Optimal Long-Term Financial Contracting with Privately Observed Cash Flows†

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***** PRELIMINARY AND INCOMPLETE *****

ABSTRACT. We consider a setting in which a risk-neutral agent/entrepreneur with limited capital seeks external financing for a project which pays stochastic cash flows over many periods. These cash flows are unobservable and unverifiable by outside investors. We identify the optimal long-term financial contract. In this contract, the agent is induced to pay investors via the threat of the loss of control of the project. After solving for the contract as an optimal mechanism, we then demonstrate that it can be implemented by a combination of standard forms of long-term debt contracts. We also extend our analysis to allow for renegotiation of the contract.

The model allows for both exogenous and endogenous determination of payoffs in the event of project termination. Our analysis is also general enough to allow for cash flows that arrive continuously. This allow us to address the optimal frequency with which payments to investors are made. Interestingly, we show that it may be optimal for payments to occur at regular, discrete intervals.

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1. Introduction

This paper analyzes optimal dynamic financial contracting. The scenario under consideration involves an agent who has a profitable business opportunity but must raise external capital to finance the opportunity. Among an external investor's concerns in funding a business is that the agent who manages the business might divert funds to himself at the expense of the investor. Our analysis incorporates this possibility. Our analysis also incorporates the possibility that control of the business can be transferred from the agent to the investor. This threat is the key to inducing the agent to pay the investor according to the financial contract.

Specifically, the model entails a risk-neutral agent who seeks funding from a risk-neutral investor. The funding will finance a business that requires an up-front investment in assets. Together, the assets and the agent can generate a series of risky cash flows over the next T periods. The agent observes the realizations of these cash flows but the investor does not. The investor must rely on the agent to pay him out of these unobservable cash flows. At any time during the life of the business, control of the assets can be transferred from the agent to the investor. If the assets are transferred, the agent is left to pursue his best alternative and the investor is free to dispose of the assets for cash. A financial contract specifies payments between the agent and the investor and it specifies the circumstances under which control of the project is transferred from the agent to the investor.

Our analysis generalizes and extends a number of prior analyses (to be discussed in detail). We fully characterize the optimal contract for an arbitrary number of periods, finite- and infinite-horizon. We assume that the business cash flows are independent over time but beyond that we make no distribution assumptions of consequence. We also present the results for both discrete-time and continuous-time versions of the model.

After characterizing the optimal contract, we show that it can be implemented with a combination of long-term coupon debt and a line of credit (which the agent may make an immediate draw on). So in each period, the agent faces a fixed charge, principal plus interest -- this is the payment on the long-term coupon debt. As long as the agent makes the payments on the coupon debt he retains control of the business. Once the agent has made sufficient payments to the investor, he can begin to pay himself dividends. If the agent cannot make the coupon debt payments out of the business' cash flow, he can draw on the line of credit. If needed, the agent can continue to draw on the line of credit until it is exhausted. After that, if the agent cannot make a payment, control is transferred to the investor with some probability. This probability is increasing in the extent of the agent's shortfall.

The coupon debt and the line of credit play different roles in implementing the optimal contract. The coupon debt is effective for financing early consumption for the agent (if given the market interest rate the agent's time preferences are such that he prefers early consumption). The line of credit provides the flexibility needed given that the business' cash flow are risky. For the special case in which the agent's subjective discount rate equals the market interest rate, the optimal financial contract can be implemented with just a line of credit. In this case, the agent funds the up-front investment from the line of credit and then pays it down to zero as soon as possible---there is no further motive for borrowing. For the limiting case in which the business cash flows are riskless, only the coupon debt is needed. With no cash flow risk, coupon debt with its fixed payments can simultaneously fund the up-front investment as well as fund the agent's early consumption (if desired).

The threat that induces the agent to pay the investor is that control of the assets can be transferred to the investor. If control is transferred, the investor and the agent receive terminal payoffs. Our modeling here covers a variety of situations of interest. It covers the possibility that a transfer of control involves liquidating the assets, selling them piecemeal at market prices. For this case, a transfer of control involves a payoff for the investor that is exogenous and may vary over the life of the business. Alternatively, a transfer of control could involve the sale of the business as a going concern. Here the sale price, and hence the payoff available from a transfer of control, is endogenous and depends on the remaining life of the business and on the wealth of potential buyers. For if a buyer must finance the purchase too, then we have the same problem again -- the price that a buyer can pay is determined as part of the solution to the optimal contracting problem. For the agent, a transfer of control may result in his accepting his next best employment alternative. For this case, his termination payoff is exogenous and may vary over the life of the business. Alternatively, a transfer of control could result in the agent borrowing to start a new business. Here the agent's terminal payoff is endogenous and determined as part of the solution to the optimal contracting problem. After characterizing the optimal contract, we explore these and other interpretations of a transfer of control.

Debt financing is typically seen as problematic because of asset-substitution problems. That is, equity-holders may have the incentive to make the firm riskier, thereby transferring wealth from debt-holders. Here, even though the optimal contract involves debt, there is no asset-substitution problem. The agent does not benefit from an increase in cash flow risk because the optimal contract has the feature that in every period, the agent's expected continuation payoff equals the expected cash flow. This arises because the contract must satisfy an incentive-compatibility constraint in order for the agent to pay the investor. To satisfy this constraint, the agent must pay a cost if he makes too low of a payment to the investor. This cost takes the form of a possible transfer of control. With the optimal contract this leads to a continuation payoff that equals the current period's cash flow plus a constant.

Throughout the analysis we consider both the case in which the agent and investor can commit to a contract and the case in which contracts can be renegotiated. With the possibility of renegotiation, the contract must be Pareto optimal at any point in time during the life of the contract. Otherwise the agent and investor would renegotiate. Of course, the possibility of renegotiation effectively entails more constraints on the contracting problem and hence leads to worse outcomes.

There have been a number of analyses of this contracting problem. Bolton and Scharfstein (1990) analyze a two-period setting in which an investor finances an agent's project in the first period and depending on the first-period outcome, the investor may finance a second project in the second period. The threat to withhold second-period financing is what induces the agent to pay in the first period. Gromb (1999) analyzes a similar model with an arbitrary number of periods. Again, in each period, the threat facing the agent is that funding for future projects will be withheld. In both, it is assumed that in each period, the agent's business cash flows have a two-state (i.i.d.) distribution. We will show that both models correspond to special cases of our model.

Allen (1983) analyzes a model in which an agent seeks funding for a riskless business and in which the agent and investor can commit to a contract. Bulow and Rogoff (1989), Hart and Moore (1994), and Hart (1995) analyze financial contracting for settings in which the cash flows are riskless and for which contracts can be renegotiated. The key difference among these analyses is in the way they model renegotiation. We will show that the Bulow and Rogoff (1989) model is a special case of ours.

Hart and Moore (1998) introduce risky cash flows into a two-period version of their model. They assume that all uncertainty is resolved immediately after the agent and investor sign the contract. By contrast, in our model the uncertainty is resolved over time. Though given our assumption of independent cash flows across periods, the Hart and Moore (1998) model has a feature that ours lacks – early cash flows convey information about later cash flows. Bolton and Scharfstein (1990) also have this possibility.

Renegotiation in Hart (1995) and Hart and Moore (1998) is modeled as a procedure that is exogenously imposed on the contracting problem.[1] In principle, though, a contract could specify an optimal renegotiation procedure; the exogenous one they impose is not necessarily optimal. Harris and Raviv (1995) reconsider (an earlier working paper version of) the Hart and Moore (1998) model. But rather than utilize an exogenously specified renegotiation game, Harris and Raviv solve for the optimal contractual arrangement. With observable but unverifiable states, they show that the optimal contract involves a game in which the agent and the investor simultaneously report the state and in which the contract specifies the outcome as a function of the reports.

In all of these analyses, as well as the analysis here, a threat to reduce future cash flows is what induces the agent to repay the investor. Bulow and Rogoff (1989) consider a sovereign debt setting and so the threat involves the interruption of a country's trade. The other analyses consider private business settings and so the threat involves either a seizure of business assets or a withholding of future funding.

There is a large literature on optimal multi-period contracting for settings in which risk-averse agents receive risky, privately-observed cash flows and seek to share the risk. To mention a few, Green (1987) characterizes the optimal contract in an infinite-horizon model, where in each period, each agent's income has a two-state (i.i.d.) distribution. Townsend (1979) and Mookherjee and Png (1989) analyze one-period problems in which there can be a (costly) audit; with an audit, the cash flow is made publicly observable and hence contractible. In the one-period setting, the threat of an audit is the key to inducing the agent to truthfully report the cash flow. Wang (1999) combines these ideas and analyzes a special case of Green's model with deterministic auditing (meaning that the low cash flow is audited and the high cash flow is not). Our analysis does not incorporate risk sharing and we do not allow for audits. Both would be important extensions, especially the former.

[1] Hart and Moore (1994) also makes this type of renegotiation assumption. That paper also differs in that the agent cannot divert the cash flows.

Section 2 presents the basic model and Section 3 presents an illustrative example. In Section 4, we solve for the optimal financial contract and in Section 5 we show how the optimal contract can be implemented with a combination of long-term coupon debt and a line of credit. In Section 6 we show how by varying the specification of terminal payoffs in the event of a transfer of control, we can accommodate a variety of situations of interest and nest some prior analyses of this issue. Section 7 discusses the asset substitution problem and the standard moral hazard problem of privately observed and costly agent effort. Section 8 has concluding remarks.

2. The Model

There is an agent and an investor. The investor is risk neutral, has unlimited capital, and values a cash flow stream $\{c_t\}_{t\in \mathcal{T}}$ as $\sum_{t\in \mathcal{T}} e^{-rt} E[c_t]$, where r is the riskless interest rate. The agent is also risk neutral, has limited capital, and values a cash flow stream $\{c_t\}_{t\in \mathcal{T}}$ as $\sum_{t\in \mathcal{T}} e^{-\gamma t} E[c_t]$, where $\gamma \geq r$ is the subjective discount rate.

The agent has a prospective project that requires an initial investment in assets of I, at date $t = 0$. The agent has initial wealth of $Y_0 \geq 0$. If $I > Y_0$, the agent must borrow to finance the project. Alternatively, even if $Y_0 \geq I$, if $\gamma > r$ the agent would like to borrow for consumption purposes. The investor has unlimited resources and can lend the funds to the agent to initiate the project. If funded, the project generates non-negative risky cash flows. The cash flow at date $t \in \mathcal{T}$ is given by the random variable Y_t. We will consider both cases in which the set of dates $\mathcal{T}$ is finite and cases in which it is infinite. Assume that the project has a positive net present value, $\sum_{t\in \mathcal{T}} e^{-rt} E[Y_t] > I$. Also, the cash flows $\{Y_t\}$ are jointly independent and for $s < t$, $E_s[Y_t] = E[Y_t] = \mu_t$; i.e., there is no learning about future cash flows. Finally, for all t, the minimum element of the support of Y_t is 0.[2]

There is a moral hazard problem in that the agent privately observes the project's cash flows. Specifically, at each date $t > 0$, the agent privately observes the realization of Y_t. The investor must rely on the agent to report this realization. Of course, the agent might lie about the cash flow in order to cheat the investor.

Control of the project can be transferred from the agent to the investor at any time. We refer to this transfer of control as termination. Specifically, after the cash flow Y_t is realized, the agent may choose to quit the firm or, if the contract allows, the investor may fire the agent. In either case, the assets of the firm can be sold at a price of $L_t \geq 0$, and the agent receives a payoff of $R_t \geq 0$ representing the agent's outside option. The liquidation value and the agent's reservation value could be stochastic (in which case interpret L_t and R_t as expected values) but in this case, as with the cash flows, we assume that there is no learning about their values prior to their realization. After termination, the cash flows from the project cease, so that $Y_s = 0$ for $s > t$.

