would like to enter.

(d) In the long–run, firms can enter and exit. Derive the the long–run equilibrium price, the output of each firm ($q$), the number of firms in the industry ($J$), the total industry output ($Q$), and the profits of each firm ($\pi$).
Solution

(a) Given an input price \( r \), the firm’s cost function is

\[
c(q) = rz = rf^{-1}(q) = r(q^2 + 100)
\]

Average cost is

\[
AC(q) = rq + 100r q^{-1}
\]

Differentiating, this is minimised at \( r = 100rq^{-2} \). Rearranging, \( q = 10 \). The price therefore equals,

\[p = AC(q) = 20r\]

When \( r = 4, p = 80 \). The output of each firm is 10. The total demand is \( X = 1800 - 800 = 1000 \). As a result, there are \( J = 100 \) firms. By construction, each firm makes zero profits.

(b) Output is \( Q = 1000 \). The price is still \( p = 80 \).

(b) The firm’s marginal cost is

\[
MC = 2rq = 2q
\]

Profit maximisation implies \( p = MC \). Hence the short run supply curve is \( q^* = p/2 \). Summing over firms, industry supply is \( Q = 50p \). The equilibrium is

\[50p = 1800 - 10p\]

Rearranging, \( p = 30 \). The output of each firm is \( q = 15 \). The industry output is \( Q = 1500 \). The profit of each firm is

\[\pi = pq - c(q) = 30 \times 15 - (15)^2 - 100 = 125\]

Which will induce entry.

(d) As in (a), average cost is minimised when \( q = 10 \). The price equals \( p = 20 \). Demand is \( X = 1800 - 200 = 1600 \). Hence there are \( J = 160 \) firms. By construction, each firm makes zero profits.
Economics 11: Homework 10

December 1, 2009

1. Profit Maximisation: Perfect Substitutes

A firm has production function \( f(z_1, z_2) = (z_1 + z_2)^{1/2} \) where the prices of the inputs are \( r_1 < r_2 \). The output price is \( p \).

(a) The firm wishes to attain target output \( q \). Derive the cost minimising inputs \( z_i^*(r_1, r_2, q) \).

(b) Derive the cost function \( c(r_1, r_2, q) \).

(c) Using the cost function, derive the profit-maximising output and the firm’s optimal profits.

(d) Using the one-step method, directly solve for the firm’s optimal inputs, the resulting output and the optimal profit. [When using the one-step method the firm chooses \((z_1, z_2)\) to maximise \( pf(z_1, z_2) - r_1 z_1 - r_2 z_2 \) subject to \( z_i \geq 0 \).

2. Profit Maximisation: Perfect Complements

A firm has production function \( f(z_1, z_2) = (\min\{z_1, z_2\})^{1/2} \) where the prices of the inputs are \( r_1 < r_2 \). The output price is \( p \).

(a) The firm wishes to attain target output \( q \). Derive the cost minimising inputs \( z_i^*(r_1, r_2, q) \).

(b) Derive the cost function \( c(r_1, r_2, q) \).

(c) Using the cost function, derive the profit-maximising output and the firm’s optimal profits.

(d) Using the one-step method, directly solve for the firm’s optimal inputs, the resulting output and the optimal profit. [When using the one-step method the firm chooses \((z_1, z_2)\) to maximise \( pf(z_1, z_2) - r_1 z_1 - r_2 z_2 \).]
3. Profit Maximisation

(a) A firm has cost function \( c(q) = 20q - 10q^2 + q^3 \). The output price is \( p = 8 \). Solve for the profit maximising output.

(b) A firm has cost function \( c(q) = 40q - 10q^2 + q^3 \). The output price is \( p = 8 \). Solve for the profit maximising output.

[Hint: you may wish to plot these functions]

4. Equilibrium and Inputs Markets

A firm has a Cobb–Douglas production function \( f(z_1) = 10z_1^{1/2} \). The price of the input is \( r_1 \).

(a) Find the supply function of the firm and its input demand (as a function of \( p \) and \( r_1 \)).

