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Partnership in a Dynamic Production System

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Abstract

This paper considers two firms that engage in joint production. The prospect of repeated interaction introduces dynamics in that actions that firms take today influence the costliness and effectiveness of actions in the future. Repeated interaction also facilitates the use of informal agreements (relational contracts) that are sustained not by the court system, but by the ongoing value of the relationship. We characterize the optimal relational contract in this dynamic system with double moral hazard. We show that an optimal relational contract has a simple form that does not depend on the past history. The optimal relational contract may require that the firms terminate their relationship with positive probability following poor performance. This may occur even when the firms observe an independent signal for the action of each firm that allows them to assign blame. If, however, the buyer's action does not influence the dynamics, the need for termination is eliminated. The paper applies the method to the issue of sequential versus parallel collaborative product development.

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1 Introduction

The success of a buyer-manufacturer relationship in creating a product often depends on the actions of both parties. Each firm has comparative strengths in understanding different aspects of the product's design and the technical aspects related to its production. Accordingly, the buying firm may provide critical technical expertise to assist the supplier in design, engineering and production. Although a manufacturer typically provides the bulk of the infrastructure for production, a buying firm may provide critical inputs, either in the form of specialized equipment or raw materials. The success of the production process, then, depends on the diligence with which both firms provide the associated physical and managerial inputs.

When a buyer and manufacturer interact, they rarely do so anticipating that they will with certainty never interact again. The prospect of potential future interaction shapes how firms behave in two ways. First, the prospect of future interaction facilitates the development of trust and cooperation. Firms will be more hesitant to behave opportunistically if they anticipate that doing so will damage their prospects for engaging trading partners in the future. Second, the prospect of future interaction introduces dynamics as conditions and the relationship itself evolves over time. In particular, a firm's actions today impact the costliness and effectiveness of actions in the future. For example, a manufacturer may invest in technologies that reduce the costs of producing certain types of products in the future. Such an investment could impact the effectiveness of the buying firm's action positively or negatively. For example, if the buyer is unfamiliar with the technologies, this may reduce the utility of the buyer's production expertise. Further, exogenous factors, such as general economic conditions, evolve over time, and these also shape the costliness and effectiveness of the firms' actions.

When firms engage in joint production, each firm observes the output of the process but does not directly observe the full scope the actions taken by its partner. Because there is uncertainty in how actions influence output, it is difficult for the firms to assess who bears what portion of responsibility for the relative success or failure of the output. This introduces a temptation for the firms to "free-ride." Each firm knows that it can assert that a bad outcome is due to the failure of the other firm or simple bad luck. Further, even if the firms could assess who was responsible ex post, given the complexity of the firms' interaction and the production process, it would be difficult or impossible ex ante to specify in a contract payments that accurately reflect the true allocation of responsibility. We focus on the setting where it is difficult to specify ex ante in terms that are verifiable ex post the quality of the output.

Biopharmaceutical manufacturing provides an example of joint production. Drug developers (buyers) contract with contract manufacturers for capacity during the drug development process, at least one year in advance of production. At the time of production, the buyer provides the manufacturer with genetically modified mammalian cells and the manufacturer ferments these cells to produce a target protein. The success of the production process depends both on the quality of the raw materials and the skill with which the production process is managed. Because the specifics of the production process (e.g., temperature, pressure) are unknown at the time of contracting, the firms contractually specify the price and the rate of “batch fermentation starts” but do not attempt to make the contract contingent on the protein yield.

Firms can provide stronger incentives for action by developing informal agreements that make payments contingent on nonverifiable output (see §2 for an example in the biopharmaceutical context). Because such payments are discretionary, they must be enforced by the value of the ongoing cooperative relationship rather than the court system. Our objective is to characterize how firms should optimally structure informal agreements in the face of dynamics and the temptation to free-ride.

The primary vehicle in economic theory for studying long-term relationships where trust, cooperation and reputation are important is the repeated game, in which players face the same “stage game” in every time period and each player seeks to maximize the discounted sum of his payoffs. Typically, a repeated game has many possible Nash equilibria, but the players are assumed to coordinate on one that is mutually advantageous. Cooperation is enforced by the threat of transition to an undesirable Nash equilibrium in the continuation game.

Klein and Leffler (1981) and Taylor and Wiggins (1997) consider settings where product quality is noncontractible and is solely a function of the manufacturer’s effort. Klein and Leffler show that in a competitive market, buyers will pay a premium above variable production cost to firms that maintain a reputation for high quality. In Taylor and Wiggins (1997) a buyer inspects every shipment from his manufacturer and rejects faulty items. Taylor and Wiggins show how the buyer can avoid costly inspection by paying a premium for every shipment and threatening to terminate this practice if he later discovers faulty items.

Baker et al. (2001, 2002) emphasize that players may shape their repeated game through transfer payments. They have popularized the term *relational contract* for an informal agreement regarding actions and voluntary payments, enforced by reputational concerns, between parties that interact repeatedly. They study a repeated game with relationship-specific investment by one party (“hold up”) and derive insights regarding optimal ownership structure. Levin (2003) examines relational contracting in a principal-agent model with moral hazard or hidden information. He proves that simple stationary relational contracts are optimal. In particular, with moral hazard, the relationship is never terminated on the equilibrium path, and the voluntary payment to the agent is "one-step": a bonus if output exceeds a threshold. Our structural analysis of relational contracts adopts techniques from Levin, but our formulation is substantively distinct.

This paper characterizes the optimal relational contract for supply chain partners in a Markov decision process, where actions influence the output in the current period and the cost structure in subsequent periods. The action of one firm cannot be observed by the other, but the state of the system, cost structure, feasible action set, and transition probabilities are common information. That is, we have a dynamic system with double moral hazard. §2 formulates the model, and §3 shows that an optimal relational contract is characterized by an unusual sort of dynamic program. §4 describes structural properties of this optimal relational contract. First, actions depend only on the current state and payments depend only on the observed transition. Second, the optimal relational contract might require that the firms terminate their relationship with positive probability in the event of an undesirable transition. This may occur even when the firms observe an independent signal for the action of each firm that allows them to assign blame. If, however, the buyer's action in period t influences the output in period t but not subsequent periods, then the relationship is never terminated. Third, a simple one-step payment scheme is optimal if the first best expected discounted profit starting from the worst state is sufficiently large. §5 applies the method to the issue of sequential versus parallel collaborative product development. §6 provides concluding remarks.

Two recent working papers also consider double moral hazard and relational contracting, but in a stationary environment with common observation of an independent signal for the effort of each firm. The paper that is closest in spirit to ours, Doornik (2004), was developed independently. Doornik shows that the optimal relational contract requires terminating the cooperative relationship when the signal for both firms is low and, if the relationship continues, a one-step payment analogous to Levin (2003). Doornik's formulation is more general in that it allows both firms to receive a portion of the output produced, whereas we consider the case where a buying firm receives the output. Our formulation is more general in that it allows for dynamics and considers the case where independent signals of effort are not available. Rayo (2004) characterizes optimal ownership structure and optimal (within a limited class) relational contracts in repeated team production. In particular, he restricts attention to non-terminating relational contracts. Ownership determines the allocation of profit in the event of a disagreement, i.e., refusal to execute the transfer payments specified in the relational contract. Rayo shows that when the signals are very noisy, ownership of 100% of joint output should be assigned to a single player. Although we focus on the case where a buyer receives the output, all theoretical results in §3 and §4, with the exception of Propositions

4 and 5, hold with a general division of ownership of output. Structural differences between the results in Rayo (2004) and Doornik (2004) and this paper show how supply chain partners should adapt their relational contracts to a dynamic business environment.

The aforementioned papers all assume that firms have common knowledge of cost structure and how effort influences the distribution of output and signals. For analysis of collaborative production under asymmetric information, we refer the reader to Iyer et al. (2002) and references therein.

2 Model

Joint production is modeled as a dynamic game. The state of the system in period t, X_t, takes values in a finite, discrete state space $\mathcal{X}$. The state reflects both external factors, such as economic conditions, and internal factors, such as the capabilities of the firms. At the beginning of each period t, the buyer and the manufacturer observe X_t and decide whether or not to transact. If both parties agree to transact, then they sign a formal contract under which the buyer contracts to pay p_t in return for the output Y_t from joint production in period t. The manufacturer undertakes a noncontractible, productive action $a_m \in A_m(X_t)$ and incurs cost $c_m(a_m, X_t)$. Similarly, the buyer chooses noncontractible action $a_b \in A_b(X_t)$ and incurs cost $c_b(a_b, X_t)$. The feasible action sets $A_m(x)$ and $A_b(x)$ are closed and bounded for each state $x \in \mathcal{X}$. The actions determine the state transition probabilities through the transition matrix $P(a_m, a_b)$ with elements

$$P_{xz}(a_m, a_b) = \Pr\{X_{t+1} = z \mid X_t = x \ ; \ a_m, a_b\}.$$

For ease of exposition, we formulate the state space so that conditional on the transition (X_t, X_{t+1}), the output in period t is independent of the actions of buyer and manufacturer in period t:

$$Y_t = Y(X_t, X_{t+1}),$$

where $Y : \mathcal{X} \times \mathcal{X} \rightarrow R^+$. If either firm refuses to transact in period t, then both buyer and manufacturer incur zero cost, $Y_t = 0$, and the distribution of X_{t+1} is governed by transition matrix $P(0, 0)$.

Each party seeks to maximize his infinite horizon discounted expected profit, using discount factor $\delta \in (0, 1)$. Throughout we assume that both parties observe the state of the system X_t, but neither party observes the other's action. The action sets, cost functions, and transition matrix are common information. However, although the output and state-transition are observable, these are not formally contractible. Therefore, the manufacturer and buyer enter into a relational contract to provide incentives for effort. The *relational contract* consists of four parts:

i. A formal (court-enforced) contract. If both parties agree to transact in period t, the buyer contracts to pay p_t to the manufacturer in return for the output Y_t. The payment p_t may be contingent on the public history at the beginning of period t.

ii. A discretionary transfer payment at the end of period t, d_t. A positive payment d_t corresponds to the buyer paying the manufacturer, while a negative payment corresponds to the reverse. The payment d_t may be contingent on the public history at the end of period t.

iii. A strategy for the manufacturer which specifies, for each period $t = 1, 2, ...$, whether or not to transact with the buyer $\tau_{mt} \in \{0, 1\}$ and, in the event that both parties agree to transact, action a_{mt} and whether or not to execute the discretionary transfer payment $e_{mt} \in \{0, 1\}$.

iv. A strategy for the buyer which specifies, for each period $t = 1, 2, ...$, whether or not to transact with the manufacturer $\tau_{bt} \in \{0, 1\}$ and, in the event that both parties agree to transact, action a_{bt} and whether or not to execute the discretionary transfer payment $e_{bt} \in \{0, 1\}$.