As we will see, the threat that the agent might lose control of the project will be what induces the agent to pay back the investor. Alternatively, the agent's threat to quit the

[2] This is a convenient but not important assumption. If the support of Y_t is bounded below by some $\underline{y}>0$, the optimal contract coincides with the limiting one derived by setting $Pr(Y_t \in [0,\underline{y}))$ arbitrarily small.

firm will limit the amount the investor can demand. Later we will show how (L_t, R_t) may be endogenously determined based on the investor's ability to hire a new manager to run the firm, and the agent's ability to raise capital and start a new firm. For now, it is convenient to think of (L_t, R_t) as exogenous parameters. We assume without loss of generality that

$$L_t = \max_{s \geq t} e^{-r(s-t)} L_s, \quad \text{and} \quad R_t = \max_{s \geq t} e^{-\gamma(s-t)} R_s \tag{1}$$

since the assets can always be stored prior to liquidation, and the agent can always wait for the best outside option.

Importantly, we assume that the cash flows Y_t that the agent receives cannot be observed or collected by the investor. However, liquidation of the business assets is observable and can be contracted on. In particular, the division of the proceeds L_t can be contractually specified.

Suppose the investor funds the agent. Because of the above restrictions, a contract between the two cannot mandate payments from the agent to the investor, since the agent's resources are not observable.[3] Thus, a contract can only specify payments made from the investor to the agent as a function of past payments made and messages sent by the agent to the investor. The contract can also specify circumstances under which control of the project passes from the agent to the investor, who may then terminate the project.

Thus, we can describe the timing as follows. Each period, the agent receives a cash flow Y_t. Then the agent makes a payment $d_t \geq 0$ and report m_t to the investor. The "message space" or domain of m_t is arbitrary, and we assume it is sufficiently large so as not to constrain communication between the parties. After making the payment d_t to the investor, the agent receives a payment w_t from the investor. The agent then chooses a quantity $c_t \geq 0$ to consume. This consumption is not observed by the investor. Any remaining cash balance accumulates at the continuously compounded return $\rho \leq r$ (all positive NPV investments are already included in the project). It may be that $\rho < r$ if it is costly to save funds in a form which is not observable and collectible. Finally, before the next period cash flow, the investor may terminate the project, if the contract allows. If the investor does not terminate the project, then the agent may choose to continue ($q_t = 0$) or quit ($q_t = 1$) the firm.[4]

Let $h_t = (d_s, m_s, \omega_s)_{s \leq t}$ denote the history of agent's payments and reports up to t, together with the indicator ω_s indicating whether the project is active ($\omega_s = 0$) or has been terminated by the investor ($\omega_s = 1$) or terminated by the agent ($\omega_s = 2$) prior to s. The contract then specifies the payment made from the investor to the agent given the history of reports on date t, denoted by the function $w_t(h_t) \geq 0$. In addition, the contract also

[3] Stated another way, even if the contract specified such a payment, it would also have to specify what happens if the agent fails to pay. Hence, the contract can only specify the investor's reactions to potential actions by the agent. (As an example consider a mortgage: the borrower is not truly required or "forced" to pay the monthly payment. Rather, if the monthly payment is not made, foreclosure and liquidation occurs.)

[4] Note that there is no loss of generality in assuming that termination can only occur at the end of the period. For instance, since the agent can always pay $d_t = 0$, there is no loss to the agent from waiting until the end of the period to quit.

specifies the circumstances under which the project is terminated and the assets are sold. Let $p_t(h_t)$ denote the contractually specified probability that the project is terminated at date $t \geq 0$, specified as a function of the history of reports. [5]

In period 0, the timing is identical except that initially, the investor commits I to start the project, with the understanding that the agent will contribute d_0. If d_0 is not contributed, the investor has the option to terminate immediately (prior to investing), and recover I. That is, that the investor can withhold funding if the agent fails to contribute d_0 can be thought of as a termination value $L_0 \geq I$. If d_0 is contributed, the investor gives the agent some amount w_0 for immediate consumption.

In what follows, we will consider two contractual environments. In the first, the contract signed at date 0 remains in force for the life of the project. In the second, at any time during the life of the project, the contract can be renegotiated and replaced with a new contract if both parties agree to do so.

3. An Illustration of Optimal Security Design

We will preview our results on security design with an example. This is intended to make the analysis easier to follow and to highlight the nature of the results.

A project requires an investment of I in assets, and then generates perpetual risky cash flows that are i.i.d. and uniformly distributed on {0,1,2,…20}. The project's liquidation value is $L_t = 80$. The agent has capital $Y_0 = 5$ and has a reservation wage $R_t = 0$. The riskless rate of interest is $r = 10\%$ and the agent's subjective discount rate for consumption is $\gamma = 10.5\%$. So the agent would like to raise funds to finance current consumption as well as to start the business. Assume that the financial market is competitive so that in initially financing the project, the agent receives all of the rents.

In a perfect capital market, the agent would be able to raise funds of 95.1, the perpetuity discounted at 10% (continuously compounded). The agent would invest I, immediately consume $95.1 + 5 - I$, and give the investor all of the business cash flow (if $I > 95.1$, the project is not worth funding). For instance, the agent sells all of the equity to the investor, pays out all future cash flow as dividends to the investor, and liquidation never occurs.

Now introduce the agent's incentive problem: the agent privately observes the realization of each period's cash flow and the agent can divert the cash flow for his own consumption. The above financial arrangement will fail. The agent will report a cash flow of 0 every period keeping all of the cash flow for himself, and the investor will receive no dividends. To induce the agent to pay the investor, there must be a threat that the investor will liquidate the business. So what is the best contract that the agent can offer the investor and how much funding can the agent raise?

Given the history of agent payments and reported cash flows, h_t, the contract specifies the payment made from the investor to the agent, $w_t(h_t)$ and it specifies the probability that the project is liquidated, $p_t(h_t)$. Our analysis derives the optimal contract (w, p) and then

[5] If both the project and the liquidation activity are constant returns to scale technologies, probabilistic liquidation could be reinterpreted as deterministic liquidation of the fraction p_t of the assets.

shows that the optimal contract can be implemented with a combination of two standard securities, coupon debt and a line of credit.

Based on our analysis, in this example, the best the agent can do is raise 79.5. Given the agent's initial capital of 5, the project can be financed if $I \leq 84.5$. Because of the incentive problem, if $84.5 < I \leq 95.1$, a positive NPV project is not financed. The optimal contract can be implemented with perpetual coupon debt with a periodic coupon of 6.4 and a line of credit with a limit of 23.4 and an interest rate of 10.5%. If a coupon payment is missed or the accumulation of interest on the credit line would put the agent over the limit, then the agent is in default. If the agent is in default, the unmade payments "convert" to control rights according to a notional value of 9. That is, if the agent defaults on payments totaling z, then the investor liquidates with probability $z/9$. Note that in this example, the liquidation value is high enough so that the renegotiation constraints do not bind.

Depending on I, the agent may either contribute some of his capital or he may finance some immediate consumption. Specifically, if $I < 60.9$, then the agent contributes no capital to fund the project and the agent is provided with an additional $60.9 - I$ for immediate consumption. If $60.9 < I \leq 65.9$, then the agent contributes $I - 60.9$ of his own capital and is provided no additional funding for immediate consumption. If $I > 65.9$, then the agent contributes all of his own capital, is provided no additional funding for immediate consumption, and takes an initial draw on the credit line to finance the project.

Say $I \leq 65.9$ and consider the example of the cash flow history in the following table.

t	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Y	5	10	12	0	0	0	2	0
LT Debt	-6.4	-6.4	-6.4	-6.4	-6.4	-6.4	-6.4	-6.4
Credit Line	0.0	-1.6	0.0	0.0	-7.1	-15.0	-23.8	-26.0
Net	-1.4	2.0	5.6	-6.4	-13.5	-21.4	-28.2	-32.4
Excess							4.8	9.0
Term Prob							53%	100%
Divs		2.0	5.6					
Credit Line	-1.4	0.0	0.0	-6.4	-13.5	-21.4	-23.4	-23.4

A coupon payment of 6.4 is due in period 1. With a cash flow of 5, the agent must draw 1.4 on the line of credit to pay the coupon. In period 2, another coupon of 6.4 is due and with interest, the credit line is up to $1.4 * e^{.105} = 1.6$. Since the cash flow is 10, the agent pays the coupon, pays down the credit line, and dividends himself the remaining 2. In period 3, the cash flow of 12 is enough to pay the coupon and the agent dividends himself the remaining 5.6. The cash flow is 0 in periods 4-6 and so the agent keeps drawing on the credit line. In period 7, another coupon is due and with interest, the credit line is up to 21.4*$e^{.105}$ = 23.8, which would put the agent over the credit limit. To stay out of default a cash flow of at least 6.4 + (23.8 – 23.4) = 6.8 is required. Given a cash flow of 2, the agent is in default on payments totaling 4.8 and the investor is given control rights worth 4.8. That is, given the contractually specified notional value of 9, the investor is allowed to liquidate the business with probability 4.8/9 = .53. Suppose, though, the project is not liquidated and the agent maintains control. In period 8, interest puts the credit line up to 23.4*$e^{.105}$ = 26, which again would put the agent over the credit limit. To stay out of default a cash flow of at least 6.4 + (26 – 23.4) = 9 is required. With a cash flow of 0 in period 8, the agent is now in default on payments totaling 9 and the investor liquidates with probability 9/9 = 1. In liquidation, the investor receives 80 and the agent receives 0.

If I > 65.9 the contract is the same. The only difference is that, as described above, the agent will begin with a draw on the credit line.

Coupon debt and a credit line implement the optimal contract. The key property of this combination of securities is that they induce three regions for the contract. If cash flows are high, the agent is provided with current consumption and faces no threat of liquidation. This corresponds to the situation in which the agent has no balance on the credit line. With intermediate cash flows, the agent is provided with no current consumption but still faces no threat of liquidation. This corresponds to the situation in which the agent has a balance on the credit line but is below the limit. If cash flows are low, the agent is provided with no current consumption and faces a threat of liquidation. This corresponds to the situation in which the agent is in default.

4. Optimal Contract Design

In this section, we outline a methodology for solving for an optimal contract. We begin with the case in which renegotiation of the contract is not possible and T is finite.

Given a contract $\sigma = (w, p)$, the agent will adopt an optimal strategy $\alpha = (d, m, c, q)$, where the payment, report, consumption and quit decision at date t are functions of the history of cash flow realizations $y^t = (y_s)_{s\le t}$, and the termination history $\omega^t = (\omega_s)_{s\le t}$.[6] When choosing an optimal strategy, the agent must respect his resource constraints; that is, he can never pay or consume more than his available cash. We denote by K_t the agent's cash balance at the start of period t, given by

$$K_t = \sum_{0\le s<t} e^{\rho(t-s)}\,(Y_s - d_s + w_s - c_s).$$

Let τ be the (random) date at which liquidation occurs. The agent will choose the strategy optimally to maximize the payoff:

$$A^*(\alpha \mid \sigma) = E[\,\textstyle\sum_t e^{-\gamma t}\,c_t + e^{-\gamma t}\,R_\tau \mid \alpha, \sigma\,],$$

[6] Since the contractual response to his decisions is fixed, this is essentially a single-agent decision problem, which is why the strategy can be written as a function of (y^t, ω^t) alone.

subject to the resource constraints,[7]

$$0 \le d_t \le K_t + Y_t,$$

$$0 \le c_t \le K_t + Y_t - d_t + w_t \,.$$

Given the agent's strategy, the investor's payoff is then

$$B^*(d, m, c \mid \sigma) = -I + E[\textstyle\sum_{t\ge 0} e^{-rt}(d_t - w_t) + e^{-rt} L_\tau \mid \alpha, \sigma],$$

where τ is the (random) date at which liquidation occurs.

We are interested in solving for an optimal contract which maximizes the investor's payoff subject to the agent receive a given payoff level. The choice among such optimal contracts is then determined by the relative market power of the parties.

Before developing the model further, note that by using a Revelation Principle-type argument, we can establish that private saving by the agent is not necessary in an efficient contract, and that messages are extraneous.

PROPOSITION 1. Given any contract σ with optimal strategy $\alpha = (d, m, c, q)$ for the agent, there exists a contract σ^* and optimal strategy $\alpha^* = (d^*, m^*, c^*, q)$ such that

$$A^*(\alpha \mid \sigma) = A^*(\alpha^* \mid \sigma^*), \qquad B^*(\alpha \mid \sigma) \le B^*(\alpha^* \mid \sigma^*),$$

$$d^*_t = Y_t, \;\; m^*_t = 0, \text{ and } \; c^*_t = w^*_t.$$

Under the contract σ^*, $K_t = 0$ for all t.

