Exercise 8: Taylor's Thm and Implicit Function Thm

September 18, 2007

1. Using induction, complete the proof of Taylor's theorem from class. [Hint: When integrating by parts, let $dv = (x-t)^n/n!$.]

Here is a second proof of Taylor's theorem. Fix x_0. Let

$$P_n(x;x_0) := \sum_{k=0}^{n} \frac{(x-x_0)^k}{k!} f^{(n)}(x_0)$$

Let the remainder be defined by $R_n(x;x_0) := f(x) - P_n(x;x_0)$. Now consider the function

$$g_n(t) = P_n(x;t) + \frac{(x-t)^{n+1}}{(x-x_0)^{n+1}} R_n(x;x_0)$$

2. (i) Fix $n=1$. Show that $g_1(x_0) = g_1(x) = f(x)$.
(ii) Observe that there exists $c \in [x_0, x]$ such that $g_1'(c) = 0$. [How do we know this?] Solving this equation, show that

$$R_1(x;x_0) = \frac{1}{2} f''(c)(x-x_0)^2$$

3. (i) Continuing the proof of Taylors theorem, consider a general n. Show that $g_n(x_0) = g_n(x) = f(x)$.
(ii) Observe that there exists $c \in [x_0, x]$ such that $g_n'(c) = 0$. Solving this equation, show that

$$R_n(x;x_0) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-x_0)^{n+1}$$

4. Suppose an agent has utility function $u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ for $\alpha \in (0,1)$. Use the implicit function theorem to derive the slope of the agent's indifference curve.

5. Suppose $f(x,t) = t^2 + x^2 - 1$. Consider the implicit function $\eta(t)$ such that $f(\eta(t), t) = 0$.
(i) Solve for $\eta(t)$. [Hint: plot $f(x,t)$.]
(ii) Is the full rank condition satisfied?
(iii) Pick $\bar{t} \in (-1,1)$ and a solution $\bar{x}$. Is $\eta(\bar{t})$ uniquely defined near $(\bar{x}, \bar{t})$? What is $\eta'(\bar{t})$?
(iv) Pick $\bar{t} = 1$ and a solution $\bar{x}$. Is $\eta(\bar{t})$ uniquely defined near $(\bar{x}, \bar{t})$? What is $\eta'(\bar{t})$?