Note that the buyer must pay the price p_t even if the output yield Y_t turns out to be low or zero. Formal contracts of this nature are common in the semiconductor industry, where the buyer purchases "wafer starts" but his yield on these wafers is stochastic. Similarly, biopharmaceutical contract manufacturers sell "batch fermentation starts" rather than actual output. An example of a discretionary payment in the biopharmaceutical industry was described to the authors by managers at a large contract manufacturer: The manufacturer agrees informally that if the yield of a batch is low due to some error in its process control, the buyer will not be required to make the full payment. However, if the manufacturer attributes the low yield to problems with the raw material provided by the buyer, it will not give the discount. Finally, note that the firms need not transact in every period. The relational contract may stipulate $\tau_{mt} = \tau_{bt} = 0$ in states where the gain from joint production is low.

The manufacturer's discounted profit starting from the beginning of period T is given by

$$\Pi_{mT} = \sum_{t=T}^{\infty} \delta^{t-T} \tau_{bt} \tau_{mt} \left[p_t + d_t e_{bt} e_{mt} - c_m(a_{mt}, X_t) \right]. \tag{1}$$

The buyer's discounted profit starting from period T is given by

$$\Pi_{bT} = \sum_{t=T}^{\infty} \delta^{t-T} \tau_{bt} \tau_{mt} \left[Y_t - p_t - d_t e_{bt} e_{mt} - c_b(a_{bt}, X_t) \right]. \tag{2}$$

The objective for each firm is to maximize its discounted expected profit. We say that a relational contract is *self-enforcing* if, given the prices and discretionary transfer payments in (i) and (ii), the firms' strategies constitute a perfect public equilibrium (PPE) with $e_{bt} = e_{mt} = 1$ for all $t = 1, 2, ...$ That is, the firms are willing to execute the discretionary transfer payment in every period that they transact. As defined in Fudenberg et al. (1994), a profile of strategies is public if in each period t, $\{\tau_{mt}, a_{mt}, e_{mt}(X_{t+1})\}$ and $\{\tau_{bt}, a_{bt}, e_{bt}(X_{t+1})\}$ depend only on the public history at the beginning of that period $H^t = \{X_1, ..X_t; \tau_{m1}, ..\tau_{mt-1}; e_{m1}, ..e_{mt-1}; \tau_{b1}, ..\tau_{bt-1}; e_{b1}, ..e_{bt-1}\}$. A PPE is a profile of public strategies that, for each period t and history H^t, constitute a Nash equilibrium from that time onward.

In particular, a self-enforcing relational contract must satisfy, for all t

$$E[\Pi_{mt}|H^t] \geq 0 \tag{3}$$

$$E[\Pi_{bt}|H^t] \geq 0 \tag{4}$$

$$a_{mt} \in \underset{a \in A_m(X_t)}{\arg\max} \left\{ -c_m(a, X_t) + \sum_{z \in \mathcal{X}} P_{X_t z}(a, a_{bt}) \left[d_t(X_1, .., X_t, z) + \delta E[\Pi_{m(t+1)}|H^t, X_{t+1} = z] \right] \right\} \tag{5}$$

$$a_{bt} \in \underset{a \in A_b(X_t)}{\arg\max} \left\{ \begin{array}{c} -c_b(a, X_t) + \sum_{z \in \mathcal{X}} P_{X_t z}(a_{mt}, a) \times \\ \left[Y_t(X_t, z) - d_t(X_1, .., X_t, z) + \delta E[\Pi_{b(t+1)}|H^t, X_{t+1} = z] \right] \end{array} \right\} \tag{6}$$

$$d_t(X_1, ..., X_{t+1}) + E[\Pi_{m(t+1)}|H^t, X_{t+1}] \geq 0 \tag{7}$$

$$E[\Pi_{b(t+1)}|H^t, X_{t+1}] - d_t(X_1, ..., X_{t+1}) \geq 0. \tag{8}$$

Because a firm can refuse to transact in period t, he is guaranteed positive discounted expected profit ((3) and (4)). The incentives for action in period t depend on the discretionary transfer payment $d_t(X_1, ..., X_{t+1})$, but not the formal price p_t. Equation (5) specifies that the manufacturer's action maximizes his infinite horizon discounted expected profit, assuming that the buyer chooses effort a_{bt} in the current period and that both parties adhere to the relational contract in all subsequent periods. Equation (6) plays the analogous role for the buyer. (7) and (8) ensure that both parties prefer to execute the discretionary transfer payment rather than terminate the relationship. Because termination is the most severe credible punishment that can be imposed on a party that fails to execute the discretionary payment, (7) and (8) are necessary conditions for the relational contract to be self-enforcing. Intuitively, if a relational contract is self-enforcing, then neither firm wishes to deviate unilaterally. As observed by Abreu (1988), conditions (3)-(8) are sufficient for a relational contract with "trigger strategies" to be self-enforcing. A trigger strategy is to adhere to the relational contract in every period until the other firm first refuses to execute the discretionary transfer payment, and then to refuse to transact in subsequent periods. In summary, a relational contract that satisfies (3)-(8) is self-enforcing. For the remainder of the paper, we will assume that the firms use trigger strategies.

With a self-enforcing relational contract, one can adjust the initial price p in the first period in which the firms transact to achieve any division of the total expected profit between the buyer and manufacturer satisfying (3) and (4) at $t = 1$. Therefore, our objective is to maximize total expected discounted profit $E[\Pi_{b1} + \Pi_{m1}]$, subject to the constraint that the relational contract be self-enforcing.

The total expected discounted profit with perfect coordination is given by the dynamic programming recursion

$$\overline{V}(x) = \max\left[\delta \sum_{z\in\mathcal{X}} P_{xz}(0,0)\overline{V}(z);\ \max_{a_m\in A_m(X_t),\ a_b\in A_b(X_t)}\left\{-c_m(a_m,x) - c_b(a_b,x) + \sum_{z\in\mathcal{X}} P_{xz}(a_m,a_b)\left[Y(x,z)+\delta\overline{V}(z)\right]\right\}\right]. \tag{9}$$

Let $\overline{\mathcal{X}} \subset \mathcal{X}$ denote the states in which it is optimal to transact, and $\overline{a}_m(x)$, $\overline{a}_b(x)$ denote the optimal actions in state $x \in \overline{\mathcal{X}}$, obtained by solving (9). We will subsequently call these the "first best" transaction states and actions. Clearly, $\overline{V}$ provides an upper bound on the total expected discounted profit that the firms can achieve under any relational contract. If actions were contractible, the buyer and manufacturer could achieve $\overline{V}$. However, the ability to write formal contracts with state-contingent payments $p_t(X_1, ..., X_t, X_{t+1})$ would *not* enable the buyer and manufacturer to achieve $\overline{V}$. The essential problem is that incentive payments to the buyer and manufacturer must add up to zero in every period. Holmstrom (1982) proved that a third party is needed to break this "budget balance constraint" to achieve the first best. Otherwise, to create second best incentives for action in the current period, an undesirable transition must be followed by "punishment" through inefficient actions in subsequent periods. These results suggest that in our setting, with *discretionary* state-contingent payments $d_t(X_1, ...X_t, X_{t+1})$, optimal relational contracts are complex, with history-dependent payments and actions.

Fortunately, characterization of an optimal relational contract is greatly simplified by introducing a correlation device (Aumann 1974). Suppose that at the end of each period t the buyer and manufacturer commonly observe the value of a random variable u_t. The sequence of random variables $\{u_t, t = 1, 2, ..\}$ is i.i.d. uniform on [0,1] and independent of the process X_t and of the firms' actions. We will expand our definition to include relational contracts in which, for each t, the discretionary payment d_t, and continuation contract and strategies from period $t + 1$ may depend upon $\{u_s : s \le t\}$. In particular, a *correlated termination relational contract* is characterized by a termination function $Q : \mathcal{X} \times \mathcal{X} \to [0, 1]$. In the event that $u_t < Q(X_t, X_{t+1})$

$$d_t = 0$$

$$\tau_{bs} = 0 \text{ and } \tau_{ms} = 0 \text{ for all } s > t;$$

that is, the firms quit joint production. (Q is mnemonic for Quit).

Finally, observe that this model formulation allows for each firm to have a state-dependent outside alternative to joint production in each period. The manufacturer's cost function $c_m(a_m, x)$ represents actual production costs and any forgone profit from working with an alternative partner. (Increasing the value of the manufacturer's outside alternative in state x increases $c_m(a_m, x)$ by a constant for all $a_m \in A_m(x)$.) Similarly, the buyer's cost function $c_b(a_b, x)$ represents actual production costs and any forgone profit from working with an alternative partner. Then, the "profit" functions in (1)-(2) represent discounted profit *in excess of the outside alternative*, and (9) is an upper bound on the value of the relationship. However, to be consistent with our assumption each firm seeks to maximize this profit, the outside alternative should evolve exogenously rather than be influenced by the firms' actions.