PROOF: See the Appendix.

This result suggests the following approach to determining an optimal contract. First, we derive an optimal contract assuming private saving is impossible and no messages can be sent. We know from Proposition 1 that this contract is as good as the true optimal contract, though it might be better since the incentive constraints related to private savings have been ignored and need not be satisfied. If, however, they are satisfied at the solution (as we will show is the case), then we have a solution to the original contracting problem.

4.1. The Problem Without Private Saving

Thus, we now consider the simplified problem in which the agent's only choice variables are (d_t, q_t), the payment and quit decisions made on date t. The payoffs in this problem are given by

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[7] In this formulation, we have treated the payoff R_t as a utility payoff rather than a cash payment which is then consumed. This is for notational convenience only – none of the analysis would change if it were a direct cash payment.

$$A(d, q \mid \sigma) = E[\ \textstyle\sum_t e^{-\gamma t}\,(Y_t - d_t + w_t) \ + e^{-\gamma t} R_\tau \mid d, q, \sigma\ ],$$

$$B(d, q \mid \sigma) = -I + E[\ \textstyle\sum_t e^{-rt}\,(d_t - w_t) \ + e^{-rt} L_\tau \mid d, q, \sigma\ ],$$

where the contract $\sigma = (w, p)$ depends only on the history of payments d^t and whether the project has terminated. We seek to identify the best possible contract for the investor given any payoff for the agent. That is, we need to solve

$$b(a) = \max_{d,q,\sigma}\ B(d, q \mid \sigma) \quad \text{s.t.}\ \ A(d, q \mid \sigma) = a \geq A(d', q' \mid \sigma) \ \text{ for all } (d', q').$$

We call a contract and payment strategy (d, q, σ) *optimal* if it solves the above.

We solve for an optimal contract using a dynamic programming approach. Consider the subgame that begins at the end of period s, but before the period s termination decision is made. Since there is no savings, past history does not affect the continuation possibilities for an active project. That is, any contract after any history will determine a pair of continuation payoffs

$$a_s = E[\ \textstyle\sum_{t>s} e^{-\gamma t}\,(Y_t - d_t + w_t) + e^{-\gamma t} R_\tau \mid d, q, \sigma, y^s\ ],$$

$$b_s = E[\ \textstyle\sum_{t>s} e^{-rt}\,(d_t - w_t) \ + e^{-rt} L_\tau \mid d, q, \sigma, y^s\ ].$$

By varying the contract σ and optimal response (d, q), (a_s, b_s) can be chosen from a set β_s that is independent of the history y^s.

Next note that for an optimal contract, only those points in β_s should be chosen which are efficient for the investor. That is, holding the agent's continuation payoff a_s fixed, an optimal contract will give the investor the highest possible payoff b_s available in the set β_s.[8]

That is, the set of continuation payoffs that can be used by an optimal contract on the subgame s can be described by the function

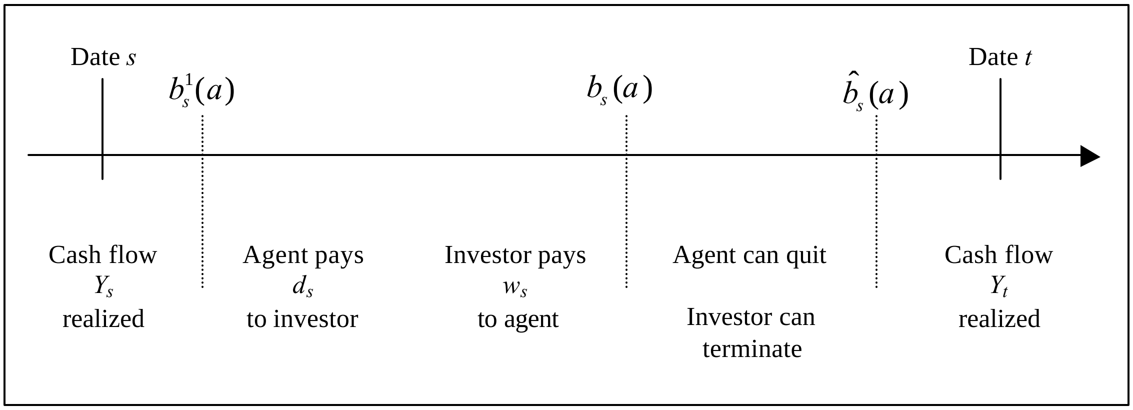
$$b_s(a) = \max\ \{\ b :\ (a, b) \in \beta_s\ \}.$$

We can now solve for b_s using dynamic programming.[9] First suppose the project has a finite economic life T, so that it would be optimal, absent incentive problems, to terminate on date T. In particular, suppose that T is the first date such that

$$L_T + R_T = \max_{s \geq T} E\left[\left(\textstyle\sum_{T<t\leq s} e^{-r(t-T)} Y_t\right) + e^{-r(s-T)}\left(L_s + R_s\right)\right]. \qquad (2)$$

[8] Note that this would not be true absent commitment on the part of the investor. If there were a moral hazard problem for the investor as well, it might be necessary to choose a lower payoff for the investor to maintain interim incentives – such is the case in standard repeated games, for example. Here, there are no interim incentives constraints for the investor, so the highest payoff is chosen.

[9] See Spear and Srivastava (1987), and Green (1987) for discussions of the use of this approach in solving dynamic contracting problems.



Thus, an optimal contract will lead to termination by date T.[10] After T, any payments to the agent must come from the investor. Since $\gamma \geq r$, it is efficient to make any such payments in the next period T^+. Thus, we have

$$b_T(a) = L_T - e^{(\gamma - r)(T^+ - T)}\left(a - R_T\right) \text{ for } a \geq R_T. \tag{3}$$

In general, suppose that we have a continuation function b_t with the following properties:

1. b_t is defined for $a \geq a^0{}_t \geq 0$,
2. b_t is concave; that is, b'_t is weakly decreasing.[11]

We now proceed to derive b_s for $s < t$ and show that it inherits the above properties. The figure above depicts the timing assumption for the continuation function b_t. Given b_t, we first solve for the optimal payments d_t and w_t, which leads to the continuation function $b^1{}_t$. Taking expectations and discounting we will then calculate the continuation function $\hat{b}_s$. Finally, analyzing the period s termination and quit decisions allows us to derive b_s.

Consider the problem faced by the agent at the start of period t, after realizing the cash flow y_t. The agent must then choose a payment d_t to make. Based on d_t, according to the contract the agent will receive a payment w_t from the investor and a continuation payoff a_t in the remainder of the game. Thus, the agent must solve

$$\max_{d \leq y_t} \; y_t - d + w_t(d) + a_t(d) \tag{4}$$

The solution to (4) implies an optimal payment for the agent of

$$d_t(y_t) = \operatorname{argmax}_{d \leq y_t} \; w_t(d) - d + a_t(d),$$

based on the contract's specification of w_t and a_t. Consider the modified contract with $\hat{w}_t(d) = d + w_t(d_t(d)) - d_t(d)$ and $\hat{a}_t(d) = a_t(d_t(d))$. Then the problem becomes

[10] To see this, note that for the agent not to quit, the agent must receive $a_T \geq R_T$ in continuation. Suppose the investor receives b_T in continuation. From (2), even without the observability constraint, $a_T + b_T \leq L_T + R_T$. Thus, by terminating immediately and paying the agent $a_T - R_T$, the investor gets $L_T + R_T - a_T \geq b_T$.

[11] Strictly speaking, b_s may not be differentiable on a set of measure 0. At these points, we interpret b'_s as an arbitrary selection from the super-gradients of b_s.

$$\max_{d \le y_t} \ \hat{w}_t(d) - d + \hat{a}_t(d) = \max_{d \le y_t} \ w_t(d_t(d)) - d_t(d) + a_t(d_t(d)),$$

which is solved with $d = y_t$, and all payoffs are unchanged. Thus, we can without loss of generality restrict attention to contracts for which $d_t(y_t) = y_t$.[12] The incentive compatibility condition is that

$$G_t(y) \equiv w_t(y) - y + a_t(y),$$

is weakly increasing in y. Note also that $G_t(0) \ge a^0_t$, the minimum possible continuation payoff for the agent.

Thus, the agent's payoff is given by $y_t + G(y_t)$, where G_t has the properties noted above and is determined by the contract. To find the optimal contract, we must consider the payoff to the investor. The investor receives

$$y_t - w_t(y_t) + b_t(a_t(y_t)) = y_t - w_t(y_t) + b_t(y_t + G_t(y_t) - w_t(y_t)).$$

Holding G_t constant, the agent's payoff is not altered by changes to w_t. Thus, w_t will be chosen to maximize the payoff to the investor. That is, the optimal contract will choose w_t to solve

$$\max_{w \ge 0} \ b_t(y_t + G_t(y_t) - w) - w.$$

This is solved with

$$w_t(y_t) = (y_t + G_t(y_t) - a^1_t)^+, \tag{5}$$

where[13]

$$a^1_t = \inf \{a : b'_t(a) \le -1\}. \tag{6}$$

Finally, we consider the optimal function G_t. From the above, the ex-ante expected payoff for the agent and the investor before Y_t is realized is given by

Agent: $E[\, Y_t + G_t(Y_t) \,] \equiv \mu_t + g_t$

Investor: $E[\, Y_t - (Y_t + G_t(Y_t) - a^1_t)^+ + b_t(\min(Y_t + G_t(Y_t), a^1_t)) \,]$

Again, holding $g_t = E[\, G_t(Y_t) \,]$ fixed, there is no change to the agent's payoff. Thus, the form of the function G_t should be chosen to maximize the investor's expected payoff given the constraints that its mean is g_t and it is weakly increasing (the incentive compatibility requirement noted previously). We have the following result:

PROPOSITION 2. $G_t(y_t) = g_t$ is optimal.

PROOF: Define $F(z) = -(z - a^1_t)^+ + b_t(\min(z, a^1_t))$. Then the problem is

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[12] This is a standard Revelation Principle argument. In fact, the result also follows from Proposition 1, but is repeated here for clarity.

[13] If b_t is not strictly concave above a^1_t, there will be indeterminacy in the optimal w_t. In particular, we could choose a^1_t as any a such that $b'_t(a) = -1$. Our choice of a^1_t corresponds to making the payment w_t as large as possible. But if $\gamma = r$, for example, the agent could instead be paid the future value of the payment at T and payoffs would be unchanged. When $\gamma > r$, this indeterminacy will not in general occur.

$$\max_G \; E[\, F(Y_t + G(Y_t))\,] \;\text{ s.t. }\; E[G(Y_t)] = g_t, \text{ and } G \text{ weakly increasing.}$$

Note that F is concave in z. Thus,

$$F(Y_t + G(Y_t)) \le F(Y_t + g_t) + F'(Y_t + g_t)\,(G(Y_t) - g_t),$$

where $F'(z)$ is a super-gradient[14] of F at z. Hence,

$$E[\, F(Y_t + G(Y_t))\,] \le E[\, F(Y_t + g_t)\,] + Cov(F'(Y_t + g_t), G(Y_t) - g_t)$$

By concavity, F' is decreasing in Y_t, and from incentive compatibility, G is weakly increasing in Y_t. Hence the covariance is weakly negative and the result follows. ◆

Thus, the agent's payment and consumption has the form $w_t(y_t) = (y_t + g_t - a^1{}_t)^+$; that is, the agent consumes cash flows in excess of $a^1{}_t - g_t$. This is equivalent to the agent having a fixed "debt" obligation of $a^1{}_t - g_t$ in period t, which is an interpretation we shall make explicit in the next section.

Before proceeding further, Proposition 2 also allows us to establish that our restriction on saving is not binding:

> **PROPOSITION 3.** In the optimal contract without private saving, the agent has no incentive to choose $K_t > 0$. This contract is therefore also optimal when private saving is possible.

PROOF: From the above analysis, the agent's continuation payoff given cash y_t in period t is equal to $y_t + g_t$. Thus, $\partial/\partial K_t$ (Agent's Payoff in Period t) = 1. Consider the agent's option to consume less and save in period $s < t$ in order to pay more in period t. The marginal payoff associated with this is

$$- e^{-\gamma s} + e^{-\gamma t}\, e^{\rho(t-s)} = e^{-\gamma s}(e^{(\rho-\gamma)(t-s)} - 1) \le 0,$$

since $\rho \le \gamma$. Note that for this result to hold it is critical that G_t is not increasing. The final conclusion then follows from the discussion following Proposition 1. ◆

4.2. Optimal Continuation Function

Having shown that saving can be ignored, we now continue solving for the optimal mechanism by characterizing the continuation function for the period s prior to t.