Suppose there are 4,000 identical firms in the market, and the supply of \( z_1 \) is given by \( z_1^S = 40r_1^2 \).

b) Assume that the output price is \( p = 1 \). Calculate the equilibrium input price \( r_1 \), the amount of \( z_1 \) each firm demands, the amount of \( z_1 \) the entire market demands and the output of each firm.

c) Suppose the output price rises to \( p = 2 \). Repeat part (b). What is the effect of an increase in the output price?

5. First Welfare Theorem

There are two goods (\( x \) and \( y \)) and two agents (A and B). The agents’ utility functions are

\[
\begin{align*}
  u_A &= v_A(x) + y \\
  u_B &= v_B(x) + y
\end{align*}
\]

where \( v_A = 2 \ln(x) \) and \( v_B(x) = 4x^{1/2} \). The agents have incomes \( m_A = 10 \) and \( m_B = 10 \). The price of good \( y \) is \( p_y = 1 \); the price of good \( x \) is to be determined.

A single firm produces good \( x \). It has cost function \( c(q) = q^2/2 \).
(a) Show that $p = 2$ is an equilibrium price. Find the equilibrium allocations $(x_A, x_B, q)$.

(b) Suppose a social planner chooses $(x_A, x_B, q)$ to maximise the total surplus,

$$v_A(x_A) + v_B(x_B) - c(q)$$

subject to $q = x_A + x_B$. Verify your allocations from part (a) satisfy the FOCs from this optimisation problem.

6. Shifts in Supply Functions and Elasticities

Suppose the supply function is $q = p$, and the demand function is $q = 10 - p$.

(a) Find the equilibrium price and quantity.

(b) Suppose the supply curve shifts up to $q = p - 2$, so each unit costs $2$ more to produce. Derive the new price and quantity.

Suppose the supply function is $q = p$, and the demand function is $q = 6 - p/5$.

(c) Find the equilibrium price and quantity.

(d) Suppose the supply curve shifts up to $q = p - 2$. Derive the new price and quantity.

Suppose the supply function is $q = p$, and the demand function is $q = 25 - 4p$.

(e) Find the equilibrium price and quantity.

(f) Suppose the supply curve shifts up to $q = p - 2$. Derive the new price and quantity.

(g) Given these results, explain how the elasticity of the demand curve affects the impact of a shift in the supply function.

7. Taxation

Suppose utility is quasilinear in good $x$ and the demand function is $q = 10 - p$. The supply function is $q = p$. 
(a) Solve for the equilibrium price. Solve for consumer and producer surplus.

(b) Suppose there is a $2 producer tax. What is the new equilibrium price? What price does the firm receive? What is the change in producer and consumer surplus? What is government revenue? What is the deadweight loss?

(c) Suppose there is a $2 consumer tax. What is the new equilibrium price? What price does the firm receive? What is the change in producer and consumer surplus? What is government revenue? What is the deadweight loss?

8. Imports

The demand for portable radios is given by: \( Q = 5000 - 100p \). The local supply curve is given by \( Q = 150p \).

a) Find the market equilibrium.

b) Suppose that radios can be imported at a price of $10 per unit. Find the market equilibrium and the amount of radios imported.

c) Suppose that the local producers convince the government to impose a tariff of $5 per radio. Find the market equilibrium, the total revenue of the tariff and the effect on the consumer and producer surplus, and the deadweight loss.
1. Profit Maximisation: Perfect Substitutes

A firm has production function \( f(z_1, z_2) = (z_1 + z_2)^{1/2} \) where the prices of the inputs are \( r_1 < r_2 \). The output price is \( p \).

(a) The firm wishes to attain target output \( q \). Derive the cost minimising inputs \( z_i^*(r_1, r_2, q) \).

(b) Derive the cost function \( c(r_1, r_2, q) \).

(c) Using the cost function, derive the profit-maximising output and the firm’s optimal profits.