3 Derivation of an Optimal Relational Contract

Our main result is that a correlated termination relational contract is optimal, and can be characterized by an unusual sort of dynamic program. Before stating the main result, we need to develop some machinery. For each $x \in \mathcal{X}$ and $v : \mathcal{X} \rightarrow R^+$ define

$$T(v)(x) = \max \left[\delta \sum_{z \in \mathcal{X}} P_{xz}(0,0) v(z); \max_{a_m \in A_m(X_t),\ a_b \in A_b(X_t)} \left\{ -C(a_m, a_b, v, x) + \sum_{z \in \mathcal{X}} P_{xz}(a_m, a_b)[Y(x,z) + \delta v(z)] \right\} \right], \tag{10}$$

where the cost function is given by

$$\begin{aligned} C(a_m, a_b, v, x) \;=\; & c_m(a_m, x) + c_b(a_b, x) \qquad (11) \\ & + \min_{V_m, V_b} \sum_{z \in \mathcal{X}} P_{xz}(a_m, a_b) Q(x,z) \delta v(z) \\ & \text{subject to:} \\ & V_m(x,z) \geq 0,\ V_b(x,z) \geq 0,\ V_m(x,z) + V_b(x,z) \leq v(z) \text{ for } z \in \mathcal{X} \\ & a_m \in \arg\max_{a \in A_m(x)} \left\{ -c_m(a,x) + \sum_{z \in \mathcal{X}} P_{xz}(a, a_b) \delta V_m(x,z) \right\} \\ & a_b \in \arg\max_{a \in A_b(x)} \left\{ -c_b(a,x) + \sum_{z \in \mathcal{X}} P_{xz}(a_m, a)[Y(x,z) + \delta V_b(x,z)] \right\} \\ & Q(x,z) = [v(z) - V_m(x,z) - V_b(x,z)]/v(z). \end{aligned}$$

The operator Tv gives the maximum total discounted expected profit under a self-enforcing relational contract with correlated termination in period 1, assuming that *if* the firms do not terminate in period 1, the total discounted expected profit at the beginning of period 2 is given by v. The cost function $C(a_m, a_b, v, x)$ has two components: the direct cost of action (c_m and c_b) and the expected cost associated with possible termination. Allowing for termination with positive probability weakly decreases the total cost of any given action (a_m, a_b); see the minimization embedded in (11). Thus, deliberately destroying value following some state transitions may increase total expected discounted profit.

The operator Tv is distinctive in that the cost of an action depends upon the ongoing value function v as well as the state x. The cost function $C(a_m, a_b, v, x)$ may take value ∞, indicating infeasibility of (a_m, a_b). For example, if $v = 0$ then the only feasible action pairs for the manufacturer and buyer in state x are

$$\left\{ (a_m, a_b) : a_m \in \arg\max_{a \in A_m(x)} \{-c_m(a,x)\},\ a_b \in \arg\max_{a \in A_b(x)} \{-c_b(a,x) + \sum_{z \in \mathcal{X}} P_{xz}(a_m, a) Y(x,z)\} \right\}.$$

These are Nash equilibria of the single-period game in which the firms transact without the value of an ongoing relationship (the potential for repeat business) to induce cooperative behavior.

Observe that the cost function $C(a_m, a_b, v, x)$ is decreasing in v. That is, the cost for the firms to implement any pair of actions is decreasing in the ongoing value of the relationship. The next proposition establishes a useful structural property of the operator T.

Proposition 1 *The operator T has a unique fixed point V^**

$$V^* = TV^*,$$

and $V^* \in [0, \overline{V}(x_1)] \times ... \times [0, \overline{V}(x_N)]$.

All proofs with the exception of that of Theorem 1 are in Plambeck and Taylor (2004a). Let $\{a_m^*(x), a_b^*(x)\}_{x \in \mathcal{X}}$ denote the actions obtained by solving (10) with $v = V^*$. Let $\{\tau_m^*(x), \tau_b^*(x)\}_{x \in \mathcal{X}}$ denote the corresponding rule for whether or not to transact:

$$\tau_m^*(x) = \tau_b^*(x) = \begin{cases} 1 & \text{if } V^*(x) > \delta \sum_{z \in \mathcal{X}} P_{xz}(0,0) V^*(z) \\ 0 & \text{if } V^*(x) = \delta \sum_{z \in \mathcal{X}} P_{xz}(0,0) V^*(z). \end{cases}$$

Finally, let $(V_m^*(x,z), V_b^*(x,z))$ denote the minimizers of $C(a_m^*(x), a_b^*(x), x, V^*)$.

Theorem 1 *A correlated termination relational contract is optimal, and it achieves total discounted expected profit of $V^*(X_1)$. The termination function is*

$$Q^*(x,z) = [V^*(z) - V_m^*(x,z) - V_b^*(x,z)]/V^*(z).$$

The firms' strategies for whether or not to transact satisfy

$$\begin{aligned}
\tau_{mt} &= \begin{cases} \tau_m^*(X_t) & \textit{if } t \leq \Upsilon \textit{ and } e_{bs} = e_{ms} = 1 \textit{ for all } s < t \\ 0 & \textit{if } t > \Upsilon \textit{ or } e_{ms}e_{bs} = 0 \textit{ for some } s < t \end{cases} \\
\tau_{bt} &= \begin{cases} \tau_b^*(X_t) & \textit{if } t \leq \Upsilon \textit{ and } e_{bs} = e_{ms} = 1 \textit{ for } s < t \\ 0 & \textit{if } t > \Upsilon \textit{ or } e_{ms}e_{bs} = 0 \textit{ for some } s < t, \end{cases}
\end{aligned}$$

where Υ is the period in which the relationship terminates

$$\Upsilon = \inf\{t : u_t < Q^*(X_t, X_{t+1})\}.$$

In each period that the firms transact, the formal price depends only on the current state, and the discretionary transfer payment depends only upon the observed transition and the correlation device

$$\begin{aligned}
p_t &= \alpha V^*(X_t) + c_m(a_m^*(X_t), X_t) - \sum_{z \in \mathcal{X}} P_{X_t z}(a_m^*(X_t), a_b^*(X_t))\delta V_m^*(z) \\
d_t &= \begin{cases} [1 - Q^*(X_t, X_{t+1})]^{-1} \delta V_m^*(X_t, X_{t+1}) - \alpha\delta V^*(X_{t+1}) & \textit{if } u_t \geq Q^*(X_t, X_{t+1}) \\ 0 & \textit{if } u_t < Q^*(X_t, X_{t+1}), \end{cases}
\end{aligned}$$

where $\alpha \in [0,1]$ is the fraction of expected total discounted profit allocated to the manufacturer; the action strategies depend only on the current state

$$a_{mt} = a_m^*(X_t), \quad a_{bt} = a_b^*(X_t) \quad \textit{for } t = 1, 2, ...$$

and each firm is willing to execute the discretionary transfer payment

$$e_{bt} = e_{mt} = 1 \quad \textit{for } t = 1, 2, ...$$

Proof of Theorem 1: The proof proceeds in three steps. To be considered as a candidate for optimality, a relational contract must have certain basic properties. The first step is to describe these properties. The second step demonstrates that for any relational contract with these basic properties, there exists a self-enforcing correlated termination contract with the same expected total discounted profit. This correlated termination relational contract is appealingly simple in that the firms' actions depend only on the current state of the system, and the discretionary transfer payment depends only on the observed transition and the correlation device. We conclude from the second step that in searching for an optimal relational contract, we can restrict attention to correlated termination contracts with this simple form. The third step constructs the optimal correlated termination contract by solving the dynamic program (10).

Step 1: Properties of a Candidate-Optimal Relational Contract

Consider a relational contract o with the following terms for the first period: formal payment $p^o(x)$, discretionary transfer payment $d^o(x,z,u)$, strategy for the manufacturer of $\{\tau_m^o(x), a_m^o(x)\}$ and strategy for the buyer of $\{\tau_b^o(x), a_b^o(x)\}$, conditional on $(X_1, X_2, u_1) = (x, z, u)$. Let $V_1^o(x)$ denote the total expected discounted profit, conditional on $X_1 = x$.

$$
\begin{aligned}
V_1^o(x) &= E^o\left[\Pi_{m1}^o + \Pi_{b1}^o \mid X_1 = x\right] \\
&= E^o\left[\sum_{t=1}^{\infty} \delta^{t-1} \tau_{bt}^o \tau_{mt}^o [Y_t - c_m(a_{mt}^o, X_t) - c_b(a_{bt}^o, X_t)] \mid X_1 = x\right],
\end{aligned}
$$

where the superscript o indicates that the expectation E^o is taken with respect to the distribution induced by the relational contract. Similarly, let $V_2^o(x)$ denote the total expected discounted profit under the optimal relational contract starting from period 2, conditional on $(X_1, X_2) = (x, z)$.

$$
V_2^o(x,z) = E^o\left[\sum_{t=2}^{\infty} \delta^{t-2} \tau_{bt}^o \tau_{mt}^o [Y_t - c_m(a_{mt}^o, X_t) - c_b(a_{bt}^o, X_t)] \mid (X_1, X_2) = (x, z)\right].
$$

To be considered as a candidate for optimality, the relational contract o must satisfy

$$
V_2^o(x,z) \le V_1^o(z) \text{ for every } x, z \in \mathcal{X}. \tag{12}
$$

If $V_2^o(x,z) > V_1^o(z)$ the firms could achieve strictly greater expected total discounted profit by starting with the continuation contract from period 2, rather than the initial contract for state z. Note that (12) may be a strict inequality, in order to create incentives for action in period 1. However, if $\tau_m^o(x) \cdot \tau_b^o(x) = 0$ then

$$
V_2^o(x,z) = V_1^o(z) \text{ for every } z \in \mathcal{X}. \tag{13}
$$

To be considered as a candidate for optimality, the relational contract o must also be self-enforcing in the first period, which implies that

$$
E^o\left[\Pi_{m1}^o \mid X_1 = x\right] \ge 0 \text{ and } E^o\left[\Pi_{b1}^o \mid X_1 = x\right] \ge 0 \text{ for every } x \in \mathcal{X}, \tag{14}
$$

and for every x such that $\tau_m^o(x) = \tau_b^o(x) = 1$ so that the firms transact in the first period:

$$a_m^o(x) = \underset{a \in A_m(x)}{\arg\max} \left\{ -c_m(a,x) + \sum_{z \in \mathcal{X}} P_{xz}(a, a_b^o) E^o[d^o + \delta\Pi_{m2}^o \mid X_1 = x,\ X_2 = z] \right\} \tag{15}$$

$$a_b^o(x) = \underset{a \in A_b(x)}{\arg\max} \left\{ -c_b(a,x) + \sum_{z \in \mathcal{X}} P_{xz}(a_m^o, a)(Y(x,z) + E^o[-d^o + \delta\Pi_{b2}^o \mid X_1 = x,\ X_2 = z]) \right\} \tag{16}$$

$$E^o[\delta\Pi_{m2}^o \mid (X_1, X_2, u_1) = (x, z, u)] \geq -d^o(x, z, u) \tag{17}$$

$$E^o[\delta\Pi_{b2}^o \mid (X_1, X_2, u_1) = (x, z, u)] \geq d^o(x, z, u). \tag{18}$$

