# Economics 326: Homework 3

23 March, 2006

## Question 1

A principal employs an agent to work on a project. The worker chooses effort $e \in \{L, H\}$ at costs $c_L = 0$ and $c_H = 1$. The project succeeds with probability

$$p_H = 1/2 \qquad \text{if} \quad e = H$$
$$p_L = 0 \qquad \text{if} \quad e = L.$$

The principal pays wage $w_0$ if the project fails and $w_1$ if it succeeds. The agent's utility if given by $u(w_s) - c_e$, where $s \in \{0, 1\}$. Utility $u(\cdot)$ is strictly increasing and concave. The agent has reservation utility $\underline{U} = 0$. The principal's profit is $x_s - w_s$ where $x_1 > x_0$.

(a) Suppose the principal can contract on effort, $e$. Assume the principal wishes to implement $e_H$. The principal maximises her profit subject to individual rationality. Write down this maximisation problem.

(b) What wages will the principal offer in the profit–maximising contract?

(c) Now suppose that the principal cannot observe effort, $e$. Again, assume the principal wishes to implement $e_H$. The principal maximises her profit subject to individual rationality and incentive compatibility. Write down this maximisation problem.

(d) What wages will the principal offer in the profit–maximising contract?

(e) How does the answer from (b) differ from that in (d)? Why?

## Question 2

Consider the setup in Question 1, except that the probability of success is

$$p_H = 1 \qquad \text{if} \quad e = H$$
$$p_L = 1/2 \qquad \text{if} \quad e = L.$$

(a) Suppose the principal can contract on effort, $e$. Assume the principal wishes to implement $e_H$. The principal maximises her profit subject to individual rationality. Write down this maximisation problem.

(b) What wages will the principal offer in the profit–maximising contract?

(c) Now suppose that the principal cannot observe effort, $e$. Again, assume the principal wishes to implement $e_H$. The principal maximises her profit subject to individual rationality and incentive compatibility. Write down this maximisation problem.

(d) What wages will the principal offer in the profit–maximising contract?

(e) How does the answer from (b) differ from that in (d)? Why?

(f) How does the second–best solution in this question compare to the second–best solution in Question 1? Why?

## Question 3

This question shows how the solutions to the continuous–action and two–action games are related. The following setup is the same as was used in the lecture (except we assume strict concavity of utility).

A principal employs an agent to work on a project. The worker chooses unobserved effort $e \in \{L, H\}$ at costs $c_L$ and $c_H$. The project succeeds with probability $p_H$ if $e = H$ and $p_L$ if $e = L$, where $p_H > p_L$. The principal pays wage $w_0$ if the project fails and $w_1$ if it succeeds. The agent's utility if given by $u(w_s) - c_e$, where $s \in \{0, 1\}$. Utility $u(\cdot)$ is strictly increasing and strictly concave. The agent has reservation utility $\underline{U}$. The principal's profit is $x_s - w_s$ where $x_1 > x_0$.

First, suppose the principal chooses to implement $e = L$.

(a) Write down the principal's maximisation problem.

(b) Ignore the (IC) constraint. Write down the Lagrangian associated with the principal's problem. Denote the multiplier associated with the (IR) constraint by $\lambda$.

(c) Derive the first–order conditions of the Lagrangian with respect to $(w_0, w_1)$. Use the strict concavity of utility to show the wage is constant. Use (IR) to derive this wage.

(d) Finally, show that the (IC) constraint is satisfied by the optimal wage.

Next, suppose the principal chooses to implement $e = H$.

(e) Write down the principal's maximisation problem.

(f) Write down the Lagrangian associated with the principal's problem. Denote the multiplier associated with the (IR) constraint by $\lambda$, and that associated with the (IC) constraint by $\mu$. Derive the first–order conditions of the Lagrangian with respect to $(w_0, w_1)$.

(g) Since (IC) is an inequality constraint $\mu \geq 0$. Using the strict concavity of utility, show that $\mu > 0$. [Hint: Prove by contradiction]. Finally, show that that $w_1 > w_0$ at the optimal solution.

## Question 4 (optional)

The following question considers the two–action continuous–output model, which is a bit harder than the stuff in the lecture. It uses the same solution technique as Question 3, expect that sums are replaced by integrals.

A principal employs an agent to work. The worker chooses unobserved effort $e \in \{L, H\}$ at costs $c_L$ and $c_H$. Output $q$ is distributed according to $f(q|e)$ if the agent takes effort $e$. Suppose a higher effort induces higher output in the sense of first order stochastic dominance, i.e. $F(q|H) \leq F(q|L)$, where $F(q|e)$ is the cumulative distribution of output. The principal pays the worker $w(q)$ if they produce output $q$.

The agent's utility if given by $u(w(q)) - c_e$, where utility $u(\cdot)$ is strictly increasing and strictly concave. Thus the expected utility of an agent who chooses $e$ is

$$\int (u(w(q)) - c_e) f(q|e)\, dq - c_e$$

The agent has reservation utility $\underline{U}$.

The principal's profit is $q - w(q)$ when output $q$ is realised. Expected profit if the agent chooses $e$ is thus

$$\int (q - w(q)) f(q|e)\, dq$$

First, suppose the principal chooses to implement $e = L$.

(a) Write down the principal's maximisation problem (i.e. maximising profit subject to (IC) and (IR)).

(b) Ignore the (IC) constraint. Write down the Lagrangian associated with the principal's problem. Denote the multiplier associated with the (IR) constraint by $\lambda$.

