Economics 326: Suggested Solutions 3

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Question 1

(a) The problem is

$$\max_{w_1,w_0} \quad \frac{1}{2}[x_1 - w_1] + \frac{1}{2}[x_0 - w_0]$$
$$s.t \quad \frac{1}{2}u(w_1) + \frac{1}{2}u(w_0) - 1 \geq 0 \qquad \text{(IR)}$$

(b) Clearly, (IR) will bind (else the principal should reduce wages). Since the agent is risk averse, $w_0 = w_1$. Thus (IR) yields $u(w_0) = u(w_2) = 1$

(c) The problem is

$$\max_{w_1,w_0} \quad \frac{1}{2}[x_1 - w_1] + \frac{1}{2}[x_0 - w_0]$$
$$s.t. \quad \frac{1}{2}u(w_1) + \frac{1}{2}u(w_0) - 1 \geq 0 \qquad \text{(IR)}$$
$$\frac{1}{2}u(w_1) + \frac{1}{2}u(w_0) - 1 \geq u(w_0) \qquad \text{(IC)}$$

(d) Following the arguments in class, both (IR) and (IC) will bind. Solving yields

$$u(w_0) = 0$$
$$u(w_1) = 2$$

(e) Wages are more spread in (d) because the principal needs to provide incentives for the agent to work.

Question 2

(a) The problem is

$$\max_{w_1,w_0} \quad [x_1 - w_1]$$
$$s.t \quad u(w_1) - 1 \geq 0 \qquad \text{(IR)}$$

(b) Clearly (IR) will bind (else the principal should reduce wages). Thus $u(w_1) = 1$. Wage w_0 is not defined.

(c) The problem is

$$\max_{w_1,w_0} \quad [x_1 - w_1]$$
$$s.t. \quad u(w_1) - 1 \geq 0 \qquad \text{(IR)}$$
$$u(w_1) - 1 \geq \frac{1}{2}u(w_1) + \frac{1}{2}u(w_0) \qquad \text{(IC)}$$

(d) Optimal wages are given by

$$u(w_1) = 1$$
$$u(w_0) = -\infty$$

(e) The answer is essentially the same. The inability of the principal to observe effort has no affect wages or utilities.

(f) This example is peculiar: if the project fails, the principal knows that the agent shirked. Thus by punishing failure sufficiently strongly the principal can stop the agent from shirking and provide full insurance to an agent who works.

Question 3

(a) The problem is

$$\max_{w_1,w_0} \quad p_L[x_1 - w_1] + (1-p_L)[x_0 - w_0]$$

$$s.t. \quad p_L u(w_1) + (1-p_L)u(w_0) - c_L \geq \underline{U} \quad \text{(IR)}$$

$$p_L u(w_1) + (1-p_L)u(w_0) - c_L \geq p_H u(w_1) + (1-p_H)u(w_0) - c_H \quad \text{(IC)}$$

(b) Ignoring (IC), the Lagrangian is

$$\mathcal{L} = p_L[x_1 - w_1] + (1-p_L)[x_0 - w_0] \quad (1)$$

$$+\lambda\Big[p_L u(w_1) + (1-p_L)u(w_0) - c_L - \underline{U}\Big] \quad (2)$$

(c) The FOC with respect to w_0 can be rearranged to obtain

$$\frac{1}{u'(w_0)} = \lambda$$

The FOC with respect to w_1 can be rearranged to obtain

$$\frac{1}{u'(w_1)} = \lambda$$

By the strict concavity of utility, $u'(w)$ is strictly decreasing and the optimal wage is constant, $w_1 = w_0 =: w$. Using (IR), $u(w) = c_L + \underline{U}$.

(d) Since the wage is constant the agent is happy to do no work. One can easily check (IC) holds.

(e) The problem is

$$\max_{w_1,w_0} \quad p_H[x_1 - w_1] + (1-p_H)[x_0 - w_0]$$

$$s.t. \quad p_H u(w_1) + (1-p_H)u(w_0) - c_H \geq \underline{U} \quad \text{(IR)}$$

$$p_H u(w_1) + (1-p_H)u(w_0) - c_H \geq p_L u(w_1) + (1-p_L)u(w_0) - c_L \quad \text{(IC)}$$

(f) The Lagrangian is

$$\begin{aligned}\mathcal{L} &= p_H[x_1 - w_1] + (1-p_H)[x_0 - w_0] &(3)\\ &+\lambda\Big[p_H u(w_1) + (1-p_H)u(w_0) - c_H - \underline{U}\Big] &(4)\\ &+\mu\Big[p_H u(w_1) + (1-p_H)u(w_0) - c_H - p_L u(w_1) - (1-p_L)u(w_0) + c_L\Big] &(5)\end{aligned}$$