Step 2: The Equivalent Correlated Termination contract

Now we will construct a self-enforcing correlated termination relational contract with the same expected total discounted profit as the relational contract o. The termination function is given by

$$Q(x,z) = \left[V_1^o(z) - V_2^o(x,z)\right]/V_1^o(z).$$

The firms' strategies for whether or not to transact are, for $t = 1, 2, ...$,

$$\tau_{mt} = \begin{cases} 1 & \text{if } t \leq \Upsilon,\ e_{bs} = e_{ms} = 1 \text{ for } s < t \text{ and } X_t \in \{x : \tau_m^o(x) = \tau_b^o(x) = 1\} \\ 0 & \text{otherwise} \end{cases}$$

$$\tau_{bt} = \begin{cases} 1 & \text{if } t \leq \Upsilon,\ e_{bs} = e_{ms} = 1 \text{ for } s < t \text{ and } X_t \in \{x : \tau_m^o(x) = \tau_b^o(x) = 1\} \\ 0 & \text{otherwise} \end{cases}$$

where Υ is the period in which the relationship terminates:

$$\Upsilon = \inf\{t : u_t < Q(X_t, X_{t+1})\}.$$

Furthermore, action strategies are, for $t = 1, 2, ...$,

$$a_{mt} = a_m^o(X_t)$$
$$a_{bt} = a_b^o(X_t).$$

The formal price is $p^o(X_t)$ and the discretionary transfer payment is $d_t = 0$ if $u_t < Q(X_t, X_{t+1})$, and otherwise is

$$d_t = d(X_t, X_{t+1}),$$

where, for each $(x,z) \in \mathcal{X} \times \mathcal{X}$

$$d(x,z) = [1 - Q(x,z)]^{-1} E^o[d^o + \delta\Pi_{m2}^o \mid X_1 = x,\ X_2 = z] - \delta E^o[\Pi_{m1}^o \mid X_1 = z].$$

Using (14)-(18), it is straightforward to verify that this correlated termination contract is self-enforcing and achieves the same expected discounted profit for the buyer and for the manufacturer as relational contract o, for each initial state $X_1 \in \mathcal{X}$.

Step 3: The Optimal Correlated Termination Relational Contract

Based on step 2, in searching for an optimal relational contract, we can restrict attention to correlated termination contracts with strategies that depend only on the current state and discretionary transfer payments that depend only upon the observed transition. We can also assume without loss of generality that the manufacturer is allocated a fraction $\alpha \in [0,1]$ of the total discounted expected profit. Let $V(z)$ denote the maximum total discounted profit that can be achieved with such a relational contract, starting in state z. Suppose that the firms will adopt this relational contract in the second period, and would like to develop a discretionary transfer payment, action strategies for the two firms, and a termination function for the first period that are self-enforcing and maximize expected total discounted profit. Given that the system is initially in state x, this must result in expected total discounted profit of $V(x)$.

$$\begin{aligned}
V(x) = \max \Bigg[& \delta \sum_{z\in\mathcal{X}} P_{xz}(0,0)v(z); \\
& \max_{d,Q,a_m,a_b} \left\{ -c_m(a_m,x) + c_b(a_b,x) + \sum_{z\in\mathcal{X}} P_{xz}(a_m,a_b)[Y(x,z) + \delta[1-Q(x,z)]V(z)] \right\} \Bigg]
\end{aligned}$$

subject to:

$$\begin{aligned}
& a_m \in \max_{a\in A_m(x)} \left\{ -c_m(a,x) + \sum_{z\in\mathcal{X}} P_{xz}(a,a_b)[1-Q(x,z)][d(x,z) + \delta\alpha V(z)] \right\} \\
& a_b \in \max_{a\in A_b(x)} \left\{ -c_b(a,x) + \sum_{z\in\mathcal{X}} P_{xz}(a_m,a)\{Y(x,z) + [1-Q(x,z)][-d(x,z) + \delta(1-\alpha)V(z)]\} \right\} \\
& \delta\alpha V(z) \geq -d(x,z) \\
& \delta(1-\alpha)V(z) \geq d(x,z) \\
& 0 \leq Q(x,z) \leq 1.
\end{aligned}$$

This is equivalent to

$$TV = V.$$

From Proposition 1, we know that T has a unique fixed point V^* and therefore $V = V^*$. Thus, the optimal terms are as given in the statement of the Theorem. ■

The optimal relational contract in Theorem 1 involves (probabilistic) termination following periods with undesirable performance as reflected in an undesirable state transition. (This is formalized in the next section's Proposition 3.) This termination could be interpreted as resulting from a dispute over who is responsible for poor performance. However, termination occurs despite the fact that in every period in which trade occurs the buyer and supplier take the agreed upon action. Thus, the firms are not penalizing one another for presumed shirking. Rather, the purpose of termination is to provide stronger incentives for action, by jointly punishing the firms for unfavorable stochastic outcomes. These stronger incentives lead to greater expected profit in the periods in which the firms transact; however, profit is, of course, reduced in periods following termination. The optimal termination relational contract balances the near-term gain from stronger incentives for action against the eventual loss resulting from termination.

In practice, one might expect that following poor outcomes, firms would break off cooperation for a limited period of time, rather than forever. One might argue that the firms cannot credibly refuse to transact; in the event of termination, they would renegotiate the relational contract to generate some ongoing profit. The economics literature on repeated games with imperfect monitoring is subject to the same criticism that in a punishment phase, the players have an incentive to coordinate on a more favorable continuation equilibrium; see, for example, Abreu et al. (1986, 1991). Several papers explore the renegotiation of formal contracts in dynamic games; see Laffont and Tirole (1990), Rey and Salanie (1996) and references therein. They observe that allowing renegotiation is equivalent to restricting attention to long term contracts that are *immune to renegotiation* (i.e., in every period, the players cannot achieve greater profit by substituting an alternative continuation contract).

Suppose that we impose the additional constraint that the relational contract be immune to renegotiation. This requires that the operator T be modified so that

$$V_m(x,z) + V_b(x,z) \leq v(z) \text{ for } z \in \mathcal{X}$$

in (11) is replaced by

$$V_m(x,z) + V_b(x,z) = v(z) \text{ for } z \in \mathcal{X}.$$

Allowing renegotiation means that the optimal relational contract cannot involve termination: $Q^* = 0$. By extension of the proof of Theorem 2 in Levin (2003), if an optimal relational contract exists, then expected discounted profit under this optimal relational contract is the largest fixed point of the modified operator T in $[0, \overline{V}]$; Theorem 1 holds with the modified operator T and $Q^* = 0$.* This weakly reduces expected discounted profit at time zero. In the product development application in §5, for a wide range of parameters, the optimal contract has $Q^* = 0$, i.e., allowing renegotiation does not reduce expected discounted profit. For other parameters, the prospect of

*A sequel paper (Plambeck and Taylor 2004b) explores the impact of renegotiation on the existence and structure of the optimal relational contract.

renegotiation weakens incentives so that the firms have zero expected profit.

In a supply network with multiple buyers and suppliers, if failure to adhere to a relational contract is public information, then the firms can sustain more stringent relational contracts. The value of public reputation in addition to the value of the a specific relationship form the upper bound on discretionary transfer payments. Furthermore, public reputation can make termination immune to renegotiation.

As a practical matter, for any given problem parameters, specifying the optimal relational contract requires calculating the optimal value function V^*. Proposition 2 provides a theoretical basis for and guidance as to how to employ value iteration to compute the optimal value function. Define $T^0V \equiv V$ and for $n \geq 1$, $T^nV \equiv T(T^{n-1}V)$. Value iteration involves computing T^nV for successively larger values of n, starting with a given value function V.

Proposition 2 *Value iteration converges to the optimal value function V^* when one begins with the first best value function $\overline{V}$:*

$$V^* = \lim_{n\to\infty} T^n\overline{V}.$$

Furthermore, the value function after a finite number of iterations is an upper bound on the optimal value function: $T^n\overline{V} \geq V^*$.

Observe that the convergence result is dependent on the initial value function. In standard dynamic programming analyses, where the cost function does not depend on the value function, convergence is often obtained regardless of the initial value function. The usual approach is to show that the optimal value operator is a contraction and then to appeal to the Banach Fixed-Point Theorem to establish convergence. In our case, because the cost function $C(a_b, a_m, v, x)$ depends on the value function, the optimal value operator T need not be a contraction. However, beginning value iteration with the first best value function $\overline{V}$ ensures that the resulting value function in each iteration is decreasing. Using this property in conjunction with the definition of T establishes the convergence result.

4 Structural Properties of an Optimal Relational Contract

Innovative software offers the opportunity for supply chain partners to closely monitor joint production processes. For example, SigmaQuest software offers real-time visibility of detailed functional test results and quality data, to facilitate collaboration between an Original Equipment Manufacturer and contract manufacturer in product development and introduction. In a second example, biopharmaceutical manufacturers and their buyers may observe detailed process control data, which will help managers to identify problems with process control (the manufacturer's fault) or faulty materials (the buyer's fault). This section explores the implications of observing such signals for the structure of optimal relational contracts. In particular, we explore to what extent observing signals enables the firms to avoid terminating the relationship.

First, this section considers the setting where the firms only observe the output and system state. We establish that an optimal relational contract terminates only in the event of transition to an undesirable state, i.e., below a specified threshold. Furthermore, the correlation device is required only in the threshold state. This suggests that the correlation device is unnecessary in a setting with a continuous state space and transition density function. Second, we consider the setting where the firms have information technology that provides a signal for the manufacturer's action that is independent of the buyer's action. If it is possible to write a court-enforced contract with payments contingent on the signal, a properly designed contract achieves the first best. This does not hold, in general, if the signal is *not* contractible. We identify the conditions under which the signal enables the firms to avoid termination and achieve the first best total discounted expected profit with a self-enforcing relational contract. We also identify conditions under which an optimal relational contract is one-step (the manufacturer receives a fixed bonus when the signal is above threshold) as in Levin's (2003) stationary model with one-sided moral hazard.