The form of the investor's payoff in period t derived above is somewhat complicated. To simplify it, we introduce the following to represent the continuation payoffs just prior to the transfers in period t

$$b_t^1(a) = \begin{cases} b_t(a) & \text{for } \; a_t^0 \le a \le a_t^1 \\ b_t(a_t^1) - (a - a_t^1) & \text{for } \; a \ge a_t^1 \end{cases}.$$

[14] This is simply the derivative when it exists.

Note that $b^1{}_t$ is concave since b_t is. Given this representation, the expected payoff of the investor at the start of period t can be written

$$E[\, Y_t - (Y_t + g_t - a^1{}_t)^+ + b_t(\min(Y_t + g_t,\, a^1{}_t))\,] = E\left[Y_t + b_t^1(Y_t + g_t)\right]. \tag{7}$$

Define $t^- \equiv \max\{\, t' \in \mathcal{T} : t' < t\}$, so that t^- is the period prior to t in $\mathcal{T}$. Similarly define t^+ as the period following t, so that $t^{+-} = t$.

Let $s = t^-$ and now consider b_s. The present value of the expected payoffs above in period s is given by

$$\text{Agent:} \qquad e^{-\gamma(t-s)}\left(\mu_t + g_t\right), \tag{8}$$

$$\text{Investor:} \qquad e^{-r(t-s)}\left(\mu_t + E\left[b_t^1(Y_t + g_t)\right]\right).$$

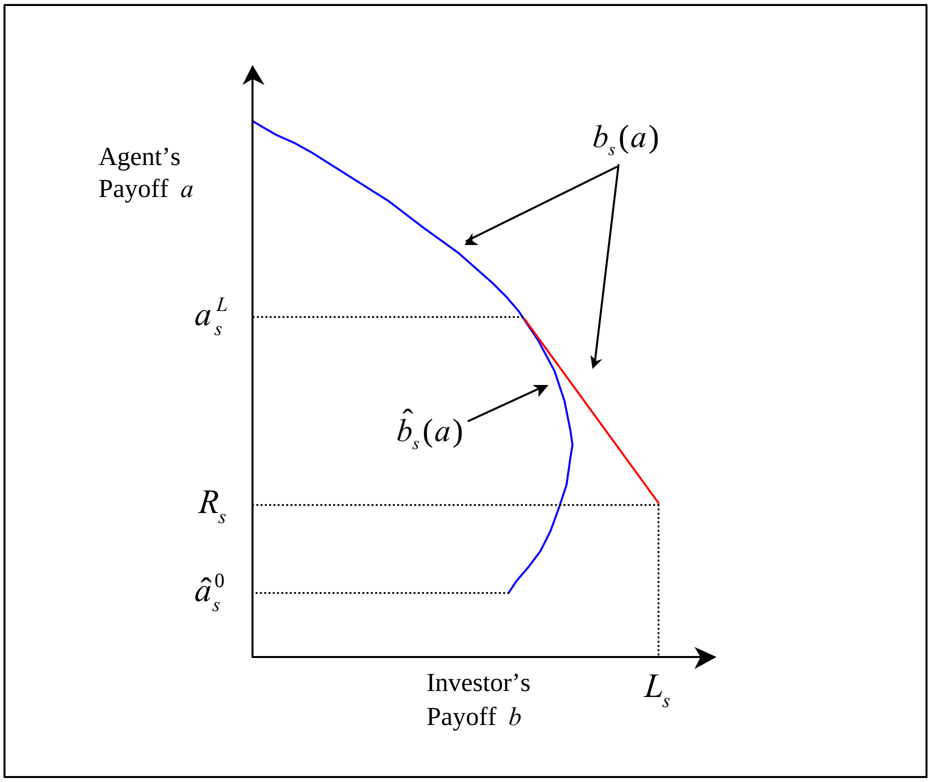
Thus, by varying $g_t \geq a^0{}_t$ (the lower bound derived above), we can trace out the frontier of available payoffs from the perspective of period s. In particular, the lowest payoff for the agent is given by

$$\hat{a}_s^0 = e^{-\gamma(t-s)}\left(\mu_t + a_t^0\right).$$

Also, by letting $a = e^{-\gamma(t-s)}(\,\mu_t + g_t\,) \geq \hat{a}_s^0$ and substituting in the investor's payoff in (8),

$$\hat{b}_s(a) = e^{-r(t-s)}\left(\mu_t + E\left[b_t^1(e^{\gamma(t-s)}a + Y_t - \mu_t)\right]\right). \tag{9}$$

The above gives the continuation possibilities as of the end of period s if the project is not terminated. Note that $\hat{b}_s$ is concave since $b^1{}_t$ is.



We now consider the possibility of project termination. If the project is terminated in period s, the agent receives R_s and the investor receives L_s. Consider first the agent's decision. The agent will quit the firm if the agent's continuation payoff is below R_s. Thus, the agent's minimum continuation payoff from date s, if the investor does not terminate, is given by

$$\max\left(\hat{a}_s^0, R_s\right).$$

Now consider the investor's termination decision. Since stochastic termination is contractible, the set of available continuation utilities is given by the convex hull of (R_s, L_s) and the payoffs $\left(a, \hat{b}_s(a)\right)$ in the region $a \geq \max\left(\hat{a}_s^0, R_s\right)$. The frontier of that set which is efficient for the investor then determines the function b_s.

Mathematically, this can be solved as follows. Define

$$l_s = \sup_{a > \max\left(\hat{a}_s^0, R_s\right)} \frac{\hat{b}_s(a) - L_s}{a - R_s}. \qquad (10)$$

Then l_s represents the slope of the tangent line from (R_s, L_s) to the frontier $\left(\hat{a}_s, \hat{b}_s(\hat{a}_s)\right)$. See the figure above. Define a^L_s to be the tangency point; that is,

$$a_s^L = \inf\left\{a : a \geq \max\left(\hat{a}_s^0, R_s\right), \quad \hat{b}_s'(a_s^L) \leq l_s\right\}. \qquad (11)$$

Then if $a \in [R_s, a^L_s)$, stochastic termination is optimal for the investor. In that case, the investor terminates the project with probability

$$p_s(a) = 1 - (a - R_s)/(a^L{}_s - R_s) = (a^L{}_s - a)/(a^L{}_s - R_s). \quad (12)$$

Adjusting for termination, this leads to a representation for the continuation frontier in period s, which we summarize below:

PROPOSITION 4. (OPTIMAL CONTINUATION FUNCTION) Given $a^0{}_t$ and b_t concave, the continuation function at $s = t^-$ is given by $a^0{}_s = R_s$ and

$$b_s(a) = \begin{cases} \hat{b}_s(a) & \text{if } a \geq a_s^L \\ L_s + l_s(a - R_s) & \text{if } a_s^0 \leq a < a_s^L \end{cases}, \quad (13)$$

where

$$a_t^1 = \inf\left\{a \geq a_t^0 : b_t'(a) \leq -1\right\}, \quad (14)$$

$$b_t^1(a) = \begin{cases} b_t(a) & \text{for } \; a_t^0 \leq a \leq a_t^1 \\ b_t(a_t^1) - (a - a_t^1) & \text{for } \; a \geq a_t^1 \end{cases}, \quad (15)$$

$$\hat{a}_s^0 = e^{-\gamma(t-s)}\left(\mu_t + a_t^0\right), \quad (16)$$

$$\hat{b}_s(a) = e^{-r(t-s)}\left(\mu_t + E\left[b_t^1(e^{\gamma(t-s)}a + Y_t - \mu_t)\right]\right), \quad (17)$$

$$l_s = \sup\left\{\frac{\hat{b}_s(a) - L_s}{a - R_s} : a > \max\left(\hat{a}_s^0, R_s\right)\right\}, \text{ and} \quad (18)$$

$$a_s^L = \inf\left\{a \geq \max\left(\hat{a}_s^0, R_s\right) : \hat{b}_s'(a_s^L) \leq l_s\right\}. \quad (19)$$

Note finally that b_s is concave.

PROOF: Follows from the preceding analysis. ◆

4.3. Renegotiating the Contract

In the analysis above, we assumed that the agent and investor can commit to a contract for the life of the project. We now consider the possibility that they cannot commit not to renegotiate the contract, and show its effect on the possible continuation payoffs in the optimal mechanism.

Consider the continuation payoff function b_t, which represents, for a given payoff of the agent, the highest payoff the investor can achieve just prior to the quit and termination decisions in period t. Suppose that for some $\tilde{a} > a$, $b_t(\tilde{a}) > b_t(a)$; i.e., suppose $(a, b_t(a))$ is Pareto inferior to $(\tilde{a}, b_t(\tilde{a}))$. Then it seems natural to suppose that if the contract were to specify the continuation path associated with $(a, b_t(a))$, the agent and investor would renegotiate the contract in favor of some mutually preferable path. If so, the continuation $(a, b_t(a))$ is infeasible.

Indeed, such renegotiation would likely be possible. It would be difficult for the parties to commit not to act in their mutual self interest. In fact, courts in the United States will not enforce contractual provisions against renegotiation. This places restrictions on what can be achieved by an optimal contract and represents a form of contract incompleteness.

Since there is full-information about the set of possible continuation payoffs, any bargaining game played in renegotiation will select some division of the surplus corresponding to a point on the Pareto frontier of the possible continuation payoffs. This amounts to a restriction that in equilibrium the continuation paths taken are always Pareto efficient. We can impose this restriction on the contract itself – the contract must only specify efficient continuation paths. Such a contract is *renegotiation-proof* in the sense that at no point is there an alternative contract which both parties would mutually prefer. The result that renegotiation-proofness is equivalent to the contract being sequentially undominated (in terms of parties payoffs) was first shown by Hart and Tirole (1988).[15] [16]

In terms of our model, the restriction that b_t be Pareto efficient is equivalent to $b'_t(a) \le 0$ for $a \ge a^0_t$. Note that, from (15) and (17), if $b'_t \le 0$ then so is $\hat{b}'_s(a)$ for $a \ge \hat{a}^0_s$. Thus, the effect of renegotiation will only be to limit the circumstances in which the project is terminated.

Recall that in termination, the agent receives R_s and the investor receives L_s. Thus, termination will be renegotiated if there exists a feasible continuation path with payoffs $\left(a, \hat{b}_s(a)\right)$ that Pareto dominates (R_s, L_s). From the definition of l_s, this is equivalent to $l_s > 0$. Thus, termination will not occur if $l_s > 0$.

This leads to the following characterization:

PROPOSITION 5. (OPTIMAL RENEGOTIATION-PROOF CONTINUATION FUNCTION) Given a^0_t and b_t concave and weakly decreasing, the continuation function at $s = t^-$ is given by

$$a_s^0 = \begin{cases} \max(\hat{a}_s^0, R_s) & \text{if } l_s > 0 \\ R_s & \text{if } l_s \le 0 \end{cases}, \tag{20}$$

together with (13) – (19).

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[15] This corresponds as well to the standard notion of renegotiation-proofness (or Pareto-Perfection) for finite horizon repeated games, developed by Bernheim and Ray (1989).

[16] Hart and Moore (1994, 1998), Hart (1995) and Bulow and Rogoff (1989) do not use this approach. They assume that one party can invoke renegotiation to a new point on the Pareto Frontier unilaterally (see, e.g., Hart and Moore (1998) page 20). This is a restriction on the power of contracts, as courts will enforce the original contract unless both parties agree to the change. As an example, suppose a contract leads to payoffs (6,4). Suppose party 2 can threaten an action that will lead to outcomes (0,0). These papers assume that this threat will provoke renegotiation to a new outcome such as (5,5). Since the original threat is not credible, however, party 1 should not agree to renegotiate and instead enforce (6,4).