(d) Using the one-step method, directly solve for the firm’s optimal inputs, the resulting output and the optimal profit. [When using the one-step method the firm chooses \((z_1, z_2)\) to maximise \( p f(z_1, z_2) - r_1 z_1 - r_2 z_2 \) subject to \( z_i \geq 0 \).]

Solution

(a) The inputs are perfect substitutes. Since \( r_1 < r_2 \), the firm chooses \( z_1^*(r_1, r_2, q) = q^2 \) and \( z_2^*(r_1, r_2, q) = 0 \).

(b) The cost function is \( c(r_1, r_2, q) = r_1 q^2 \).

(c) The firm chooses \( q \) to maximise

\[
\pi = pq - r_1 q^2
\]

Using the FOC, the optimal output is thus \( q^* (p, r_1, r_2) = p/2r_1 \). The resulting profit is \( \pi(p, r_1, r_2) = p^2/4r_1 \).

(d) The firm chooses \((z_1, z_2)\) to maximise

\[
\pi = p(z_1 + z_2)^{1/2} - r_1 z_1 - r_2 z_2
\]
subject to $z_i \geq 0$. The FOCs are

$$
\frac{\partial \pi}{\partial z_1} = \frac{1}{2} p(z_1 + z_2)^{-1/2} - r_1
$$

$$
\frac{\partial \pi}{\partial z_2} = \frac{1}{2} p(z_1 + z_2)^{-1/2} - r_2
$$

Since $r_1 < r_2$, we find that \( \partial \pi / \partial z_1 \geq \partial \pi / \partial z_2 \), so the firm sets $z_2^*(p, r_1, r_2) = 0$. The FOC for $z_1$ becomes

$$
\frac{1}{2} p(z_1)^{-1/2} - r_1 = 0
$$

Rearranging, $z_1^*(p, r_1, r_2) = (p/2r_1)^2$. The optimal output is thus $q^*(p, r_1, r_2) = z_1^{1/2} = p/2r_1$. The optimal profits are $\pi(p, r_1, r_2) = p^2/4r_1$.

2. Profit Maximisation: Perfect Complements

A firm has production function $f(z_1, z_2) = \left( \min\{z_1, z_2\} \right)^{1/2}$ where the prices of the inputs are $r_1 < r_2$. The output price is $p$.

(a) The firm wishes to attain target output $q$. Derive the cost minimising inputs $z_i^*(r_1, r_2, q)$.

(b) Derive the cost function $c(r_1, r_2, q)$.

(c) Using the cost function, derive the profit-maximising output and the firm’s optimal profits.

(d) Using the one-step method, directly solve for the firm’s optimal inputs, the resulting output and the optimal profit. [When using the one-step method the firm chooses $(z_1, z_2)$ to maximise $pf(z_1, z_2) - r_1 z_1 - r_2 z_2$.]

Solution

(a) The inputs are perfect complements, so the firm chooses $z_1 = z_2$. Given target output $q$, it chooses

$$
z_1^*(r_1, r_2, q) = z_2^*(r_1, r_2, q) = q^2
$$

(b) The cost function is $c(r_1, r_2, q) = (r_1 + r_2)q^2$. 
(c) The firm chooses $q$ to maximise

\[ \pi = pq - (r_1 + r_2)q^2 \]

Using the FOC, the optimal output is thus $q^*(p, r_1, r_2) = p/2(r_1 + r_2)$. The resulting profit is $\pi(p, r_1, r_2) = p^2/4(r_1 + r_2)$.

(d) The firm chooses $(z_1, z_2)$ to maximise

\[ \pi = p(\min\{z_1, z_2\})^{1/2} - r_1z_1 - r_2z_2. \]

When $z_1 > z_2$, the derivative is $\partial\pi/\partial z_1 = -r_1$, so the firm wants to reduce $z_1$. Hence at the optimum we have $z_1 = z_2$, and denote this common number by $z$. The firm maximises

\[ \pi = p^{1/2} - (r_1 + r_2)z \]

The FOC for $z$ becomes

\[ \frac{1}{2}p(z_1)^{-1/2} - (r_1 + r_2) = 0 \]

Rearranging, $z_1^*(p, r_1, r_2) = z_2^*(p, r_1, r_2) = (p/2(r_1 + r_2))^{2}$. The optimal output is thus $q^*(p, r_1, r_2) = z^{1/2} = p/2(r_1 + r_2)$. The optimal profits are $\pi(p, r_1, r_2) = p^2/4(r_1 + r_2)$.