To state and prove these results, we impose additional assumptions about the action sets and how actions influence the transition probabilities. Specifically, Rogerson (1985) proposed sufficient conditions to justify the "first order approach" (relaxing the constraint that the agent chooses an action that maximizes his utility to a first order necessary condition) in a static principal-agent problem. We extend these conditions to a system with Markovian dynamics. First, we assume that the feasible action set for each firm is a real interval. For every $x \in \mathcal{X}$

$$\begin{aligned} A_m(x) &= [\underline{a}_m(x), \overline{a}_m(x)] \\ A_b(x) &= [\underline{a}_b(x), \overline{a}_b(x)], \end{aligned} \tag{19}$$

and the cost functions $c_m(a_m, x)$ and $c_b(a_b, x)$ are increasing and continuously differentiable in the actions a_m and a_b, respectively. For fixed state $x \in \mathcal{X}$, we can order the states $\mathcal{X} = \{z_1, z_2, ..., z_N\}$ such that

$$Y(x, z_1) + \delta V^*(z_1) \leq Y(x, z_2) + \delta V^*(z_2) \leq ... \leq Y(x, z_N) + \delta V^*(z_N).$$

Under the optimal relational contract, starting from state x, a transition to state z_{i+1} yields greater expected total discounted profit than a transition to state z_i, for each $i = 1, .., N-1$. The second assumption is that for any $a_b \in A_b(x)$, $\{a_m, a_m^1, a_m^2\} \in A_m(x)$ and $\beta \in [0,1]$ such that $c_m(a_m, x) = \beta c_m(a_m^1, x) + (1-\beta) c_m(a_m^2, x)$, and for each $n \in \{1, .., N\}$,

$$\sum_{i=n}^{N} P_{xz_i}(a_m, a_b) \geq \beta \sum_{i=n}^{N} P_{xz_i}(a_m^1, a_b) + (1-\beta) \sum_{i=n}^{N} P_{xz_i}(a_m^2, a_b); \tag{20}$$

for any $a_m \in A_m(x)$, $\{a_b, a_b^1, a_b^2\} \in A_b(x)$ and $\beta \in [0,1]$ such that $c_b(a_b, x) = \beta c_b(a_b^1, x) + (1-\beta) c_b(a_b^2, x)$, and for each $n \in \{1, .., N\}$,

$$\sum_{i=n}^{N} P_{xz_i}(a_m, a_b) \geq \beta \sum_{i=n}^{N} P_{xz_i}(a_m, a_b^1) + (1-\beta) \sum_{i=n}^{N} P_{xz_i}(a_m, a_b^2). \tag{21}$$

Intuitively, this second assumption implies a decreasing marginal expected discounted profit for each additional dollar's worth of action. The third assumption is that $P_{xz}(a_m, a_b)$ strictly positive and continuously differentiable in (a_m, a_b), and for any $a_m \in A_m(x)$, $a_b \in A_b(x)$

$$\frac{\frac{\partial}{\partial a_m} P_{xz_i}(a_m, a_b)}{P_{xz_i}(a_m, a_b)} \quad \text{and} \quad \frac{\frac{\partial}{\partial a_b} P_{xz_i}(a_m, a_b)}{P_{xz_i}(a_m, a_b)} \quad \text{increase with } i. \tag{22}$$

Rogerson (1985) points out that this assumption is equivalent to the following statistical property. If one is given a prior over a firm's action choice, observes the transition (x, z), and then calculates a posterior distribution $G(a|(x,z))$ for the action choice, then for every a and $i = 1, .., N-1$,

$$G(a|(x, z_{i+1})) \leq G(a|(x, z_i)). \tag{23}$$

That is, observing a more desirable transition allows one to infer that the firm took greater action, in the sense of stochastic dominance. Together, these three assumptions guarantee that if a firm's ongoing expected discounted profit (including current-period output for the buyer) contingent on the transition (x, z_i) *increases* with i, then that firm's objective is a concave function of his action. This allows us to substitute the first order condition for each firm's incentive compatibility constraint in (5)-(6) and (11). The product design problem with continuous action spaces in the next section satisfies these three assumptions.

Proposition 3 establishes that there exists an optimal relational contract that requires the correlation device in at most one threshold state z_n. The firms continue to cooperate if $X_{t+1} > z_n$, termination occurs with probability 1 if $X_{t+1} < z_n$, and termination occurs with probability $Q^* \in [0,1]$ in the threshold state $X_{t+1} = z_n$. Indeed, the correlation device is unnecessary in a setting with a continuous state space and transition density function.

Proposition 3 *There exists an optimal relational contract with the following termination threshold* *property. For each* $x \in \mathcal{X}$ *with* $V^*(x) > \delta \sum_{z \in \mathcal{X}} P_{xz}(0,0) V^*(z)$, *i.e., for each state* x *in which the firms transact, there exists a threshold state* z_n *such that*

$$Q^*(x, z_i) = \begin{cases} 1 & \text{for } z_i < z_n \\ 0 & \text{for } z_i > z_n. \end{cases} \tag{24}$$

In the optimal relational contract in Proposition 3, if termination occurs at all, it occurs in the event of a transition to an undesirable state which, intuitively, allows Bayesian inference that the firms took little action (in the sense of stochastic dominance in (23)). However, although the behavior is consistent with the idea that information is being extracted from the observed state, the firms are not, in fact, making statistical inferences. In each period, both firms take the actions specified in the relational contract. The purpose of this form of termination function is to provide incentives for those actions. Finally, it is straightforward to extend the proof of Proposition 3 to show that *any* optimal relational contract must have the threshold property (24) if the following two conditions are satisfied: (22) holds in the strict sense and the optimal actions $\{a_m^*(x), a_b^*(x)\}_{x \in \mathcal{X}}$ are unique.

Independent Signals: the Value of Assigning Blame

Now we assume that in each period t that the firms transact, they observe a signal s_{mt} that conveys information about the manufacturer's action a_{mt} and is invariant with respect to the buyer's action. The signal takes values in an ordered set $s_{mt} \in \{s^1, s^2, ..., s^N\}$ where $s^i \geq s^{i-1}$ for $i = 1, .., N$. With a slight adaptation, let $P_{x(z,s^i)}(a_m, a_b)$ denote the probability of observing signal s^i and a transition to state z, given actions (a_m, a_b) and initial state x. Assume that (20)-(22) continue to hold with the substitution of $P_{x(z,s^i)}(a_m, a_b)$ for $P_{xz}(a_m, a_b)$. This generalization of (22) implies existence of a signal-threshold $\widehat{s}_m(a_m, x)$ such that

$$\frac{\partial}{\partial a_m} P_{x(z,s^i)}(a_m, a_b) \geq 0 \text{ if and only if } s^i \geq \widehat{s}_m(a_m, x).$$

Holmstrom (1982) proved that firms engaged in team production cannot, in general, use a court-enforced contract that divides the output to create incentives for the first best actions. Incentive problems arise because the transfer payments must sum to zero; a third party is needed to break this "budget balance constraint" and implement the first best actions. Proposition 4a establishes that if it is possible to write a court-enforced contract contingent on the signal, then a properly designed contract achieves the first best. Contracting on the signal breaks the budget balance constraint. When the signal is not contractible, termination plays the role of breaking the budget balance constraint. This seems to suggest that observing the independent signal for the manufacturer's action will allow the firms to avoid terminating the relationship. Proposition 4b dashes that hopeful conjecture: optimal relational contracts continue to require termination. In fact, termination can occur in the optimal relational contract even with independent signals for *both* the buyer and the manufacturer. A numerical example is given in §5 in the setting of product development; see Figure 2.

Proposition 4 (*a*) *If the signal s_{mt} is contractible, a properly designed formal contract implements the first best actions.* (*b*) *Suppose the signal is not contractible. If the buyer's action a_{bt} influences the continuation total expected discounted profit in period $t+1$, then the optimal relational contract may require termination.*

If the signal is contractible, repeated interaction (and the discretionary payments supported thereby) is inessential: a properly designed formal contract achieves the first best even if the firms anticipate interacting only once. However, in most cases a signal will not be contractible. For example, detailed process control data observed by the firms is both complex and subject to manipulation, making contracting on this information difficult. In the sequel we focus on the case where the signal is not contractible.

Proposition 5 establishes the converse to Proposition 4b: if buyer's action in period t *does not* influence the continuation profit from period $t+1$, then there exists a non-terminating optimal contract ($Q^* = 0$). For ease in presenting previous results, we adopted a state space formulation in which the output is a deterministic function of the observed transition, $Y_t = Y(X_t, X_{t+1})$. Hence the current state, X_t, contains information about the output in the previous period. To state Proposition 5 we must associate states with the same ongoing profit. Let $D : \mathcal{X} \to \mathcal{D}$ be a mapping with the property that for any $x_1,\ x_2 \in \mathcal{X}$, $D(x_1) = D(x_2)$ if and only if $A_m(x_1) = A_m(x_2)$, $A_b(x_1) = A_b(x_2)$, $c_m(\cdot; x_1) = c_m(\cdot; x_2)$, $c_b(\cdot; x_1) = c_b(\cdot; x_2)$, and $P_{x_1\cdot}(\cdot,\cdot) = P_{x_2\cdot}(\cdot,\cdot)$.

Proposition 5 *Suppose that $D(X_{t+1})$ is invariant with respect to the buyer's action a_{bt}. Then there exists a non-terminating optimal relational contract ($Q^* = 0$) with discretionary transfer payment*

$$d_t = \begin{cases} \beta(X_t)V^*(X_{t+1}) & \text{if } s_{mt} \geq \widehat{s}_m(a_m^*(X_t), X_t) \\ 0 & \text{if } s_{mt} < \widehat{s}_m(a_m^*(X_t), X_t), \end{cases} \tag{25}$$

where $\beta(x) \in [0,1]$, and court-enforced payment

$$p_t = c_m(a_m^*(X_t), X_t) - \sum_{s \geq \widehat{s}_m(a_m^*(X_t), X_t)} \sum_{z \in \mathcal{X}} P_{X_t(z,s)}(a_m^*(X_t), a_b^*(X_t)) d(X_t, z). \tag{26}$$

The first best is achieved if and only if, for every state $x \in \overline{\mathcal{X}}$,

$$\sum_{s \geq \widehat{s}_m(\overline{a}_m(x), x)} \sum_{z \in \mathcal{X}} \frac{\partial}{\partial a_m} P_{x(z,s)}(\overline{a}_m(x), \overline{a}_b(x)) \delta \overline{V}(z) \geq \frac{\partial}{\partial a_m} c_m(\overline{a}_m(x), x). \tag{27}$$

Proposition 5 establishes that the optimal relational contract has a simple form. In the special case of a stationary repeated game, $V^*(X_{t+1})$ is constant, so the discretionary payment is one-step as in Levin's (2003) stationary game with one-sided moral hazard. With dynamics, the size of the bonus depends on the ongoing value of the relationship. In particular, in a state x where the optimal relational contract has strict underinvestment by the manufacturer:

$$\frac{\partial}{\partial a_m}\left[\sum_{z\in\mathcal{X}} P_{xz}(a_m^*(x), a_b^*(x))[Y(x,z) + \delta V^*(z)] - c_m(a_m, x)\right] > 0,$$

the parameter $\beta(x) = 1$, so the manufacturer receives the maximum bonus, all of the relational capital $V^*(X_{t+1})$, when his signal exceeds the threshold. Finally, Proposition 5 provides a necessary and sufficient condition for the the first best to be achieved. This condition holds when the discounted expected value under perfect coordination, $\overline{V}$, is sufficiently large.