PROOF: From the preceding discussion, if $l_s \leq 0$ termination is not renegotiated and there is no change to the characterization of b_s. On the other hand, if $l_s > 0$ then termination is renegotiated. In this case, since the agent earns at least $\hat{a}_s^0$ by continuing, $a_s^0 = \max\left(\hat{a}_s^0, R_s\right)$ since the agent has the option to quit. Also, (19) implies that $a^L{}_s = a^0{}_s$, so that from (13), $b_s(a) = \hat{b}_s(a)$. Note finally that b_s is concave and weakly decreasing. ◆

4.4. Initial Payoffs & Project Horizon

Given the continuation function b_0 available at the end of period 0, we can now derive the agent's initial payoff as a function of his wealth Y_0. We explicitly consider both the case in which investors compete to lend to a monopolistic agent, and the converse. Intermediate cases can obviously also be considered.

PROPOSITION 6. Suppose investors are perfectly competitive. Define

$$a^*(Y_0) = \max\left\{a \geq Y_0 : b_0^1(a) \geq I - Y_0\right\} \text{ and}$$

$$d_0 = \min(Y_0, I - b_0(a^1{}_0)) \ .$$

Then if $a^*(Y_0)$ exists, the agent contributes d_0 and earns the total payoff $a^*(Y_0)$, for a net gain $g_0 = a^*(Y_0) - Y_0$.

If the investor is a monopolist, and agents are identical and perfectly competitive, then define

$$a^{**} = \operatorname{argmax}_a b_0(a).$$

The investor proceeds and earns $b_0(a^{**}) + \min(a^{**}, Y_0) - I$ if this payoff is positive. The agent earns a payoff of $\max(a^{**}, Y_0)$, or a net gain $g_0 = (a^{**} - Y_0)^+$. Note that if contracts can be renegotiated, $a^{**} = a^0{}_0$.

PROOF: See the Appendix.

In most of our applications, it is most natural to consider the first case above, in which the agent seeks financing from competitive investors. Note that in this case, if $I < b_0(a^1{}_0)$, then $d_0 < 0$ and the agent can pull cash out of the firm at date 0. Otherwise, the agent must contribute to the investment I.

The second case in which the investor is a monopolist becomes relevant if the assets themselves are the unique resource. Initially, the agent owns the resource and seeks outside financing from investors. However, if the project is terminated and the investor takes control of the assets, in liquidation the investor will seek a new agent to run the firm. Thus, the liquidation payoff will be determined as in the second case. We discuss this further in Section 6.

One source of inefficiency introduced by the incentive problems is that the project may not attract initial financing. This occurs if $b_0(a^{**}) < I - Y_0$. Another inefficiency may be that the project is terminated prior to the first-best horizon T. Below we consider the project horizon under the optimal contract.

PROPOSITION 7. Let $T^* = \min\{t : l_t \le -1\}$. Then $T^* \le T$, and under the optimal contract the project is terminated with probability 1 in period T^*. When $\gamma = r$, $T^* = T$, and as $\gamma \to \infty$, $T^* \to 0$.

PROOF: See the Appendix.

5. Interpretation and Implementation

The optimal mechanism defined above has the following features. The agent's continuation payoff in period t is given by $Y_t + g_t$ for some constant g_t. If $Y_t + g_t$ exceeds the threshold $a^1{}_t$ then the agent consumes the excess, $Y_t + g_t - a^1{}_t$. If $Y_t + g_t$ falls below the threshold $a^L{}_t$, then the agent faces a probability of termination. In this section we describe how the optimal mechanism can be implemented using standard contracts.

First we define two types of contracts:

LONG TERM DEBT: A long term debt contract is characterized by a sequence of fixed payments x_t. If a payment is not made, the project is terminated.

CREDIT LINE: A credit line is characterized by an interest rate $\hat{r}$ and a credit limit $c^L{}_t \ge 0$. No payments need be made on the credit line unless the credit limit is exceeded. If the debt exceeds the limit by $z_t > 0$ at the end of period t, then the project is terminated with probability $f_t(z_t)$. Otherwise, the excess debt z is forgiven.

We have the following characterization of the optimal contract:

PROPOSITION 8. The optimal contract is implemented by a combination of a long-term debt contract and a credit line. For $0 < t \le T^*$ and $s = t^-$ (the period prior to t), the long-term debt contract is defined by

$$x_t = \mu_t + a^1{}_t - e^{\gamma(t-s)} a^1{}_s \,.$$

The agent's initial debt obligation at date 0 is equal to $x_0 = a^1{}_0 - g_0$, and in the final period, $x_{T*} = 0$.

The credit line interest rate is $\hat{r} = \gamma$, and the credit limit for $t < T^*$ is given by

$$c^L{}_t = a^1{}_t - a^L{}_t.$$

If the credit limit is exceeded by z_t in period t, the project is terminated with probability $f_t(z_t) = z_t/(a^L{}_t - R_t)$. Also, $c^L{}_{T*} = c^L{}_{T*-} = 0$.

PROOF: First we show that the credit limit $c^L{}_t \geq 0$. Note that from the definition of T^*, for $t < T^*$, $l_t > -1$ and thus $a^L{}_t \leq a^1{}_t$. For $t = T^*$, since termination occurs with probability 1, $a^1{}_{T*} = a^L{}_T{}^* = R_T{}^*$ and thus $c^L{}_{T*} = 0$. Note also that if $t = T^{*-}$, then since $b'_{T*} \leq -1$, $\hat{b}_t' \leq -1$ and therefore

$$a_t^1 = a_t^L = \max(\hat{a}_t^0, R_t) = \max\left(e^{-\gamma(T^*-t)}(\mu_{T^*} + R_{T^*}), R_t\right) \text{ for } t = T^{*-}. \tag{21}$$

Hence, $c^L_{T^{*-}} = 0$. Also, (21) implies $x_{T*} = 0$ as we next show.

First, suppose $R_t > e^{-\gamma(T^*-t)}(\mu_{T^*} + R_{T^*})$. Since $a^0{}_T{}^* = a^1{}_T{}^*$,

$$\hat{b}_t^0 = e^{-r(T^*-t)} b^0_{T^*} = e^{-r(T^*-t)} L_{T^*} \leq L_t,$$

where we define $\hat{b}_t^0 = \hat{b}_t(\hat{a}_t^0)$ and $b^0{}_t = b_t(a^0{}_t)$.

Thus, $(R_t, L_t) \geq (\hat{a}_t^0, \hat{b}_t^0)$ which implies $l_t < -1$, contradicting the definition of T^*. Hence, $R_t \leq e^{-\gamma(T^*-t)}(\mu_{T^*} + R_{T^*})$ and so $a_t^1 = e^{-\gamma(T^*-t)}(\mu_{T^*} + R_{T^*})$. Thus, $x_{T*} = 0$.

Next note that the termination probability is well-defined; that is, that the debt never exceeds the credit limit by more than $a^L{}_t - R_t$. To see this, suppose the agent concludes the previous period $s = t^-$ with maximum debt $D_s = c^L{}_s$. Then the debt obligation in period t is

$$e^{\gamma(t-s)} D_s + x_t = a^1{}_t + \mu_t - e^{\gamma(t-s)} a^L{}_s,$$

which exceeds the period t limit $c^L{}_t$ by $z_t = a^L{}_t + \mu_t - e^{\gamma(t-s)} a^L{}_s$. Since $a_s^L \geq \hat{a}_s^0 = e^{-\gamma(t-s)}\left(\mu_t + a_t^0\right)$, we have

$$z_t \leq a^L{}_t - a^0{}_t \leq a^L{}_t - R_t, \tag{22}$$

where the last inequality follows since $a^0{}_t \geq R_t$.

Next we show that the consumption and termination profiles coincide for the debt contract above and the optimal contract σ. Suppose the agent begins period s with outstanding debt D_s on the credit line. Given a required coupon payment of x_s, the total debt is $D_s + x_s$. We show that this corresponds to $g_s = a^1{}_s - D_s - x_s$.

Given cash flow realization y_s in period s, the agent can consume

$$(y_s - D_s - x_s)^+ = (y_s + g_s - a^1{}_s)^+$$

in period s after paying off all debt, which is identical to the payment under σ as in (5) above. The agent then earns

$$a_s = \min(y_s + g_s, a^1_s)$$

in continuation under σ. If $a_s < a^L_s \le a^1_s$, then $a_s = y_s + g_s$ and the optimal contract σ terminates with probability

$$\begin{aligned} p_s &= (a^L_s - a_s)/(a^L_s - R_s) = (a^L_s - (y_s + g_s))/(a^L_s - R_s) \\ &= (a^L_s - (y_s + a^1_s - D_s - x_s))/(a^L_s - R_s) = (D_s + x_s - y_s - c^L_s)/(a^L_s - R_s) \\ &= z_s/(a^L_s - R_s). \end{aligned}$$

This corresponds with the termination probability specified by the credit line contract. Note also that if there is renegotiation and $l_s > 0$, then $a^L_s = a^0_s \le a_s$ and no termination occurs under σ; equivalently, from (22), $z_s \le 0$ and the credit line is not exceeded.

If $a_s \ge a^L_s$, there is also no termination under the optimal contract. In that case, this implies $y_s + g_s \ge a^L_s$, or

$$0 \ge a^L_s - (y_s + g_s) = D_s + x_s - y_s - c^L_s = z_s\,,$$

and again the credit limit is not exceeded.

Absent termination, the optimal contract σ then continues by giving the agent the continuation payoff $\max(a_s, a^L_s)$, which from (8) implies

$$\max(a_s, a^L_s) = e^{-\gamma(t-s)}(\mu_t + g_t),$$

so that the agent enters the next period with

$$g_t = e^{\gamma(t-s)} \max(\min(y_s + g_s, a^1_s), a^L_s) - \mu_t.$$

On the other hand, under the debt contract above, the agent begins next period with debt

$$D_t = e^{\gamma(t-s)} \min(c^L_s, (D_s + x_s - y_s)^+) = e^{\gamma(t-s)} \min(c^L_s, (a^1_s - g_s - y_s)^+).$$

Thus,

$$\begin{aligned} a^1_t - D_t - x_t &= a^1_t - D_t - (\mu_t + a^1_t - e^{\gamma(t-s)}a^1_s) \\ &= e^{\gamma(t-s)}(a^1_s - \min(c^L_s, (a^1_s - g_s - y_s)^+)) - \mu_t. \end{aligned}$$

But since

$$a^1_s - \min(c^L_s, (a^1_s - g_s - y_s)^+) = \max(a^L_s, \min(a^1_s, y_s + g_s)),$$

then $g_t = a^1_t - D_t - x_t$, and we are done. Finally, for $t = 0$, this yields $x_0 = a^1_0 - g_0$. ◆

To understand better the role of each type of contract, we consider some special cases. First, we show that the long-term debt contract is designed to accommodate the agent's impatience given $\gamma > r$. By committing to future payments, the agent can pay less now, or even take cash out of the firm. To see this, we show that when $\gamma = r$, the long-term debt contract is unnecessary.

First define the following,

$$V_s = \sum_{s<t\leq T^*} e^{-\gamma(t-s)}\,\mu_t \;+\; e^{-\gamma(T^*-s)}\,R_{T^*}, \tag{23}$$

the remaining value of the project if consumed exclusively by the agent.

PROPOSITION 9. Suppose $\gamma = r$. Then $T^* = T$ and $x_t = 0$ for all $t \leq T$. Also,

$$a_s^1 = V_s, \quad \text{and} \quad b_s(a_s^1) = e^{-r(T-s)} L_T .$$

PROOF: See the Appendix.

Our next result establishes an upper bound on the total indebtedness of the agent:

PROPOSITION 10. The agent's total indebtedness as of date s is bounded above as follows:

$$c_s^L + \sum_{s<t\leq T^*} e^{-\gamma(t-s)} x_t = V_s - a_s^L \leq e^{-\gamma(s^+-s)}\left(V_{s^+} - R_{s^+}\right).$$

PROOF: See the Appendix.

