Moral Hazard: Continuous Actions, Two Outputs

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Model

A worker chooses effort $e \in [0, \infty)$. The project then succeeds with probability $p(e)$, where $p(e)$ is increasing, concave and satisfies $\lim_{e \to 0} p'(e) = +\infty$. Wages in state $s \in \{0, 1\}$ are $\{w_0, w_1\}$.

The worker obtains payoff $u(w_s) - c(e)$. Suppose $u(w)$ is strictly increasing and strictly concave. Suppose $c(e) = e$, for simplicity. The agent has a reservation utility $\underline{U}$.

The principal's payoff is $x_s - w_s$, where $x_1 > x_0$.

First Best

Assume e is verifiable. Let (w_0, w_1) be the wages when the agent follows the seller's instructions. If the agent does anything else, he is decapitated.

The principal's problem is

$$\begin{aligned} \max_{w_1, w_0, e} \quad & p(e)[x_1 - w_1] + (1 - p(e))[x_0 - w_0] \\ s.t. \quad & p(e)u(w_1) + (1 - p(e))u(w_0) - e \geq \underline{U} \qquad \text{(IR)} \end{aligned}$$

The corresponding Lagrangian is

$$\begin{aligned} \mathcal{L} \quad = \quad & p(e)[x_1 - w_1] + (1 - p(e))[x_0 - w_0] \\ & + \lambda \left[p(e)u(w_1) + (1 - p(e))u(w_0) - e - \underline{U} \right] \end{aligned}$$

The first–order condition with respect to w_0 is

$$\frac{1}{u'(w_0)} = \lambda$$

The first–order condition with respect to w_1 is

$$\frac{1}{u'(w_1)} = \lambda$$

We claim that $\lambda > 0$. Since (IR) is an inequality constraint, $\lambda \geq 0$. Suppose, by contradiction, that $\lambda = 0$. The Lagrangian becomes

$$\mathcal{L} \quad = \quad p(e)[x_1 - w_1] + (1 - p(e))[x_0 - w_0]$$

Hence the firm should reduce wages, and the original solution cannot be optimal.

Since $u(w)$ is strictly concave, $w_0 = w_1 =: w^*$. From (IR), $u(w^*) = e + \underline{U}$ and $w^* = u^{-1}(e+\underline{U})$. We have thus solved for the optimal wages, for any choice of e. The principal must then choose e to maximise her profit

$$\max_e \quad p(e)x_1 + (1-p(e))x_0 - u^{-1}(e+\underline{U})$$

Second Best

Suppose e is not observed by the principal. Her problem is

$$\begin{aligned} \max_{w_1,w_0,e} \quad & p(e)[x_1 - w_1] + (1-p(e))[x_0 - w_0] \\ s.t. \quad & p(e)u(w_1) + (1-p(e))u(w_0) - e \geq \underline{U} && \text{(IR)} \\ & p(e)u(w_1) + (1-p(e))u(w_0) - e \geq p(\hat{e})u(w_1) + (1-p(\hat{e}))u(w_0) - \hat{e} \quad (\forall \hat{e}) && \text{(IC)} \end{aligned}$$

The first order approach replaces the continuum of (IC) constraints with the agent's first–order condition. By incentive compatibility,

$$e \in \text{argmax}_{\hat{e}} p(\hat{e})u(w_1) + (1-p(\hat{e}))u(w_0) - \hat{e}$$

Since $\lim_{e\to 0} p'(e) = +\infty$ the seller will wish to implement a strictly positive effort level and we must therefore have

$$p'(e)[u(w_1) - u(w_0) - 1 = 0 \tag{ICFOC}$$

Moreover, the agent's problem is concave in $\hat{e}$ so any solution to (ICFOC) satisfies (IC).

The Lagrangian is thus

$$\begin{aligned} \mathcal{L} \quad = \quad & p(e)[x_1 - w_1] + (1-p(e))[x_0 - w_0] \\ & + \lambda \left[p(e)u(w_1) + (1-p(e))u(w_0) - e - \underline{U}\right] \\ & + \mu \left[p'(e)[u(w_1) - u(w_0) - 1\right] \end{aligned}$$

The first–order condition with respect to w_0 is

$$\frac{1}{u'(w_0)} = \lambda - \mu \frac{p'(e)}{1-p(e)}$$

The first–order condition with respect to w_1 is

$$\frac{1}{u'(w_1)} = \lambda + \mu \frac{p'(e)}{p(e)}$$

We claim that $\lambda > 0$. Since (IR) is an inequality constrain, we know that $\lambda \geq 0$. If $\lambda = 0$

then the firm can always increase their profit by reducing wages such that $u(w_1) - u(w_0)$ remains constant. Hence this cannot be optimal.

We claim that $\mu > 0$. Suppose, by contradiction, that $\mu \leq 0$. Then the FOCs imply that $w_0 \geq w_1$, and the agent chooses $e = 0$. This contradicts the fact that the seller is trying to implement $e > 0$.

Putting this together, the FOCs imply that $w_1 > w_0$.