3. Profit Maximisation

(a) A firm has cost function $c(q) = 20q - 10q^2 + q^3$. The output price is $p = 8$. Solve for the profit maximising output.

(b) A firm has cost function $c(q) = 40q - 10q^2 + q^3$. The output price is $p = 8$. Solve for the profit maximising output.

[Hint: you may wish to plot these functions]

Solution

(a) The firm’s profit is

\[ \pi(q) = 8q - 20q + 10q^2 - q^3 = -12q + 10q^2 - q^3 \]
The FOC is 
\[ \frac{d}{dq} \pi(q) = -12 + 20q - 3q^2 = 0 \]

The SOC is 
\[ \frac{d^2}{dq^2} \pi(q) = 20 - 6q \]

Solving the quadratic, the FOC yields 
\[ q^* = \frac{20 \pm \sqrt{400 - 4 \times 3 \times 12}}{6} = \frac{20 \pm 16}{6} \]

The solutions are \( q^* = 2/3 \) and \( q^* = 6 \). Of these, only \( q^* = 6 \) satisfies the SOC.

We also need to check that there is not a boundary solution. If \( q = 0 \) then \( \pi = 0 \). If \( q = 6 \) then \( \pi = 72 \), which is better than 0. Hence the optimal solution is \( q^* = 6 \).

(b) The firm’s profit is 
\[ \pi(q) = 8q - 40q + 10q^2 - q^3 = -32q + 10q^2 - q^3 \]

The FOC is 
\[ \frac{d}{dq} \pi(q) = -32 + 20q - 3q^2 = 0 \]

The SOC is 
\[ \frac{d^2}{dq^2} \pi(q) = 20 - 6q \]

Solving the quadratic, the FOC yields 
\[ q^* = \frac{20 \pm \sqrt{400 - 4 \times 3 \times 32}}{6} = \frac{20 \pm 4}{6} \]

The solutions are \( q^* = 8/3 \) and \( q^* = 4 \). Of these, only \( q^* = 4 \) satisfies the SOC.

We also need to check that there is not a boundary solution. If \( q = 0 \) then \( \pi = 0 \). If \( q = 4 \) then \( \pi = -32 \), which is worse than 0. Hence the optimal solution is \( q^* = 0 \).

4. Equilibrium and Inputs Markets

A firm has a Cobb–Douglas production function \( f(z_1) = 10z_1^{1/2} \). The price of the input is \( r_1 \).
(a) Find the supply function of the firm and its input demand (as a function of \( p \) and \( r_1 \)).

Suppose there are 4,000 identical firms in the market, and the supply of \( z_1 \) is given by \( z_1^S = 40r_1^2 \).

b) Assume that the output price is \( p = 1 \). Calculate the equilibrium input price \( r_1 \), the amount of \( z_1 \) each firm demands, the amount of \( z_1 \) the entire market demands and the output of each firm.

c) Suppose the output price rises to \( p = 2 \). Repeat part (b). What is the effect of an increase in the output price?

**Solution**

(a) We use the two-step method. Inverting the production function, we find that

\[
z_1 = \frac{q^2}{100}
\]

The cost function is therefore

\[
c(q) = r_1 \frac{q^2}{100}
\]

The profit maximisation problem is

\[
\max_q pq - r_1 \frac{q^2}{100}
\]

This is concave. The FOC is

\[
p = r_1 \frac{q}{50}
\]

Rearranging,

\[
q^* = 50 \frac{p}{r_1}
\]

Using (1),

\[
z_1^* = 25 \frac{p^2}{r_1^2}
\]

(b) Suppose \( p = 1 \). The equilibrium price equates supply and demand:

\[
4000 \times 25 \frac{1}{r_1^2} = 40r_1^2
\]
Rearranging, \( r_1^4 = 2500 \) or \( r_1 = (50)^{1/2} \). Each firm demands \( z_1 = 25/50 = 1/2 \). The market demand \( 4000 \times 1/2 = 2000 \). Firm output is \( 50/\sqrt{50} = \sqrt{50} \).