In general, we expect that the long-term debt contract will specify a payment $x_t \geq 0$. In some circumstances, however, it is possible that $x_t < 0$; that is, the agent receives a payment rather than making one. As the proposition below shows, this only happens if R_{t-} is very large. Then $x_t < 0$ is necessary to induce the agent to continue rather than quit the firm.[17]

PROPOSITION 11. Let $s = t^-$. If $R_s \leq e^{-\gamma(t-s)}(\mu_t + a^1{}_t)$, then $x_t \geq 0$. A sufficient condition for this is

$$R_s \leq e^{-\gamma(t-s)}(\mu_t + R_t). \tag{24}$$

Alternatively, suppose $R_s > e^{-\gamma(t-s)}(\mu_t + a^1{}_t)$, then

$$x_t < 0, \quad x_s \leq 0, \quad \text{and} \quad c^L{}_s = c^L{}_{s-} = 0.$$

Finally, suppose $R_t \leq V_t$ for $t < T^*$. Then, for $s < T^*$,

$$\sum_{s<t\leq T^*} e^{-\gamma(t-s)}\, x_t \geq 0.$$

PROOF: See the Appendix.

[17] This is analogous to the vesting of benefits in standard employment contracts.

The last result of the proposition can be interpreted as follows. The condition, $R_t \le V_t$, is that the agent would not choose to terminate the project early if he could consume all of the cash flows. That is, the agent and the investor do not disagree about the optimal maturity of the project. In that case, the remaining long-term debt payments are a net liability for the agent. Thus, any $x_t < 0$ can be interpreted as the proceeds of a new debt issue.

In Proposition 9 above we established that the role of the long-term debt contract is to allow the agent to draw cash from the firm given $\gamma > r$. Next we show that the role of the credit line is to allow for flexibility given the uncertainty of the cash flows. We show this by demonstrating that as the cash flows become certain, the credit line disappears.

PROPOSITION 12. Suppose that there is no renegotiation, or that termination is efficient, and that for $s = t^-$, $\mu_s \ge e^{-\gamma(t-s)}\,\mu_t$. Then as $Pr(Y_t \ge \mu_t) \to 1$, $c^L{}_t = 0$. Also,

$$x_t \le \left(\hat{a}_t^0 - R_t\right)^+,$$

with equality if (24) holds.

PROOF: See the Appendix.

6. Termination Payoffs

In this section we discuss a number of possible interpretations for the liquidation payoffs (R_t, L_t) and their implications.

One obvious interpretation for L_t is as an exogenous liquidation value based on the value of the assets in some alternative use. Suppose there is an alternative use which generates *observable* cash flows with expected value $\nu_t < \mu_t$ each period. Then it is natural to define

$$L_s = \textstyle\sum_{t>s} e^{-r(t-s)}\,\nu_t\,. \tag{25}$$

That is, when the project is terminated, the assets are sold for use in the alternative technology. The agent is dismissed, and earns a reservation utility which it is convenient to assume is constant and normalized to zero; i.e, $R_s = 0$.

Another important interpretation of this formulation is when the project requires ongoing investment. That is, suppose that an initial investment ν_0 is required to start the firm, and that each period, an investment of ν_t is required to keep the firm in operation. Then,

$$I = \nu_0 + L_0 \tag{26}$$

is the initial capital the investor must set aside to finance the future operations. In any period s, however, the investor can "pull the plug" and thereby recover L_s.

This formulation can also be equivalently interpreted as follows. Suppose that there is a monitoring technology that can be introduced to monitor the cash flows generated by the firm. Monitoring is costly, and reduces the cash flows each period by $\mu_t - \nu_t$. Once cash flows are monitored, the agent is paid the reservation wage, as there is no longer an incentive problem.

In the above cases, the liquidation value is exogenous. An example of an endogenous liquidation value is given by the following. Suppose that upon termination and seizure of the assets, the investor can hire an equivalent, new agent to replace the old. The old agent is dismissed, and receives the reservation utility of 0.

Assume the pool of available new agents is competitive. Then when hiring the new agent, the investor can offer the agent the contract that yields the highest possible continuation utility for the investor. That is,

$$L_s = \max\left\{\hat{b}_s(a) : a \geq \hat{a}_s^0\right\}.^{18} \tag{27}$$

Note that under this specification, termination is always renegotiation-proof. This leads to the following simplification of the characterization of the optimal mechanism in Section 4.

PROPOSITION 13. Suppose L_s is given by (27) for all s. Then the optimal contract is renegotiation-proof and is given by

$$a_s^0 = 0\,,\quad a_s^L = \hat{a}_s^0\,,\quad L_s = \hat{b}_s(\hat{a}_s^0) \quad \text{and} \quad l_s = 0,$$

together with (13) – (17).

PROOF: See the Appendix.

Obvious variations on the above can be considered. For example, each period the investor may have the option of either monitoring or firing and replacing the agent. In that case,

$$L_s = \max\left(\hat{b}_s(\hat{a}_s^0),\, e^{-r(t-s)}(\nu_t + L_t)\right), \tag{28}$$

where $t = s^+$.

Another variation is to suppose that there are costs associated with hiring a new agent, and to allow for the possibility that the new agent may have capital to contribute to the project. Let $\Delta^a{}_t$ and $\Delta^b{}_t$ represent the switching cost of the new agent and the investor, respectively, and let k_t be the capital of the new agent. Then

$$L_s = \max_{a,k}\ \hat{b}_s(a) + k - \Delta_s^b \qquad \text{s.t.} \quad a \geq k + \Delta_s^a,\ k \leq k_s\,, \tag{29}$$

[18] Note that here we use the continuation schedule $\hat{b}_s$ rather than b_s since the new agent is hired after the termination decision is made.

where the constraint $a \geq k + \Delta^a{}_s$ is required for the new agent to be willing to join and contribute capital k.

In the case in which there are no switching costs and new agents have sufficient capital, there is an obvious solution – simply rent the assets to a new agent each period. This is formalized below:

PROPOSITION 14. Suppose L_s is given by (29), $\Delta^a{}_s = \Delta^b{}_s = 0$, and $k_s \geq e^{-\gamma(t-s)}\mu_t$ for all s, $t = s^+$. Then the first best is obtained by terminating the project each period, and hiring a new agent in period s who contributes $e^{-\gamma(t-s)}\mu_t$ and consumes Y_t. This is equivalent to setting $c^L{}_s = 0$ and $x_s = e^{-\gamma(t-s)}$. If k_s is inadequate, or if there are positive switching costs, the first best is not obtained unless $\gamma = r$ and $I = 0$.

PROOF: See the Appendix.

Now we turn our attention to cases in which R_s is non-trivial. For example, suppose that the agent has some unique talent which can be applied in another activity in which cash flows are observable. This would lead to the exogenous specification:

$$L_s = \textstyle\sum_{t>s} e^{-r(t-s)}\, \nu_t\,,$$

$$R_s = \textstyle\sum_{t>s} e^{-\gamma(t-s)}\, \eta_t\,. \tag{30}$$

Generally, we would suppose $\eta_t \leq \mu_t$ (the alternative activities is the second best use of the specific talents), so that (24) holds. Indeed, we would also expect $\eta_t + \nu_t \leq \mu_t$.

In the special case in which production in termination is efficient (i.e., when $\eta_t + \nu_t \geq \mu_t$), we can characterize completely the solution. Note that termination is not efficient in this case if $\gamma > r$. Nevertheless, as we now show, the optimal contract can do no better.

PROPOSITION 15. Suppose (30) holds and $\eta_t + \nu_t \geq \mu_t$ for all t. Then it is optimal to terminate the project each period. Thus, the agent's initial borrowing capacity is L_0. In fact a necessary and sufficient condition for this result is

$$\eta_t + \nu_t \geq \mu_t - (e^{(\gamma-r)(t-s)} - 1)\, \nu_t,$$

where $s = t^-$.

PROOF: See the Appendix.

As an immediate application of Proposition 15, consider the model of sovereign debt of Bulow and Rogoff (1989). In that model, a consortium of banks (the investor) lend to an LDC (the agent). If the LDC defaults, it is possible for the banks to monitor the LDC's exports Y_t and seize up to some fraction β of them. To prevent seizure, the LDC is willing to pay β Y_t each period to the banks. Thus, seizure represents a type of "monitoring" technology in which there is an upper bound on the amount the agent can be forced to repay. The natural question then is whether a contract can be used to achieve superior results than continuous monitoring with the threat of seizure. Proposition 15 shows that since monitoring/seizure can be implemented without destroying cash flows, there is no superior contract. In fact, the result holds as long as the monitoring/seizure costs are small enough relative to β and the relative impatience of the LDC. If these costs are large, then an optimal contract can be used to save monitoring costs.

Another example in which R_s is non-trivial is when it is the agent's human capital that is the scarce resource, and the physical assets are easily replaceable. In this case, L_s corresponds to the liquidation/replacement value of the assets. If the project is terminated, the agent looses control of the existing assets, but can start a new firm and continue the project with new assets. This implies that, for $t = s^+$,

$$R_s = \max\left\{a \geq \hat{a}_s^0 : \hat{b}_s(a) \geq L_s\right\} \cup \left\{e^{-\gamma(t-s)} R_t\right\}. \tag{31}$$

That is, the agent earns the highest possible payoff consistent with being able to purchase new assets by borrowing L_s from a new investor. If it would not be possible for the agent to finance a new project this period, the agent's must wait until next period to attempt financing.

> **PROPOSITION 16.** Suppose R_s is given by (31) for all s. Then the optimal contract is renegotiation-proof.

PROOF: See the Appendix.

This model is closely related to the model of Hart (1995; Chapter 5). There, however, cash flows are deterministic so that $Y_t = \mu_t$. The key emphasis of this model is the "inalienability of human capital," or the agent's right to quit. Here we model that by assuming the agent can quit and start a new firm if new capital can be raised to purchase equivalent assets for price L_s.[19] The difference between our model and Hart (1995) is the following. Since we assume that asset purchases are observable, if the agent quits and starts a new firm in period s, the entire amount L_s must be financed externally. Any funds contributed by the agent could be seized by the creditors of the initial firm. Hart (1995) instead has a specification which is equivalent to assuming that the agent can contribute capital to start the new firm.

This is best illustrated by an example. Suppose $Y_t = \mu_t = 10$, $\nu_t = 6$, $T = 6$, $r = 0$ and γ is very close to zero, but positive.[20] Since there is no uncertainty, from Proposition 12 the credit line is zero. The chart below calculates the optimal long-term debt contract x_t, as well as the payments from Hart (1995).

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[19] In Hart (1995), there is no such outside option, but instead the agent is assumed to have sufficient bargaining power. The end result is the same.

[20] This assumption on γ selects Hart's "slowest" repayment policy.

t	0	1	2	3	4	5
Y_t		10	10	10	10	10
L_t	30	24	18	12	6	0
x_t		10	0	8	10	0
$H95$		6	6	6	6	0

To fund the project at date 0, the agent must invest 30 to buy the initial assets. Under our solution, the agent can borrow 28 and so must contribute 2. Under Hart (1995), the entrepreneur can borrow at most 24 and so must contribute 6.

To understand the difference, consider the payment in period 4. Here, $x_4 = 10$ since the agent is threatened with losing the project in period 5, which is worth 10. The agent would prefer to pay 6 to buy new assets, but these assets will be seized by the initial creditors. The agent cannot externally finance the new assets for 6, since the agent cannot commit to repay any loan in period 5. This differs from Hart (1995), since there the agent will never pay more than the current value of the assets.

In period 3, $x_3 = 8$. Note that the threat of termination could induce the agent to pay up to 10. However, $x_3 + x_4$ cannot exceed 18 since otherwise at the end of period 2 the agent would quit and start a new firm by raising 18 and promising 8+10 to the new creditor. This also implies $x_2 = 0$, since if $x_2 > 0$ the agent will again default, consume x_2 and start a new firm. This cycle then repeats as more periods are added to the model.

The above example highlights the distinction between our model and Hart (1995). Since creditors can seize new assets, our contracts leads to superior outcomes. Of course, our model also generalizes to the case of uncertain cash flows and different discount rates.

Another variation of our model subsumes the model of Gromb (1999). Gromb considers an environment in which participation of both the investor and the agent are necessary (neither party has an outside option). There is a sequence of available positive NPV projects. The project in period t requires investment ν_t and produces cash flows Y_t. (Gromb restricts this further by assuming stationarity and Y_t binary.) Each period a decision must be made whether to finance the current project, or whether to "mothball" and remain idle.

Absent renegotiation, Gromb's model coincides with that discussed in (26) above, since the investor can commit to permanent liquidation and refuse to finance any future projects (which Gromb shows is optimal).

