Exercise 7: Convexity

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1. Show that the intersection of two convex sets is convex. Is the union of two convex sets necessarily convex?

2. Show that \( f : \mathbb{R} \to \mathbb{R} \) is concave if and only if

\[
f(\sum_i \alpha_i x_i) \geq \sum_i \alpha_i f(x_i)
\]

for \( \alpha_i \in [0, 1] \) such that \( \sum_i \alpha_i = 1 \). [Hint: Induction.]

3. Show that the sum of 2 concave functions is concave.

4. Show that the pointwise minimum (i.e. \( \min\{f(x), g(x)\} \)) of two concave functions is concave.

5. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is concave, and \( g : \mathbb{R} \to \mathbb{R} \) is increasing and concave. Show that \( g(f(x)) \) is concave.

6. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is concave, and \( g : \mathbb{R} \to \mathbb{R} \) is increasing. Show, by example, that \( g(f(x)) \) may not be concave.

7. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is quasi–concave, and \( g : \mathbb{R} \to \mathbb{R} \) is increasing. Show that \( g(f(x)) \) is quasi–concave.