(c) Derive the first–order condition of the Lagrangian with respect to $w(q)$. Use the strict concavity of utility to show the wage is constant. Use (IR) to derive this wage. [Technical note: the derivative of $u(w(q_1)) + u(w(q_2))$ with respect to $w(q_1)$ is $u'(w(q_1))$. Similarly, the derivative of $\int u(w(q))\, dq$ with respect to $w(q)$ is $u'(w(q))$ $(\forall q)$. This is known as differentiating "pointwise"].

(d) Finally, show that the (IC) constraint is satisfied by the optimal wage.

Next, suppose the principal chooses to implement $e = H$.

(e) Write down the principal's maximisation problem.

(f) Write down the Lagrangian associated with the principal's problem. Denote the multiplier associated with the (IR) constraint by $\lambda$, and that associated with the (IC) constraint by $\mu$. Derive the first–order condition of the Lagrangian with respect to $w(q)$.

(g) Since (IC) is an inequality constraint $\mu \geq 0$. Argue that $\mu > 0$. [Hint: Prove by contradiction].

## Question 5

A principal employs a risk–neutral agent to work on a project. The worker chooses effort $e \in [0, \infty)$ at cost $c(e) = e$. The project succeeds with probability $p(e)$, which is increasing and concave. The principal pays wage $w_0$ if the project fails and $w_1$ if it succeeds. The agent's utility if given by $w_s - c(e)$, where $s \in \{0, 1\}$. The agent has reservation utility $\underline{U}$. The principal's profit is $x_s - w_s$ where $x_1 > x_0$.

(a) Suppose the principal can observe the agent's effort. Write down the principal's problem.

(b) Argue that the (IR) constraint must bind.

(c) Substitute (IR) into the principal's profit. Derive the first–order condition that characterises the first–best effort level. [Note that, since both objective and constraint are linear we don't have to set up a Lagrangian.]

(d) Next, suppose that the principal cannot observe the worker's effort. Derive the first–order condition associated with the agent's optimisation problem.

(e) Show that the principal can implement the first–best solution. [Hint: compare the two first order conditions.]

(f) Intuitively, why is the first–best attainable?

## Question 6

A principal employs a risk–neutral agent to work on a project. The worker chooses effort $e \in [0, \infty)$ at cost $c(e) = e$. The project succeeds with probability $p(e)$, which is increasing and concave. The principal pays wage $w_0$ if the project fails and $w_1$ if it succeeds. The agent's utility if given by $w_s - c(e)$, where $s \in \{0, 1\}$. The principal's profit is $x_s - w_s$ where $x_1 > x_0$.

Suppose the agent has no reservation utility, but does have limited liability. That is, the agent cannot be paid a negative sum, $w_s \geq 0$ for $s \in \{0, 1\}$.

First suppose the agent's effort is verifiable.

(a) The principal maximises profit subject to the two limited liability constraints. Write down this problem.

(b) What are the first–best wages and effort?

Next, suppose the agent's effort is not observable.

(c) Using the first–order approach, the principal maximises profit subject to the two limited liability constraints and the first order condition from the incentive compatibility constraint. Write down this problem.

(d) First argue that the limited liability constraint for $w_0$ will bind, while that for $w_1$ can be ignored. Next, substitute (IC) into the principal's profit maximisation problem.

(e) Differentiate the principal's profit to find the optimum effort level. Why does this differ from the first–best?

## Question 7

Consider the competitive screening model from class.

(a) Show that a pooling equilibrium cannot exist.

(b) Under what conditions does a separating equilibrium also not exist?

(c) Do you view this result as troubling? How might we be able to alter the model to get a more reasonable prediction?

## Question 8

Consider the competitive screening model from class, except that the single crossing condition does not hold, i.e. $c_{t\theta}(t,\theta) \geq 0$. For example, we are looking to employ an accountant, and $t$ is the number of pushups required for the job.

(a) Show that in any equilibrium, both firms make zero profits.

(b) Show that no pooling equilibrium can exist where $t > 0$.

(c) Can a pooling equilibrium exist where $t = 0$?

## Question 9

Consider the competitive screening model from class. Suppose that utility is given by

$$u(w,t|\theta) = w - \frac{t}{\theta}$$

and that $\theta_L = 10$, $\theta_H = 20$. The proportion of high types is $\lambda = 1/2$.

Consider the following contracts. Are there equilibria? If not, why not?

(a) Firm $A$ offers $(w, t) = (8, 10)$, while firm $B$ offers $(w, t) = (18, 120)$.

(b) Firm $A$ offers $(w, t) = (15, 10)$, while firm $B$ offers $(w, t) = (10, 0)$

(c) Firm $A$ offers $(w, t) = (15, 0)$, while firm $B$ offers $(w, t) = (20, 60)$

(d) Firm $A$ offers $(w, t) = (10, 10)$ and $(w, t) = (20, 110)$, while firm $B$ offers no contract.

(e) Firm $A$ offers $(w, t) = (10, 0)$, while firm $B$ offers $(w, t) = (18, 100)$.

(f) Firm $A$ offers $(w, t) = (10, 0)$, while firm $B$ offers $(w, t) = (20, 110)$.

(g) Firm $A$ offers $(w, t) = (10, 0)$ and $(w, t) = (20, 100)$, while firm $B$ offers no contract. [Note: this is slightly subtle].

(h) Firms $A$ offers $(w, t) = (10, 0)$ and $(w, t) = (20, 100)$. Firm $B$ offers $(w, t) = (10, 0)$, $(w, t) = (20, 100)$ and $(w, t) = (15, 90)$.