With renegotiation, this environment can be modeled with L_s as in (28), and

$$R_s = \begin{cases} \hat{a}_s^0 & \text{if} \quad \hat{b}_s(\hat{a}_s^0) > e^{-r(t-s)}(\nu_t + L_t) \\ e^{-\gamma(t-s)} R_t & \text{otherwise} \end{cases},$$

that is, the investor gets the highest possible continuation payoff with the agent employed, or mothballs and recovers ν_t. Correspondingly, the agent only earns rents when the project is not mothballed.

7. Asset Substitution and Agency

The previous section investigated observable investment decisions. In this section consider briefly some of the consequences of unobserved investment decisions by the agent.

First we consider a pure form of the asset substitution problem; namely, that the agent can influence the riskiness of future cash flows, but not their mean. Specifically, suppose on date s, the agent can choose a parameter ϕ_s such that for $t > s$,

$$E[\, Y_t \mid \phi_s \,] = E[\, Y_t \,] = \mu_t. \tag{32}$$

Of course, other moments of the distribution of the cash flows are potentially affected by ϕ_s, so that for example $Var(\, Y_t \mid \phi_s \,)$ need not be constant in ϕ_s.

Generally speaking, most models of debt contracts have the consequence that the agent has an incentive to increase the riskiness of cash flows (Jensen and Meckling (1976)). This results in a transfer of wealth from debt holders to the agent as the residual claimant or equity holder. This problem is only avoided if debt is riskless. Since debt is risky in our model, one might expect to find this same incentive for asset substitution. Interestingly, this is not the case as shown below.

> **PROPOSITION 17.** Under the optimal contract of Section 4, the agent is indifferent with respect to the choice of the pure asset substitution parameter ϕ_s satisfying (32). Thus, the agent can be assumed to choose ϕ_s optimally for the investor, which implies the elimination of any mean-preserving spreads. Thus, the contract of Section 4 remains optimal.

PROOF: See the Appendix.

The intuition for the above result is the following. Increasing risk generally benefits equity holders since they participate in the gains, but only partially in the losses. In our model, incentive compatibility requires that the agent pay for losses through forfeiture of control of the assets. Thus there is no gain for the agent from increasing risk. There is a loss for the debt holders, however, since generally there may be dead weight costs associated with seizing the assets and terminating the project (i.e., the full information value of the project exceeds $R_t + L_t$).[21]

Consider next unobserved effort provision by the agent. Suppose the agent can spend effort $e_s \in \xi_s$ in period s, where e_s is denominated in consumption-equivalent units and we assume $0 \in \xi_s$. Suppose that this effort affects only output Y_s, but not future output. Since the agent's payoff under the contract in period s is then

$$E[\ Y_s \mid e_s\ ] + g_s - e_s,$$

the agent will choose e^*_s to solve

$$\max_{e \in \xi_s} E\left[Y_s \mid e_s = e\right] - e \ . \tag{33}$$

Note that this effort choice is identical to optimal choice in a first-best world with complete contracts and symmetric information. However, though effort is first best under the contract derived in section 4, the contract of section 4 is no longer optimal. To see why, note that the investor's payoff is also increasing in the project's output. This externality is not included in problem (33). That is, in this second best world, the optimal effort level generally exceeds the first best level.[22] Intuitively, high output allows the parties to avoid inefficient termination (which is not a problem in a first best world). An optimal contract will induce higher effort by rewarding high output.[23]

There is one special case in which the contract of this paper is optimal even when the agent takes an unobserved effort decision. This occurs if the first best level of effort is already the highest level of effort possible. While such a corner solution is unlikely in the general case, it is common in models with binary (high/low) effort choices.

> **PROPOSITION 18.** Suppose an increase in e_s increases Y_s in the sense of First Order Stochastic Dominance. If the solution e^*_s to (33) is such that $e^*_s = \max\{ e \in \xi_s \}$, then the contract of Section 4 remains optimal.

PROOF: See the Appendix.

The above analysis considered contemporaneous effort; i.e., effort that affects current output. If effort also affects future output, then the contract of section 4 no longer induces the first best effort level. There are two reasons for this. First, the agent

[21] This same argument holds in other models in which incentives are provided strictly through non-pecuniary penalties, as introduced by Diamond (1984).

[22] To be precise, it will always exceed the first best level if effort increases output in the sense of First Order Stochastic Dominance, and if the first best effort is interior. On the other hand, if effort also increases risk, then it is possible that optimal effort is below the first best level.

[23] In terms of the notation of section 4, an optimal contract in the presence of moral hazard will use a function $G_s(Y_s)$ that is increasing rather than constant to induce effort that exceeds the first best. Note that rewarding the agent in this way is costly due to the concavity of the investor's continuation function b. Thus, G must be chosen to balance incentives and risk. We plan to explore this problem further in future work.

discounts future cash flows at rate $\gamma > r$. Second, there is a positive probability in general that the agent will be terminated prior to receiving the benefit of the effort.

8. Concluding Remarks

We have assumed that the project is run at a fixed scale. In DeMarzo and Fishman (2001) we extend the analysis to cover more general agency problems and we allow the scale to be determined as part of the optimal contract. A key result here is that even though the profitability of current investment is independent of the profitability of past investment, the optimal contract entails more current investment if the past business cash flows are high. Like the threat of a transfer of control, the promise of funding for new investment can be used to induce the agent to pay the investor. This result matches the empirical finding that, after controlling for investment opportunities, firms' investment decisions are positively correlated with cash flow; see Fazzari, Hubbard and Petersen (1988). Moreover our model also matches their finding that the correlation between cash flow and investment is higher for firms that are not paying dividends. We also show that investment will be positively serially correlated over time.

To be completed.

9. Appendix

PROOF OF PROPOSITION 1: Let (y^t, ω^t) be a cash flow and termination history. Then given the agent's strategy (d, m), we can infer the history $h_t = (d_s, m_s, \omega_s)_{s\leq t}$ that would occur given y^t. Let this mapping be given by $H(y^t, \omega^t) = h_t$. If for some s, y_s is not in the support of Y_s, then $H(y^t, \omega^t)$ is not well-defined by the above. In that case, define $H(y^t, \omega^t)$ by computing h_t as though $y_s = 0$, which is in the support by assumption.

Define the contract σ^* as follows. Given the history h_t, let d^t be the payments made by the agent. Then let $p^*_t(h_t) = p_t(H(d^t, \omega^t))$; that is, the probability of termination is the same as under the original contract if $y^t = d^t$. Also $w^*_t(h_t) = c_t(d^t, \omega^t)$; i.e., the consumption he would have under strategy α and contract σ if $y^t = d^t$.

Clearly, under that strategy, $A^*(\alpha \mid \sigma) = A^*(\alpha^* \mid \sigma^*)$. We now argue that α^* above is an optimal strategy given σ^* for the agent. To see that it is optimal, suppose that given σ^* the agent plays some feasible strategy $\alpha' = (d', m', c', q')$ instead. Let

$$e'_t(y^t, \omega^t) = c'_t(y^t, \omega^t) - w^*_t(d'(y^t, \omega^t), m'(y^t, \omega^t), \omega^t) = c'_t(y^t, \omega^t) - c_t(d'(y^t, \omega^t), \omega^t),$$

the agent's consumption in excess of his payment from the investor after any history. The net savings of the agent in period t are thus $y_t - d'_t(y^t, \omega^t) - e'_t(y^t, \omega^t)$. We now show that there is an equivalent feasible deviation under σ.

Let $y'^t = d'(y^t, \omega^t)$ and define the strategy $\alpha'' = (d'', m'', c'', q')$ under contract σ, with $d''_t(y^t, \omega^t) = d_t(y'^t, \omega^t)$, $m''_t(y^t, \omega^t) = m_t(y'^t, \omega^t)$, and $c''_t(y^t, \omega^t) = c_t(y'^t, \omega^t) + e'_t(y^t, \omega^t)$. That is, the agent behaves as though $y^t = d'(y^t, \omega^t)$, and also consumes the excess amount e'_t. Clearly, $A^*(\alpha' \mid \sigma^*) = A^*(\alpha'' \mid \sigma)$.

To see that (d'', m'', c'') is feasible under σ, imagine the agent maintains two private accounts. The agent deposits $Y_t = y_t$ in account 1, and then transfers $d'_t(y^t, \omega^t) = y'_t$ from account 1 to account 2. Then he pays $d_t(y'^t, \omega^t)$ from account 2 to the investor. The agent then receives $w_t(d(y'^t, \omega^t), m(y'^t, \omega^t), \omega^t)$ into account 2. Finally, the agent consumes $c_t(y'^t, \omega^t)$ from account 2 and $e'_t(y^t, \omega^t)$ from account 1. The net cash flows to each account in period t are thus:

Account 1: $\quad y_t - d'_t(y^t, \omega^t) - e'_t(y^t, \omega^t)$

Account 2: $\quad y'^t - d_t(y'^t, \omega^t) + w_t(d(y'^t, \omega^t), m(y'^t, \omega^t), \omega^t) - c_t(y'^t, \omega^t)$

Feasibility of α' under σ^* implies that the balance in account 1 does not fall below zero. The balance in account 2 does not fall below zero by feasibility of α under σ given realization y'^t.

Thus, by optimality of α under σ, we have

$$A^*(\alpha' \mid \sigma^*) = A^*(\alpha'' \mid \sigma) \leq A^*(\alpha \mid \sigma) = A^*(\alpha^* \mid \sigma^*).$$

Finally, we need to show that $B^*(\alpha \mid \sigma) \leq B^*(\alpha^* \mid \sigma^*)$. To see this, note that for any contract and strategy of the agent, the feasibility constraints plus the fact that $\rho \leq r$ implies that

$$B^* \leq -I + E[\textstyle\sum_{t\geq 0} e^{-rt} (Y_t - c_t) + e^{-r\tau} L_\tau].$$

(If $\rho = r$ this holds with equality.) Since this holds with equality under the new contract, and the liquidation policy and consumption is unchanged, the result follows. ◆

PROOF OF PROPOSITION 6: First suppose investors are competitive. If the agent contributes d, the investor must receive at least $I - d$ in continuation. Thus, the agent's continuation payoff a satisfies $b_0(a) \geq I - d$. Since investors are competitive, they will offer $\max\{a : b_0(a) \geq I - d\}$. This is increasing in d as long as $d \leq I - b_0(a^1_0)$. Thus, the agent will contribute d_0. Given this, the agent's payoff is $\max\{a : b_0(a) \geq I - d_0\}$, and the agent only earns a profit if this exceeds d_0. Thus, the agent's payoff is

$$\max\{a \geq d_0 : b_0(a) \geq I - d_0\} + Y_0 - d_0.$$

Since $d_0 = Y_0$ is equivalent to $a \leq a^1_0$, this is equivalent to $a^*(Y_0)$.

Next suppose the investor is a monopolist. Then he will choose the continuation with the highest payoff for the investor, which implies that the agent receives a^{**}. Thus, the agent will be willing to pay up to a^{**} to participate. If there is renegotiation, $b'_0 \leq 0$, so that $a^{**} = a^0_0$. ◆

PROOF OF PROPOSITION 7: From (2), $a + \hat{b}_T(a) \le L_T + R_T$. Thus, $l_T \le -1$ and $T^* \le T$. Recall that the agent's continuation payoff at T^* is $\min(y_{T^*} + g_{T^*}, a^1{}_{T^*})$. Since $a^1{}_{T^*} = R_{T^*} = a^0{}_{T^*} \le g_{T^*}$, this continuation payoff is R_{T^*}. Thus from (12), $p_{T^*} = 1$, and termination by T^* occurs under the optimal contract σ.

As $\gamma \to \infty$, the agent's continuation payoff from any path goes to zero. Hence, the agent will consume all cash immediately, and cannot be induced to pay anything to the investor. The investor will therefore choose to terminate immediately (not start the project) and receive L_0.

If $\gamma = r$, then $b'_T = -1$. Similarly, for $s < T$, $\hat{b}_s' \ge -1$ and thus $b'_s \ge -1$. This implies that $b_t \equiv b^1{}_t$ for all t.