(c) Suppose \( p = 2 \). The equilibrium price equates supply and demand:

\[
4000 \times 25 \frac{4}{r_1^2} = 40r_1^2
\]

Rearranging, \( r_1^4 = 10000 \) or \( r_1 = 10 \). Each firm demands \( z_1 = 100/100 = 1 \). The market demand \( 4000 \times 1 = 4000 \). Firm output is \( 100/10 = 10 \). Hence the higher output price raises the demand for \( z_1 \) and raises the equilibrium price.

5. First Welfare Theorem

There are two goods (\( x \) and \( y \)) and two agents (A and B). The agents’ utility functions are

\[
\begin{align*}
  u_A &= v_A(x) + y \\
  u_B &= v_B(x) + y
\end{align*}
\]

where \( v_A = 2 \ln(x) \) and \( v_B(x) = 4x^{1/2} \), The agents have incomes \( m_A = 10 \) and \( m_B = 10 \). The price of good \( y \) is \( p_y = 1 \); the price of good \( x \) is to be determined.

A single firm produces good \( x \). It has cost function \( c(q) = q^2/2 \).

(a) Show that \( p = 2 \) is an equilibrium price. Find the equilibrium allocations \( (x_A, x_B, q) \).

(b) Suppose a social planner chooses \( (x_A, x_B, q) \) to maximise the total surplus,

\[
v_A(x_A) + v_B(x_B) - c(q)
\]

subject to \( q = x_A + x_B \). Verify your allocations from part (a) satisfy the FOCs from this optimisation problem.

Solution

(a) Substituting her budget into her utility, A maximises

\[
2 \ln(x) + (m - px)
\]
The FOC implies that demand is given by \( x = 2/p \). Similarly, B maximises

\[ 4x^{1/2} + (m - px) \]

The FOC implies that demand is given by \( x = 4/p^2 \).

The firm’s marginal cost is \( MC = q \). Hence the supply curve is \( q = p \). The equilibrium price thus solves

\[ \frac{4}{p^2} + \frac{2}{p} = p \]

Substituting in, \( p = 2 \) solves this equation. At this price the allocations are

\( (x_A, x_B, q) = (1, 1, 2) \)

(b) Using the fact that \( q = x_A + x_B \), the social planner wishes to maximise

\[ S = 2 \ln(x_A) + 4x_B^{1/2} - \frac{1}{2}(x_A + x_B)^2 \]

(where the S stands for ‘surplus’). The FOCs are

\[ \frac{dS}{dx_A} = 2x_A^{-1} - (x_A + x_B) = 0 \]
\[ \frac{dS}{dx_B} = 2x_B^{-1/2} - (x_A + x_B) = 0 \]

Substituting in \( x_A = 1 \) and \( x_B = 1 \) we see that both are satisfied.

6. Shifts in Supply Functions and Elasticities

Suppose the supply function is \( q = p \), and the demand function is \( q = 10 - p \).

(a) Find the equilibrium price and quantity.

(b) Suppose the supply curve shifts up to \( q = p - 2 \), so each unit costs $2 more to produce. Derive the new price and quantity.

Suppose the supply function is \( q = p \), and the demand function is \( q = 6 - p/5 \).

(c) Find the equilibrium price and quantity.
(d) Suppose the supply curve shifts up to \( q = p - 2 \). Derive the new price and quantity.

Suppose the supply function is \( q = p \), and the demand function is \( q = 25 - 4p \).

(e) Find the equilibrium price and quantity.

(f) Suppose the supply curve shifts up to \( q = p - 2 \). Derive the new price and quantity.

(g) Given these results, explain how the elasticity of the demand curve affects the impact of a shift in the supply function.