Together, Propositions 4b and 5 demonstrate one of our main insights: The presence of dynamics and the need for termination are tightly interconnected. If the system does not exhibit dynamics (or more precisely, if the buyer does not influence the dynamics of the system), then an optimal relational contract does not require termination. If dynamics are present and are influenced by the buyer's action, termination may be required.

The intuition behind these diverging results is the following. When the buyer's action does not influence the dynamics of the system, the buyer has incentives for efficient action when the discretionary transfer payment does not depend on his own action. The discretionary transfer payment in (25) is constructed so that, indeed, it only depends on the manufacturer's action, as reflected in the signal s_{mt} and the continuation value $V^*(X_{t+1})$. Providing incentives for the manufacturer to take the optimal action does not require destroying value, because any value not allocated to the manufacturer can be transferred as a windfall gain to the buyer, without distorting the buyer's incentives. When the buyer's action *does* influence the dynamics of the system, this logic breaks down. Providing incentive for optimal actions requires that the discretionary transfer payment depends on both firms' actions. This introduces the free-rider problem, which can be addressed by the joint punishment of termination.

Proposition 5 provides a strong result for the case where the buyer does not influence the dynamics. Proposition 6 provides a weaker result for the more general case where the buyer may influence the dynamics.

Proposition 6 *Suppose that for every state $x \in \overline{\mathcal{X}}$,*

$$\sum_{s\geq \widehat{s}_m(\overline{a}_m(x),x)} \sum_{z\in\mathcal{X}} \frac{\partial}{\partial a_m} P_{x(z,s)}(\overline{a}_m(x), \overline{a}_b(x)) \min_{z\in\mathcal{X}} \overline{V}(z) \geq \frac{\partial}{\partial a_m} c_m(\overline{a}_m(x), x). \tag{28}$$

Then every optimal relational contract is non-terminating: $Q^* = 0$. Furthermore, the first best is achieved in a self-enforcing relational contract with one-step discretionary transfer payment

$$d_t = \begin{cases} d(X_t) & \text{if } s_{mt} \geq \widehat{s}_m(\overline{a}_m(X_t), X_t) \\ 0 & \text{if } s_{mt} < \widehat{s}_m(\overline{a}_m(X_t), X_t), \end{cases} \tag{29}$$

where

$$d(x) = \frac{\partial}{\partial a_m} c_m(\overline{a}_m(x), x) / \left(\sum_{s \geq \widehat{s}_m(\overline{a}_m(x), x)} \sum_{z \in \mathcal{X}} \frac{\partial}{\partial a_m} P_{x(z,s)}(\overline{a}_m(x), \overline{a}_b(x)) \right), \tag{30}$$

and with a court-enforced payment

$$p_t = c_m(\overline{a}_m(X_t), X_t) - \sum_{s \geq \widehat{s}_m(\overline{a}_m(X_t), X_t)} \sum_{z \in \mathcal{X}} P_{X_t(z,s)}(\overline{a}_m(X_t), \overline{a}_b(X_t)) d(X_t). \tag{31}$$

Proposition 6 allows for the buyer's action to influence the continuation value, but imposes a stronger version of condition (27), substituting $\min_{z \in \mathcal{X}} \overline{V}(z)$ for $\overline{V}(z)$ in the left hand side. When the worst-case continuation value is sufficiently high, the first best can be implemented with a simple one-step discretionary payment. The size of the manufacturer's bonus for generating a signal above the threshold depends only on the initial state X_t. This is not the unique optimal relational contract. However, if the worst-case continuation value is sufficiently high, every optimal relational contract is non-terminating.

Recall that increasing the value of the manufacturer's outside alternative in state z increases $c_m(a_m, z)$ by a constant for all $a_m \in A_m(z)$. Similarly, increasing the value of the buyer's outside alternative in state z increases $c_b(a_b, z)$ by a constant for all $a_b \in A_b(z)$. This reduces the first-best value of the relationship $\overline{V}(z)$ and, by violating (28), may prevent the firms from achieving the first best and cause termination with positive probability in the second best optimal relational contract.

5 Product Design

This section provides an application of the general model developed in §2 and demonstrates how embedding the model in a particular context allows for the development of additional insights. An important trend in new product development is the shift towards collaborative product design. Historically, in developing new products, buying firms often embraced a "I design, you build" approach in dealing with suppliers. As buying firms have recognized the areas in which their suppliers have comparative advantages (e.g., understanding of certain aspects of technology and manufacturability), they have moved towards designing new products collaboratively with their suppliers (Hanfield et al. 1999). Joint product development is also pursued by firms that are "peers"; examples of such alliances include GM and Susuzuki, Motorola and Toshiba, and Intel and AMD (Amaldoss et al. 2000).

An important issue in product design is the degree to which development work is done in parallel versus sequentially. Parallel development holds out the promise of more quickly identifying a successful design. However, to the extent that a firm is seeking to identify a single design, parallel development will typically entail actions that ex post will be revealed to be redundant and wasteful. Hence, there is a trade-off between speed to market and the expected cost of development (Loch et al. 2001). This trade-off and the surrounding issues are enriched when firms develop products collaboratively.

Consider two firms that seek to jointly develop a product. Each firm focuses on a distinct, but essential, component of the end product. The success of the end product design depends on each of the firms developing components that work together successfully. Both firms observe whether the end product is successful, but neither firm observes the level of design effort exerted by the other firm. For consistency, we label one firm the buyer and one firm the manufacturer, but the analysis applies to any two firms. The system begins in state 0, which denotes that a successful end product has not been developed. When a successful product is developed, the system transitions to state 1 and remains in this state for all subsequent periods: $P_{1,1} = 1$.

In each period, each firm decides the number of prototypes it will explore. For the manufacturer, the cost of exploring a prototype is c_m, and the probability that any particular prototype is successful is p_m. Let N_{mt} denote the number of prototypes the manufacturer explores in period t. The probability that at least one of the manufacturer's prototypes is successful is

$$1 - (1 - p_m)^{N_{mt}}.$$

Let c_b, p_b and N_{bt} denote the analogous quantities for the buyer. The end product is successful with probability

$$P_{0,1}(N_{mt}, N_{bt}) = [1 - (1 - p_m)^{N_{mt}}][1 - (1 - p_b)^{N_{bt}}],$$

and is a failure with probability $P_{0,0}(N_{mt}, N_{bt}) = 1 - P_{0,1}(N_{mt}, N_{bt})$. The probability that either firm's design process produces a successful component is increasing in the number of firm's prototypes, but there are diminishing returns from incremental prototypes. Further, N_{mt} and N_{bt} are complements: the impact of one firm's exploring an incremental prototype on the end product success probability is increasing the number of prototypes explored by the other firm. The probability of success depends only on the actions of the firms in the current period. This is appropriate, for example, when technological standards are evolving so that previous unsuccessful design efforts do not influence the probability of success in the future.[†] Nevertheless, the formulation developed in §2 is general enough to capture this type of dependence. The discounted expected profit generated by a successful end product is G. This is captured in our framework with $Y(0,0) = Y(0,1) = 0$ and $Y(1,1) = (1-\delta)G$ (the firms optimally explore 0 prototypes in state 1). Alternatively, G may represent expected total discounted profit under an optimal relational contract for joint production of the new product, involving noncontractible production activities that are not explicitly modeled here.

One alternative is for the firms to pursue a sequential approach to product design: each firm explores a single prototype in each period. However, because developing a successful product requires that both firms concurrently develop successful components, parallel development–in which one or both firms explores multiple prototypes–may be more attractive. We say that the *degree of parallelism* is increasing in the number of prototypes per period explored by each firm. When the success of the end product depends on a single component, greater parallelism reduces the expected time to market at the expense of additional expected development cost. When the success of the end product depends on multiple components, this trade-off is more complex. Greater parallelism reduces the expected time to market and can reduce the expected development cost, due to the complementarity in the number of prototypes explored by each firm.

To obtain analytical results, we focus on the case where the production technology is symmetric, i.e., $c_m = c_b = c$ and $p_m = p_b = p$, and relax the restriction that N_m and N_b be integer-valued. We refer to p as the component success probability; let $\bar{p} = 1 - p$. We exclude the uninteresting case in which the firms' profit under an optimal relational contract is zero. We begin by considering the benchmark setting where a single firm determines the number of prototypes (N_m, N_b) to explore. Because the production technology is stationary, it is optimal to explore the same number of prototypes in each period prior to identifying a successful end product design. The discounted expected profit in state 0 when the firms explore (N_m, N_b) prototypes in each period is

$$\frac{\delta P_{0,1}(N_m, N_b)G - c(N_m + N_b)}{1 - \delta P_{0,0}(N_m, N_b)}. \tag{32}$$

The optimal (N_m, N_b) maximizes (32). Proposition 7 characterizes the optimal number of prototypes to explore.

[†]Alternatively, this assumption is appropriate when the design space is unstructured so that successful end product designs are dispersed over the design space (Terwiesch and Loch 2004). In this case, the firms seek in each period to explore a specific area of the design space jointly. If they are unsuccessful, in subsequent periods they will explore previously unexplored regions of the design space.

Proposition 7 *In the centralized system, the optimal number of prototypes to explore is symmetric and is given by*

$$\overline{N}_m = \overline{N}_b = \overline{N} = \max\left\{N : -[1-\delta\bar{p}^N(2-\bar{p}^N)]c - \delta\bar{p}^N(1-\bar{p}^N)[(1-\delta)G + 2cN]\log(\bar{p}) = 0\right\}. \tag{33}$$

Further, $\overline{N}$ is increasing in G and decreasing in c.

It is intuitive that as the payoff from a successful product design increases or the cost of exploring a prototype decreases, the firm should explore more prototypes. The impact of the component success probability and discount factor on the optimal number of prototypes is less obvious. As the component success probability p improves, the value of an incremental prototype increases. On the other hand, fewer prototypes are required to achieve the same end product success probability. In extensive numerical study we observed that this second effect almost always dominates, so that the optimal number of prototypes is decreasing in p; however, in some cases, $\overline{N}$ is increasing in p. Similarly, the impact of the discount factor on the optimal number of prototypes is ambiguous: depending on the problem parameters, increasing the discount factor can either increase or decrease the optimal number of prototypes.

