## Exercise 3: Sequences

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1. Show the set $\{\ldots,-2,-1,0,1,2, \ldots\}$ is countable.
2. Suppose $x_{k} \rightarrow x$ and $x_{k} \rightarrow y$. Show $x=y$.
3. Show $\frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$.
4. Prove the sandwich property.
5. Prove that a convergent sequence is bounded.
6. Suppose $x_{n} \rightarrow x$ and $y_{n} \rightarrow y$. Show $x_{n}+y_{n} \rightarrow x+y$.
7. [Hard] Suppose $x_{n} \rightarrow x$ and $y_{n} \rightarrow y$. Show $x_{n} y_{n} \rightarrow x y$. [Hint: Notice that $x_{n} y_{n}-x y=$ $x_{n} y_{n}-x y_{n}+x y_{n}-x y$.]
8. Let $x_{n}$ be an increasing sequence with $x_{n}-x_{n-1} \leq \frac{1}{n}$. Does $x_{n}$ necessarily converge? [Hint: If you're stuck, look up "harmonic sequence".]
9. Show $\sqrt{n^{2}+1} / n!\rightarrow 0$ as $n \rightarrow \infty$. [Hint: Approximate the numerator from above, and show this new fraction converges to zero].