**Solution**

(a) The equilibrium price and quantity are \( p = 5 \) and \( q = 5 \).

(b) The equilibrium price and quantity are \( p = 6 \) and \( q = 4 \).

(c) The equilibrium price and quantity are \( p = 5 \) and \( q = 5 \).

(d) The equilibrium price and quantity are \( p = 6\frac{2}{3} \) and \( q = 4\frac{2}{3} \).

(e) The equilibrium price and quantity are \( p = 5 \) and \( q = 5 \).

(f) The equilibrium price and quantity are \( p = 5\frac{2}{5} \) and \( q = 3\frac{2}{5} \).

(g) When demand is inelastic (part (d)) then the shift in the supply curve has little effect on quantity but increases the price a lot. When demand is elastic (part (f)) then the shift in the supply curve has a large effect on quantity but a small effect on price.

7. **Taxation**

Suppose utility is quasilinear in good \( x \) and the demand function is \( q = 10 - p \). The supply function is \( q = p \).

(a) Solve for the equilibrium price. Solve for consumer and producer surplus.
(b) Suppose there is a $2 producer tax. What is the new equilibrium price? What price does the firm receive? What is the change in producer and consumer surplus? What is government revenue? What is the deadweight loss?

(c) Suppose there is a $2 consumer tax. What is the new equilibrium price? What price does the firm receive? What is the change in producer and consumer surplus? What is government revenue? What is the deadweight loss?

Solution

(a) Setting supply equal to demand

\[ 10 - p = p \]

implies that \( p = 5 \). The quantity traded is \( q' = 5 \). Consumer surplus and producer surplus are both \( 5 \times 5/2 = 12 \frac{1}{2} \).

(b) Suppose there is a $2 producer tax. The new supply curve is \( q = p - 2 \). The equilibrium price is \( p' = 6 \), although the firm receives \( p' - t = 4 \). The quantity traded is \( q' = 4 \).

With the tax, the consumer and producer surplus are both \( 4 \times 4/2 = 8 \), a change of \( 4 \frac{1}{2} \). Government revenue is \( 4 \times 2 = 8 \). The deadweight loss therefore equals 1.

(c) Suppose there is a $2 consumer tax. The new demand curve is \( q = 8 - p \). The equilibrium price is \( p' = 4 \), although the consumer pays \( p' + t = 6 \). The quantity traded is \( q' = 4 \).

With the tax, the consumer and producer surplus are both \( 4 \times 4/2 = 8 \), a change of \( 4 \frac{1}{2} \). Government revenue is \( 4 \times 2 = 8 \). The deadweight loss therefore equals 1.

8. Imports

The demand for portable radios is given by: \( Q = 5000 - 100p \). The local supply curve is given by \( Q = 150p \).

a) Find the market equilibrium.

b) Suppose that radios can be imported at a price of $10 per unit. Find the market equilibrium and the amount of radios imported.
c) Suppose that the local producers convince the government to impose a tariff of $5 per radio. Find the market equilibrium, the total revenue of the tariff and the effect on the consumer and producer surplus, and the deadweight loss.

Solution

a) Equating supply and demand, \(150p = 5000 - 100p\). Hence \(p = 20\) and \(Q = 3000\).

b) The price drops to 10. As a result the quantity demanded is \(Q = 4000\). Domestic production is \(150 \times 10 = 1500\), while imports \(4000 - 1500 = 2500\).

c) The price rises to 15, and demand falls to \(Q = 3500\). Domestic production is \(150 \times 15 = 2250\) and imports are \(3500 - 2250 = 1250\).

Tariff revenues are \(1250 \times 5 = 6250\). The consumer surplus without the tariff is \((50 - 10) \times 4000/2 = 80,000\). The CS with tariff is \((50 - 15) \times 3500/2 = 61,250\), resulting in a loss of 18,750. The transfer to producers is \(5 \times 1500 + (15 - 10) \times (2250 - 1500)/2 = 9375\). The deadweight loss is \(18750 - 9375 - 6250 = 3125\).