If the number of prototypes explored is contractible, then the firms can achieve the first best by specifying that each firm explore the number of prototypes that is optimal for the integrated system. However, typically each firm will not directly observe the level of design effort exerted by its partner, much less would this level of effort be verifiable by a third party. A firm can easily claim to have asserted more effort that it did and then claim that an unfavorable outcome of an end product design is due to either the failure of the partner firm's development process or "bad luck." In the context of parallel product design, a firm can present unsuccessful component prototypes and untruthfully claim that these were the result of diligent product development efforts. Typically, it will be difficult or impossible for the partner firm to ascertain the veracity of such assertions.

Our objective is to characterize the optimal relational contract and to compare this with the first best solution. The firms can provide stronger incentives for action by imposing joint punishment following a failure to develop a successful end product via (probabilistic) termination. However, such termination is costly as it destroys value. An important question is whether the optimal relational contract ever requires termination, and if so, under what circumstances. Proposition 8 characterizes when the optimal relational contract requires termination as well as the optimal number of prototypes to explore. Let

$$\begin{aligned}
\kappa &= \left[3+\sqrt{5}-2\log\left(\left[1-\sqrt{1-4/(2+\sqrt{5})}\right]/2\right)\right]c/\log(\bar{p}) \\
\underline{G} &= -(4+2\sqrt{5})c/[\delta\log(\bar{p})] \\
\overline{G} &= \left(\underline{G}+\kappa\right)/(1-\delta),
\end{aligned}$$

and note $0<\underline{G}<\overline{G}$.

Proposition 8 *In the optimal relational contract, the optimal number of product prototypes to explore is symmetric: $N_m^*=N_b^*=N^*$. The optimal relational contract does not terminate following the development of a successful end product: $Q^*(0,1)=Q^*(1,1)=0$. If $G\le\underline{G}$, then the optimal relational contract terminates with probability one following a failure to develop a successful end product:*

$$Q^*(0,0)=1, \tag{34}$$

and the optimal number of prototypes is

$$N^*=\log\left(\left[1-\sqrt{1+8c/[\delta G\log(\bar{p})]}\right]/2\right)/\log(\bar{p}). \tag{35}$$

If $G\in\left(\underline{G},\overline{G}\right)$, then the optimal relational contract terminates with nonzero probability following a failure to develop a successful end product:

$$Q^*(0,0)=1-(G-\underline{G})/(\delta G+\kappa), \tag{36}$$

and the optimal number of prototypes is

$$N^*=\log((3-\sqrt{5})/2)/\log(\bar{p}). \tag{37}$$

Otherwise, the optimal relational does not require termination:

$$Q^*(0,0)=0, \tag{38}$$

and the optimal number of prototypes is

$$N^*=\max\left\{N:-2[1-\delta\bar{p}^N(2-\bar{p}^N)]c-\delta\bar{p}^N(1-\bar{p}^N)[(1-\delta)G+2cN]\log(\bar{p})=0\right\}. \tag{39}$$

Further, $Q^*(0,0)$ is decreasing in G and p and increasing in c. N^* is increasing in G and decreasing in c.

The thresholds $\underline{G}$ and $\overline{G}$ are increasing in c and decreasing in p. Consequently, the optimal relational contract involves termination with positive probability following the failure to develop a successful product if and only if the product design technology is sufficiently poor: the component success probability is small, the cost to explore a prototype is large, and the value of a successful end product is small. Further, the optimal termination probability increases as the product design technology degrades. When this technology is very poor, the product development effort endogenously emerges as a one-shot interaction.

To see the intuition, suppose the relational contract is non-terminating. In this case, the firms have an incentive to underinvest in prototypes because each firm receives only a portion of its marginal contribution to the total system. This effect is most severe when the probability that a firm's partner will develop a successful component is small, which occurs when the design technology is poor. Imposing the joint punishment of termination strengthens each firm's incentive to exert effort, which is further strengthened because efforts are complements. The immediate gain in expected profit more than offsets the expected future cost of possible termination. When the product design technology is strong, the tendency to underinvest is mitigated, and the cost of potential termination exceeds the benefit of further ameliorating underinvestment.

The intuition that as the payoff from a successful product design increases or the cost of exploring a prototype decreases, the firms should explore more prototypes extends from the centralized to the decentralized case. However, the impact of the component success probability on the optimal number of prototypes may diverge. For some parameter settings with $\overline{N} \gg N^*$, increasing p simultaneously decreases $\overline{N}$ and increases N^*.

The next proposition establishes that the performance of the decentralized system is always strictly worse than the centralized system. Decentralization distorts the actions of the firms in two ways. When G is small, decentralization introduces termination, which never occurs in the centralized system. This first distortion affects the second, the degree of parallelism in development. By imposing the punishment of termination, a relational contract can provide incentives for the firms to pursue greater parallelism than in the centralized case. Proposition 8 establishes that doing so is never optimal.

Proposition 9 *Decentralization leads to less parallelism in development:*

$$N^* < \overline{N}.$$

The optimal relational contract calls for the firms to explore fewer prototypes per period than in the centralized case. The intuition the following: As noted above, without the prospect of termination, the firms have an incentive to underinvest in exploring prototypes. Although termination can ameliorate this tendency to underinvest, it is counterproductive to make the punishment so strong as to cause overinvestment because this entails both excessive development costs and the costly prospect of discontinued cooperation.

We have considered the case where the firms only observe whether the end product is successful. This is appropriate when the components that the firms develop are tightly integrated and interdependent. If the components are more loosely coupled it may be possible to attribute the failure in an end product design to a failure of a particular component. More generally, firms may be able to invest in testing technology that can evaluate the effectiveness of each component independently. Observing whether a firm's component is successful provides a signal of that firm's efforts that is unclouded by the partner firm's actions. Figure 1 illustrates the impact of having this signal information on the optimal relational contract and its performance. The upper two panels

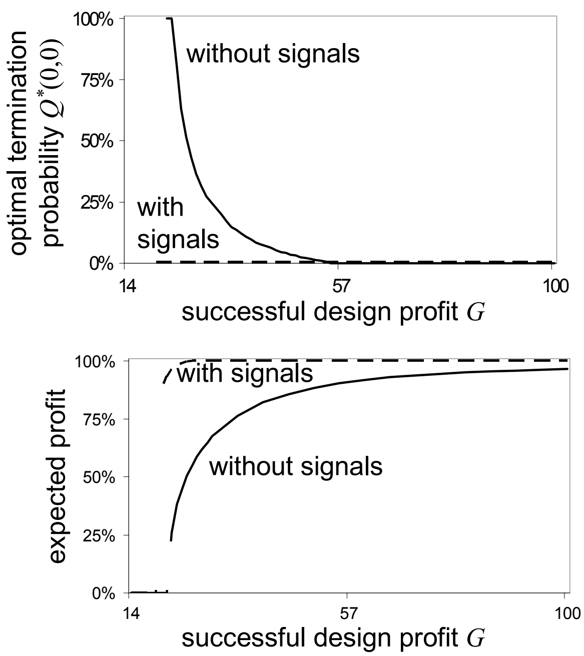

Figure 1: System performance and optimal relational contracts when signals of effort are or are not observed. System parameters are: $c = 1$, $p = 0.33$ and $\delta = 0.9$. On the top left, termination probability in the optimal relational contract, $Q^*(0, 0)$. On the top right, optimal number of prototypes per firm N in the centralized system and in the optimal relational contract. On the bottom left, discounted expected system profit, as a percentage of the first best profit.

depict the optimal relational contract as a function of the discounted expected profit generated by a successful end product G. Proposition 8 characterizes the optimal relational contract when the firms only observe the end product's success, i.e., without signals, and the panels illustrate this result. The optimal relational contract requires termination if and only if G is sufficiently small, and the optimal number of prototypes is increasing in G. When the firms only observe the end product's success, imposing the joint punishment of termination is optimal because it strengthens incentives for effort. When the firms observe signals, the need for termination is typically eliminated, as depicted in Figure 1. Observing signals allows the firms to provide stronger incentives for effort because in the event that only one firm's component is successful, that firm can be rewarded with a larger portion of the continuation profit. Consequently, observing signals results in greater parallelism, without necessitating costly termination.

The lower panel reports the discounted expected system profit when an end product has not been developed, as a percentage of the first best profit, $V^*(0)/\overline{V}(0)$ and $V_s^*(0)/\overline{V}(0)$, where $V_s^*(0)$ denotes the discounted expected profit of the system when signals are observed. When the firms only observe the end product's success, the loss in system efficiency can be substantial. The loss in system efficiency is due both to less parallelism (Proposition 9) and to endogenous termination. When G is small, providing incentives for effort requires imposing the joint punishment of termination, and it is here that the loss of system efficiency is largest.

As noted above, an immediate implication of Proposition 9 is that the optimal relational contract in the setting without signals never achieves the first best profit. Observing signals can overcome this inefficiency. Indeed, for $G > 28$, the optimal relational contract in the setting with signals achieves the first best. Even when the optimal relational contract fails to achieve the first best, the loss in system efficiency is quite small, provided that joint product development can be sustained. Observing signals can substantially increase profit. Both the absolute gain, $V_s^*(0) - V^*(0)$, and the relative gain, $[V_s^*(0) - V^*(0)]/V^*(0)$ from observing the signal are decreasing in G. Thus, perhaps surprisingly, firms that gain the least by developing a successful product gain the most from obtaining information about the success of each firm's individual design process.