The FOC with respect to w_0 can be rearranged to obtain

$$\frac{1}{u'(w_0)} = \lambda + \mu\left[1 - \frac{1-p_L}{1-p_H}\right]$$

The FOC with respect to w_1 can be rearranged to obtain

$$\frac{1}{u'(w_1)} = \lambda + \mu\left[1 - \frac{p_L}{p_H}\right]$$

(g) Suppose $\mu = 0$. By the strict concavity of utility, the wage is flat. Yet this would induce the worker to choose e_L. Hence $\mu > 0$ and $w_1 > w_0$.

Question 4

(a) The problem is

$$\begin{aligned}\max_{w_1,w_0} \quad & \int [q - w(q)] f(q|L)\, dq \\ s.t. \quad & \int u(w(q)) f(q|L)\, dq - c_L \geq \underline{U} &\text{(IR)}\\ & \int u(w(q)) f(q|L)\, dq - c_L \geq \int u(w(q)) f(q|H)\, dq - c_H &\text{(IC)}\end{aligned}$$

(b) Ignoring (IC), the Lagrangian is

$$\begin{aligned}\mathcal{L} &= \int [q - w(q)] f(q|L)\, dq &(6)\\ &+\lambda\left[\int u(w(q)) f(q|L)\, dq - c_L - \underline{U}\right] &(7)\end{aligned}$$

(c) The FOC with respect to $w(q)$ is

$$-f(q|L) + u'(w(q))f(q|L) = 0$$

This can be rearranged to obtain

$$\frac{1}{u'(w(q))} = \lambda$$

By strict concavity of utility, the optimal wage is constant, $w_1 = w_0 =: w$. Using (IR), $u(w) = c_L + \underline{U}$.

(d) Since the wage is constant the agent is happy to do no work. One can easily check (IC) holds.

(e) The problem is

$$\begin{aligned} \max_{w_1,w_0} \quad & \int [q - w(q)]f(q|H)\,dq \\ s.t. \quad & \int u(w(q))f(q|H)\,dq - c_H \geq \underline{U} && \text{(IR)} \\ & \int u(w(q))f(q|H)\,dq - c_H \geq \int u(w(q))f(q|L)\,dq - c_L && \text{(IC)} \end{aligned}$$

(f) The Lagrangian is

$$\begin{aligned} \mathcal{L} \quad = \quad & \int [q - w(q)]f(q|L)\,dq && (8) \\ & +\lambda\left[\int u(w(q))f(q|H)\,dq - c_H - \underline{U}\right] && (9) \\ & +\mu\left[\int u(w(q))f(q|H)\,dq - c_H - \int u(w(q))f(q|L)\,dq + c_L\right] && (10) \end{aligned}$$

The FOC with respect to $w(q)$ is

$$-f(q|L) + \lambda u'(w(q))f(q|L) + \mu[u'(w(q))f(q|H) - u'(w(q))f(q|L)] = 0$$

this can be rearranged to obtain

$$\frac{1}{u'(w(q))} = \lambda + \mu\left[1 - \frac{f(q|L)}{f(q|H)}\right]$$

(g) Suppose $\mu = 0$. By strict concavity of utility, the wage is flat. Yet this would induce the worker to choose e_L.

To better understand the shape of the optimal wage, see MWG, Section 14.B.

Question 5

(a) The problem is

$$\begin{aligned} \max_{w_1, w_0, e} \quad & p(e)[x_1 - w_1] + (1 - p(e))[x_0 - w_0] \\ s.t. \quad & p(e)w_1 + (1 - p(e))w_0 - e \geq \underline{U} \qquad \text{(IR)} \end{aligned}$$

(b) If the (IR) doesn't bind, then increase wages.

(c) The principal maximises

$$p(e)x_1 + (1 - p(e))x_0 - e - \underline{U}$$

The FOC is

$$p'(e)(x_1 - x_0) - 1 = 0$$

(d) The agent's problem is to choose e to maximise

$$p(e)w_1 + (1 - p(e))w_0 - e$$

This FOC is

$$p'(e)(w_1 - w_0) = 1$$

(e) The problem is

$$\begin{aligned} \max_{w_1, w_0, e} \quad & p(e)[x_1 - w_1] + (1 - p(e))[x_0 - w_0] \\ s.t. \quad & p(e)w_1 + (1 - p(e))w_0 - e \geq \underline{U} \qquad \text{(IR)} \\ & p'(e)(w_1 - w_0) = 1 \qquad (11) \end{aligned}$$

Using the fact (IR) binds, this can be rewritten as

$$\max_{w_1,w_0,e} \quad p(e)x_1 + (1-p(e))x_0 - e - \underline{U}$$

$$p'(e)(w_1 - w_0) = 1 \tag{IC}$$

This can be solved by setting $w_1 = x_1 - k$ and $w_0 = x_0 - k$, where (w_0, w_1) satisfy the (IR) constraint.

