Exercise 5: Continuity

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1. Let $f(x) = x^2$, with $x \in \mathbb{R}$. Show f is continuous. [Hint: Fix ϵ, and find an appropriate δ.]

2. Consider $f : [0, \infty) \to \mathbb{R}$, where $f(0) = 0$ and $f(x) = x \sin(1/x)$ for $x > 0$. Is f continuous at 0? [Hint: plot the function]

3. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and an open set $A \subset \mathbb{R}$ such that $f(A)$ is not open.

4. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a closed set $B \subset \mathbb{R}$ such that $f(B)$ is not closed. Is this possible if B is also bounded?

5. Give an example of a continuous and bounded function on $\mathbb{R}$ that does not attain a maximum or minimum.

6. Let $f : [0, 1] \to [0, 1]$ be continuous. Prove that f has a fixed point. That is, there exists $x \in [0, 1]$ such that $f(x) = x$. [Hint: Intermediate value theorem.]

7. Prove there is no continuous map taking $[0, 1]$ onto $(0, 1)$.