Exercise 4: Topology

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1. Suppose $x, y \in \mathbb{R}^N$ are orthogonal under the Euclidian metric. Show that $||x - y||^2 = ||x||^2 + ||y||^2$.

2. Show that the union of an arbitrary number of open sets is open.

3. Show that the intersection of an arbitrary number of open sets may not be open. [Hint: just find a counterexample.]

4. Provide an example of a set which is neither open or closed.

5. Provide an example of a set which is both open and closed.

6. Show that any single point in $\mathbb{R}$ is closed. [Note: Since a finite union of closed sets is closed, a corollary is that any finite set is closed.]

7. Show the ϵ–disk, $D(x, \epsilon) = \{y | d(y, x) < \epsilon\}$ is open. [Hint: For $y \in D(x, \epsilon)$, consider $D(x, \epsilon')$ with $\epsilon' = \epsilon - d(x, y)$.]

8. Suppose $S \subset \mathbb{R}$ is bounded and closed. Show that $\sup(S) \in S$.

9. Let $A = \{x \in \mathbb{R} | \sin(x) = 0.2\}$. Is A closed? Is A bounded?