To show $T^* = T$, first define the following,

$$V_s = \sum_{s<t\le T} e^{-\gamma(t-s)}\,\mu_t + e^{-\gamma(T-s)}\,R_T.$$

We now show that $a^1{}_s = V_s$ by induction. Note that $a^1{}_T = a^0{}_T = R_T = V_T$. Thus suppose $a^1{}_t = V_t$ for $t = s^+$. Then

$$V_s = e^{-r(t-s)}(\mu_t + a^1{}_t).$$

Hence, from the definition of $a^1{}_t$, $\hat{b}_s'(V_s) = E\left[b_t'\left(a_t^1 + Y_t\right)\right] = -1$ and, since 0 is in the support of Y_t, $\hat{b}_s'(a) > -1$ for $a < V_s$. Thus, $a^1{}_s = V_s$ as long as $V_s \ge a^L{}_s$. To see this holds, first note that

$$\hat{b}_s(V_s) = e^{-r(t-s)}\left(\mu_t + E\left[b_t\left(a_t^1 + Y_t\right)\right]\right) = e^{-r(t-s)} b_t\left(a_t^1\right).$$

Iterating the above and using the fact that $b_T(a^1{}_T) = b_T(R_T) = L_T$, we have

$$\hat{b}_s(V_s) = e^{-r(T-s)} L_T. \tag{34}$$

Next, from the definition of V_s and T,

$$V_s + \hat{b}_s(V_s) = \sum_{s<t\le T} e^{-r(t-s)}\mu_t + e^{-r(T-s)}\left(R_T + L_T\right) > R_s + L_s.$$

This implies that $l_s > -1$ and that, from (1),

$$R_s < V_s + \hat{b}_s(V_s) - L_s = V_s + e^{-r(T-s)} L_T - L_s \le V_s.$$

Thus,

$$a^1{}_s = V_s > a^L{}_s \ge R_s, \tag{35}$$

verifying the induction hypothesis.

Since $l_s > -1$, we have shown that $T^* = T$. ◆

PROOF OF PROPOSITION 9: This follows from the case $\gamma = r$ in Proposition 7. For $a^1{}_s$ and $b_s(a^1{}_s)$, see (34) and (35). Finally, $a^1{}_s = V_s$ implies $x_t = \mu_t + a^1{}_t - e^{r(t-s)}\, a^1{}_s = 0$. ◆

PROOF OF PROPOSITION 10: Note that from the definition of x_t and the fact that $a^1{}_{T*} = R_{T*}$,

$$\sum_{s<t\leq T^*} e^{-\gamma(t-s)}\, x_t = V_s - a^1{}_s. \tag{36}$$

Using the definition of $c^L{}_s$ then gives the first result. Since

$$a_s^L \geq \hat{a}_s^0 = e^{-\gamma(t-s)}\left(\mu_t + a_t^0\right),$$

where $t = s^+$, and $a^0{}_t \geq R_t$,

$$V_s - a^L{}_s \leq V_s - e^{-\gamma(t-s)}(\mu_t + R_t) = e^{-\gamma(t-s)}(V_t - R_t).$$

◆

PROOF OF PROPOSITION 11: Recall that

$$\hat{b}_s{}'(a) = e^{(\gamma-r)(t-s)} E\left[\max(b_t{}'(e^{\gamma(t-s)}a + Y_t - \mu_t), -1)\right],$$

Since $\gamma \geq r$, $Y_t \geq 0$, and b'_t is decreasing, $\hat{b}'_s\left(a\right) \leq -1$ if $e^{\gamma(t-s)}a - \mu_t \geq a^1{}_t$. Thus, since $b'_s \leq \hat{b}'_s$, $b'_s(a) \leq -1$ if $a \geq e^{-\gamma(t-s)}(a^1{}_t + \mu_t)$. Thus, if $R_s \leq e^{-\gamma(t-s)}(a^1{}_t + \mu_t)$, then $a^1{}_s \leq e^{-\gamma(t-s)}(a^1{}_t + \mu_t)$ and $x_t \geq 0$. The sufficient condition follows immediately since $a^1{}_t \geq R_t$.

Suppose instead that $R_s > e^{-\gamma(t-s)}(a^1{}_t + \mu_t)$. Then, from the definition of T^*, liquidation is not efficient: $\hat{b}_s(R_s) > L_s$. Hence,

$$a^1{}_s = a^L{}_s = a^0{}_s = R_s. \tag{37}$$

Then $x_t < 0$ and $c^L{}_s = 0$ follow immediately.

Note that

$$a^1_{s^-} \geq \hat{a}^0_{s^-} = e^{-r(s-s^-)}\left(\mu_s + a_s^0\right) = e^{-r(s-s^-)}\left(\mu_s + a_s^1\right).$$

Thus, $x_s \leq 0$. Also, this implies

$$a^1_{s^-} = a^L_{s^-} = \max\left(\hat{a}^0_{s^-}, R_{s^-}\right),$$

so that $c^L{}_{s-} = 0$.

Now we prove the last result. From (36), we need to show that $a^1{}_s \leq V_s$. This holds for T^*; suppose it holds for $t \leq T^*$. Then, by the same arguments as above,

$$R_s \leq a^1{}_s \leq e^{-\gamma(t-s)}(a^1{}_t + \mu_t) \leq e^{-\gamma(t-s)}(V_t + \mu_t) = V_s.$$

◆

PROOF OF PROPOSITION 12: Note that $c^L{}_{T*} = 0$ immediately. We now show that for $t < T^*$,

$$a_t^1 = a_t^L = \max(\hat{a}_t^0, R_t) \text{ and } c^L{}_t = 0.$$

The case $t = T^{*-}$ follows from (21). We show it holds for $s < t$ by induction.

Suppose for now that $Y_t = \mu_t$. Recall that

$$\hat{a}_s^0 = e^{-\gamma(t-s)}\left(\mu_t + a_t^0\right).$$

Thus,

$$\hat{b}_s'(\hat{a}_s^0) = e^{(\gamma-r)(t-s)} E\left[\max\left(b_t'(e^{\gamma(t-s)}\,\hat{a}_s^0 + Y_t - \mu_t), -1\right)\right]$$
$$= e^{(\gamma-r)(t-s)} \max\left(b_t'(\mu_t + a_t^0), -1\right).$$

This yields the intermediate result

$$\mu_t + a^0{}_t \geq a^1{}_t \text{ implies } \hat{b}_s'(\hat{a}_s^0) \leq -1. \tag{38}$$

Suppose $\hat{b}_s'(\hat{a}_s^0) \leq -1$. Then since $l_t > -1$,

$$a_s^1 = a_s^L = \max(\hat{a}_s^0, R_s) \text{ and } c^L{}_s = 0.$$

Hence, it remains to show that $\mu_t + a^0{}_t \geq a^1{}_t$, or

$$\mu_t \geq a_t^1 - a_t^0 = \max(\hat{a}_t^0, R_t) - a_t^0.$$

Since $a^0{}_t \geq R_t$ and $\mu_t \geq 0$, it is sufficient to show that

$$\mu_t \geq \hat{a}_t^0 - R_t = e^{-\gamma(t^+-t)}\left(\mu_{t^+} + a_{t^+}^0\right) - R_t.$$

Since termination is not renegotiated, $a^0{}_{t+} = R_{t+}$. But then, from our assumption on μ_t, and also using (1),

$$e^{-\gamma(t^+-t)}\left(\mu_{t^+} + a_{t^+}^0\right) - R_t = e^{-\gamma(t^+-t)}\left(\mu_{t^+} + R_{t^+}\right) - R_t \leq e^{-\gamma(t^+-t)}\mu_{t^+} \leq \mu_t.$$

which completes the induction.

Note that to derive (38) above, we assumed $Y_t = \mu_t$. It is easy to verify that (38) requires only that

$$\Pr(Y_t \geq \mu_t) \geq \frac{e^{(r-\gamma)(t-s)} + b_t'(a_t^0)^+}{1 + b_t'(a_t^0)^+}.$$

For x_t, note that

$$x_t = \mu_t + a_t^1 - e^{\gamma(t-s)} a_s^1 = \mu_t + \max\left(\hat{a}_t^0, R_t\right) - e^{\gamma(t-s)} \max\left(\hat{a}_s^0, R_s\right)$$

$$\le \mu_t + \max\left(\hat{a}_t^0, R_t\right) - e^{\gamma(t-s)}\hat{a}_s^0 = \mu_t + \max\left(\hat{a}_t^0, R_t\right) - \left(\mu_t + a_t^0\right)$$

$$= \max\left(\hat{a}_t^0, R_t\right) - a_t^0 \le \left(\hat{a}_t^0 - R_t\right)^+ . \tag{39}$$

Finally, if (24) holds and *if termination is not renegotiated*, then $R_s \le \hat{a}_s^0 = e^{-\gamma(t-s)}(\mu_t + a_t^0)$ and $a^0{}_t = R_t$, and so the above holds with equality. ◆

PROOF OF PROPOSITION 13: This specification implies $b'_t \le 0$ for all t. Hence, $\hat{b}_s' \le 0$, so that $\hat{b}_s$ is maximized at $\hat{a}_s^0$. The rest is immediate. ◆

PROOF OF PROPOSITION 14: It is obvious that the above solution attains the first best. To see how it relates to our earlier characterization, suppose the project is terminated for sure next period. Then $\hat{a}_s^0 = e^{-\gamma(t-s)}\mu_t$ and $\hat{b}_s'(\hat{a}_s^0) \le -1$. From (29), $L_s = \hat{b}_s^0 + \hat{a}_s^0$ with so that $l_s = -1$ and the project is terminated in period s. Alternatively, since $l_s = -1$, we can instead set $a_s^1 = a_s^L = \hat{a}_s^0$ (see footnote 13), which gives the second implementation. Finally, if k_s is inadequate or switching costs are positive, then first best cannot be attained unless there is no possibility termination. However, without termination, the agent's incentive constraint cannot be satisfied unless the agent consumes all cash flows. This is consistent with first best only if $\gamma = r$ and $I = 0$. ◆

PROOF OF PROPOSITION 15: Suppose termination occurs for sure in period t. Then for $s = t^-$,

$$\hat{a}_s^0 = e^{-\gamma(t-s)}(\mu_t + R_t) \quad \text{and} \quad \hat{b}_s^0 = e^{-r(t-s)}L_t ,$$

where we use the fact that $a^0{}_t = R_t$, $b^0{}_t = L_t$, and $b'_t(a^0{}_t) \le -1$. Since $\gamma \ge r$,

$$\hat{a}_s^0 + \hat{b}_s^0 = e^{-\gamma(t-s)}\mu_t + e^{-\gamma(t-s)}R_t + e^{-r(t-s)}L_t$$

$$\le e^{-\gamma(t-s)}\eta_t + e^{-\gamma(t-s)}R_t + e^{-r(t-s)}\nu_t + e^{-r(t-s)}L_t = R_s + L_s .$$

Thus, $l_s \le -1$ and termination is optimal in period s. The necessary and sufficient condition follows from the above inequality. ◆

PROOF OF PROPOSITION 16: Clearly, if $R_s > e^{-\gamma(t-s)}R_t$, then $a_s^0 = R_s \ge \hat{a}_s^0$. Also, $b_s'(a_s^0) = \hat{b}_s'(R_s) \le 0$. If $R_s = e^{-\gamma(t-s)}R_t$, then $L_s > \max_{a \ge R_s} \hat{b}_s(a)$ and thus $l_s < 0$. ◆

PROOF OF PROPOSITION 17: From the analysis in section 4, the agent's continuation payoff in period t is equal to $Y_t + g_t$ for some constant g_t. Thus, the agent's expected payoff is unaffected by ϕ_s. On the other hand, the investor's continuation payoff in period t is given by (see (7))

$$Y_t + b^1{}_t(Y_t + g_t).$$

Since $b^1{}_t$ is concave, the investor has an induced risk aversion regarding the cash flows of the firm. Thus, the investor will prefer that ϕ_s be chosen to eliminate mean-preserving-spreads. ◆

PROOF OF PROPOSITION 18: Recall from section 4 that the agent's incentive compatibility constraint on reporting output implies that agent's payoff in period s is given by

$$Y_s + G_s(Y_s) - e_s,$$

where G_s is weakly increasing. Given such a payoff, by FOSD and the fact that $e^*{}_s = \max\{e \in \xi_s\}$ solves (33), the agent will choose effort $e^*{}_s$ under any feasible contract. Thus, we can regard this effort choice as exogenous and solve for the optimal contract as in the previous analysis. ◆

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