Quiz: Solutions

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1. Only (e) is correct.

2. (i) True. Suppose, by contradiction, that $x < a$. Since $x_n \to x$, $x_n \in D(x, \frac{a-x}{2})$ for $n \geq N$. But then, $x_n < a$ for $n \geq N$.
(ii) False. Consider $x_n = a + 1/n$.

3. Let $x_H > x_L$. Consider the points $x = x_H$ and $x' = 0$. Define α such that $\alpha x_H = x_L$. By convexity,

$$\frac{c(x_L)}{x_L} = \frac{c(\alpha x_H)}{\alpha x_H} \leq \frac{\alpha c(x_H)}{\alpha x_H} = \frac{c(x_H)}{x_H}$$

4. (a) Write the constraints as

$$\begin{aligned} g_1(x,y) &= 5 - (x+1)^2 - (y+1)^2 \\ g_2(x,y) &= x \\ g_3(x,y) &= y \end{aligned}$$

The gradient of the inequality constraints are

$$\nabla g_1(x,y) = \begin{pmatrix} 2(x+1) \\ 2(y+1) \end{pmatrix} \quad \nabla g_2(x,y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \nabla g_3(x,y) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Since only two constraints can bind at once, and $x, y \geq 0$, these are linearly independent. Alternatively, one can use the Slater condition.

(b) The objective function is concave and the constraint function is convex, so the Kuhn-Tucker conditions are sufficient.

(c) The Kuhn-Tucker conditions are

$$\begin{aligned}
3 - 2\lambda(x+1) + \mu_x &= 0 \\
1 - 2\lambda(y+1) + \mu_y &= 0 \\
[5 - (x+1)2 - (y+1)2]\lambda &= 0 \\
\mu_x x &= 0 \\
\mu_y y &= 0 \\
5 - (x+1)2 - (y+1)2 &\geq 0 \\
x &\geq 0 \\
y &\geq 0 \\
\lambda, \mu_x, \mu_y &\geq 0
\end{aligned}$$

The unique solution of these conditions is $(x, y, \lambda, \mu_x, \mu_y) = (1, 0, 3/4, 0, 1/2)$. The solution is hence $(x, y) = (1, 0)$.