(f) Intuitively, the agent is risk neutral, so the principal must only provide the right incentives for them to work.

Question 6

(a) The problem is

$$\begin{aligned} \max_{w_1,w_0,e} \quad & p(e)[x_1 - w_1] + (1-p(e))[x_0 - w_0] \\ s.t. \quad & w_1 \geq 0 && (LL_1) \\ & w_0 \geq 0 && (LL_0) \end{aligned} \tag{12}$$

(b) The principal should pay $w_0 = w_1 = 0$ and choose $e = \infty$.

(c) The problem is

$$\begin{aligned} \max_{w_1,w_0,e} \quad & p(e)[x_1 - w_1] + (1-p(e))[x_0 - w_0] \\ s.t. \quad & w_1 \geq 0 && (LL_1) \\ & w_0 \geq 0 && (LL_0) \\ & p'(e)(w_1 - w_0) = 1 && (IC) \end{aligned}$$

(d) In order to get the agent to do any work, we require $w_1 \geq w_0$. Hence (LL_0) implies (LL_1). Next, observe that (LL_0) will bind—else the firm should reduce both w_0 and w_1 by ϵ. Thus we have the problem

$$\max_{w_1,e} \quad p(e)[x_1 - w_1] + (1-p(e))[x_0]$$
$$s.t. \quad p'(e)w_1 = 1 \tag{IC}$$

Substituting (IC) into the profit,

$$\max_{w_1,e} \quad p(e)\left[x_1 - \frac{1}{p'(e)}\right] + (1-p(e))[x_0]$$

(e) The first order condition with respect to e yields

$$p'(e)(x_1 - x_0) = 1 - \frac{p(e)p''(e)}{[p'(e)]^2}$$

This is obviously less than the first best level of effort. Intuitively, since $w_0 \geq 0$, the firm needs to increase w_1 to induce extra effort. Yet this is costly for the firm.

Question 7

(a) Suppose A is offering a pooling contract. B can make money by skimming off the high types.

(b) A separating equilibrium is beaten by a pooling contract if the high type prefers $(w, t) = (E[\theta], 0)$ to the least cost separating equilibrium.

(c) There is no right answer. Personally, I don't think this result is particularly troubling: separating equilibria don't exist when they are Pareto dominated, which is no bad thing. In practice, it seems the ability for one firm to skim off another is somewhat limited. One approach, due to Wilson (1979), is to allow the pooling firm to withdraw his pooling contract after the separating firm has skimmed off the top. This means that the separating firm is then lumbered with both types. One also may be able to obtain existence by allowing mixed strategies or more types.

Question 8

(a) Same as Lemma 1 in class. Let (w_L, t_L) and (w_H, t_H) be the proposed equilibrium contracts, and suppose industry profits are positive. The firm making less profits should undercut and offer $(w_L - \epsilon, t_L)$ and $(w_H - \epsilon, t_H)$.

(b) Suppose the equilibrium contract is $(E[\theta], t)$, where $t > 0$. Then one firm can offer $(E[\theta] - \epsilon, 0)$ and make a profit.

(c) Yes. It is now impossible to skim off the high type.

Question 9

Below I given one deviation in the cases that are not equilibria. There may be other perfectly good deviations.

(a) Firm B can offer $(w, t) = (9, 10)$ and steal the low type.

(b) Firm B can offer $(w, t) = (16, 25)$ and steal the high type.

(c) Firm A should withdraw his contract, since he's making a loss.

(d) Firm B can offer $(w, t) = (9.5, 0)$ and steal the low type.

(e) Firm A can offer $(w, t) = (19, 100)$ and steal the high type.

(f) Firm A can offer $(w, t) = (19.6, 100)$ and steal the high type.

(g) Since B offers no contracts, A can unilaterally lower his wages and offer $(w, t) = (9, 0)$ and $(w, t) = (19, 100)$ and make a positive profit.

(h) This is an equilibrium. Observe that B's third contract, $(w, t) = (15, 90)$, is not accepted by either agent.