Among automakers, Toyota has particularly deep long-term relationships with its suppliers. In its product development process, Toyota and its suppliers pursue an approach with a substantial degree of parallelism, exploring many more prototypes than other automakers. Further, while other automakers closely monitor their suppliers' development processes, Toyota follows a "hands-off" approach. Ward et al. (1995) describe Toyota's product development process as a paradox: On the one hand, Toyota and its suppliers explore so many prototypes that the degree of parallelism strikes outside observers as wasteful and inefficient. On the other hand, Toyota is the industry's fastest and most efficient product developer. To the extent that our simple model accurately reflects collaborative product development in the auto industry, our results shed some light on these phenomena. First, although the degree of parallelism in a joint product design process may appear excessive, Figure 1 suggests that it is, in fact, (weakly) insufficient: Total system expected profit

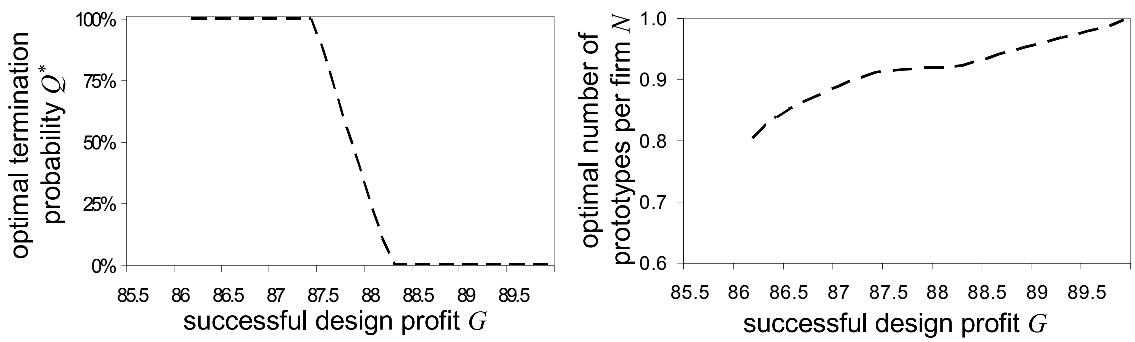

Figure 2: Optimal relational contract when signals of effort are observed. System parameters are: $c = 1$, $p = 0.6$ and $\delta = 0.1$. On the left, optimal termination probability when both firms fail to produce a successful component, Q^*. On the right, the optimal number of prototypes per firm N.

would (weakly) increase if each firm appropriately increased the number of prototypes it explored.‡ To the extent that Toyota is able to observe whether it supplier's component is successful, there is little or no distortion from the first best. In contrast, the other automakers' practice of monitoring is expensive and effectively increases the marginal cost of exploring a prototype. This leads to larger downward distortions from the first best, which is consistent with the observation that Toyota pursues greater parallelism.

Although observing signals typically eliminates the need for termination, the optimal relational contract may still require termination. Figure 2 depicts the optimal relational contract for an example in which the discount factor is small and the discounted expected profit generated by a successful end product is small relative to the cost of a prototype. The optimal relational contract only calls for termination if both firms fail to produce a successful component. As in the case without signals depicted in Figure 1, the optimal relational contract requires termination with positive probability if and only if G is sufficiently small. If $G < 87.4$, then the optimal relational contract requires termination with probability one.

6 Discussion

An integral aspect of long-term interaction is the introduction of dynamics, as conditions and the relationship itself evolve over time. While some aspects of these dynamics may be outside the

‡In fact, Proposition 9 shows that for the case where the components are tightly interconnected, the degree of parallelism is strictly insufficient.

control of the firms, as with general economic conditions, other aspects may be shaped by the firms themselves, as with investments or divestments of human and physical assets.

This paper shows how firms engaged in joint economic activity (e.g., production, product design) should structure informal agreements in the face of dynamics and the temptation to free-ride. We demonstrate that an optimal relational contract has a simple, memoryless form. To address dynamics, it is sufficient that the agreed upon actions depend only the current state and payments depend on the observed transition. The relational contract discourages free-riding by imposing termination to jointly punish the firms following a transition associated with poor performance. We show how process visibility can improve system performance by reducing the need for termination. A key question is to what extent optimal relational contracts nonetheless require termination.

One of our main insights is that dynamics and the need for termination are tightly interlinked. If the buyer does not influence the dynamics, then an optimal relational contract does not require termination. If the buyer does influence the dynamics, termination may be required. We consider an application of our model to collaborative product development, where the actions of both firms influence the dynamics. The optimal relational contract requires termination following the failure to produce a successful end product if and only if the product design technology is sufficiently poor. Without the threat of termination, incentives to invest in product development are weak. When the design technology is poor, the expected gain from the stronger incentives imposed by termination outweighs the expected loss from discontinued cooperation.

The framework we provide for addressing dynamic, joint economic activity is quite general. For example, the framework applies to progressive investment by a manufacturer in technology or capacity. Such relationship-specific investments may have the effect of making opportunism less attractive to the investor, but more attractive to the other firm. More generally, the framework applies to any action that affects the value of output from joint economic activity. An example is a buyer or manufacturer's investment in branding a manufacturer's component in the end consumer market, such as in the "Intel Inside" advertising campaign for personal computers. We are optimistic that the framework can be applied to a number of specific problem contexts (e.g., in operations or at the operations/marketing interface) to obtain sharper insights about dynamics and relationships, and we hope that future work will follow.

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Technical Appendix to Partnership in a Dynamic Production System

Lemma 1 is useful in the proofs of Propositions 1 and 2. Define $\rho(v_1, v_2) \equiv \sup_{x \in \mathcal{X}} |v_1(x) - v_2(x)|$.

Lemma 1 *The operator T is isotone. Further,*

$$\textit{if } v_1 \geq v_2, \; \textit{then } \rho(Tv_1, Tv_2) \leq \delta\rho(v_1, v_2). \tag{40}$$

Proof of Lemma 1: First we establish that T is isotone: if $v_1 \geq v_2$ then

$$C(a_m, a_b, v_1, x) \leq C(a_m, a_b, v_2, x) \text{ for every } x \in \mathcal{X},\ a_m \in A_m(x),\ a_b \in A_b(x),$$

and therefore

$$Tv_1 \geq Tv_2.$$

It remains to establish (40). We have shown that $v_1 \geq v_2$ implies $Tv_1 \geq Tv_2$. Define

$$(\hat{a}_b, \hat{a}_m) = \underset{(a_b, a_m) \in A_b(x) \times A_m(x)}{\arg\max} \left\{ -C(a_b, a_m, v_1, x) + \sum_{z \in \mathcal{X}} P_{xz}(a_m, a_b)[Y(x,z) + \delta v_1(z)] \right\}.$$

Suppose that

$$\delta \sum_{z \in \mathcal{X}} P_{xz}(0,0) v_1(z) \leq -C(\hat{a}_b, \hat{a}_m, v_1, x) + \sum_{z \in \mathcal{X}} P_{xz}(\hat{a}_b, \hat{a}_m)[Y(x,z) + \delta v_1(z)]. \tag{41}$$

This implies

$$\begin{aligned} 0 \leq Tv_1(x) - Tv_2(x) \quad &\leq \quad -C(\hat{a}_b, \hat{a}_m, v_1, x) + \sum_{z \in \mathcal{X}} P_{xz}(\hat{a}_b, \hat{a}_m)[Y(x,z) + \delta v_1(z)] \\ &\quad\quad + C(\hat{a}_b, \hat{a}_m, v_2, x) - \sum_{z \in \mathcal{X}} P_{xz}(\hat{a}_b, \hat{a}_m)[Y(x,z) + \delta v_2(z)] \\ &= \quad C(\hat{a}_b, \hat{a}_m, v_2, x) - C(\hat{a}_b, \hat{a}_m, v_1, x) + \delta \sum_{z \in \mathcal{X}} P_{xz}(\hat{a}_b, \hat{a}_m)[v_1(z) - v_2(z)] \\ &\leq \quad \delta \sum_{z \in \mathcal{X}} P_{xz}(\hat{a}_b, \hat{a}_m)[v_1(z) - v_2(z)] \\ &\leq \quad \delta \sup_{z \in \mathcal{X}} [v_1(z) - v_2(z)]. \end{aligned} \tag{42}$$

A similar argument establishes that (42) holds if the inequality in (41) is reversed. Thus,

$$|Tv_1(x) - Tv_2(x)| = Tv_1(x) - Tv_2(x) \leq \delta \sup_{z \in \mathcal{X}}[v_1(z) - v_2(z)] = \delta \sup_{z \in \mathcal{X}}|v_1(z) - v_2(z)|.$$

Taking the supremum over x in the expression above yields (40). ■

Proof of Proposition 1: As defined in (9), $\overline{V}(x)$ is the maximum discounted expected profit starting from state x. Therefore, for any $v \in [0, \overline{V}(x_1)] \times ... \times [0, \overline{V}(x_n)]$

$$Tv \in [0, \overline{V}(x_1)] \times ... \times [0, \overline{V}(x_N)].$$

Because T is isotone (by Lemma 1), from Tarski's fixed point theorem, the operator T has a greatest fixed point. We next establish that the fixed point is unique. Suppose T has more than one fixed point. Let v_1 denote the greatest fixed point and v_2 denote another fixed point, so $v_1 \geq v_2$. From Lemma 1,

$$\rho(v_1, v_2) = \rho(Tv_1, Tv_2) \leq \delta\rho(v_1, v_2).$$

Because $\delta \in (0, 1)$, this implies that $\rho(v_1, v_2) = 0$, which implies the result. ■

Proof of Proposition 2: By induction, because the operator T is isotone (from Lemma 1),

$$T^n\overline{V} \leq T^{n-1}\overline{V} \quad \text{for } n = 1, 2, ... \tag{43}$$

By induction on (43) and (40),

$$\begin{aligned} \rho(T^{n+1}\overline{V}, T^n\overline{V}) &\leq \delta\rho(T^n\overline{V}, T^{n-1}\overline{V}) \\ &\leq \delta^n\rho(T\overline{V}, \overline{V}) \\ &\leq \delta^n \left\|\overline{V}\right\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

so $\left\{T^n\overline{V}\right\}$ forms a Cauchy sequence, which has a limit

$$\hat{V} = \lim_{n\rightarrow\infty} T^n\overline{V}$$

satisfying

$$\hat{V} \leq T^n \overline{V}. \tag{44}$$

To see that $\hat{V}$ is a fixed point of T, choose $\epsilon > 0$ and select $N \geq 0$ such that $n \geq N$ implies $\rho(T^n\overline{V}, \hat{V}) < \epsilon/2$. Then for any $n \geq N$,

$$\begin{aligned} \rho(\hat{V}, T\hat{V}) &\leq \rho(\hat{V}, T^{n+1}\overline{V}) + \rho(T^{n+1}\overline{V}, T\hat{V}) \\ &\leq \frac{\epsilon}{2} + \rho(T^{n+1}\overline{V}, T\hat{V}) \\ &\leq \frac{\epsilon}{2} + \delta\rho(T^n\overline{V}, \hat{V}) \\ &\leq \frac{\epsilon}{2} + \delta\frac{\epsilon}{2} \\ &\leq \epsilon \end{aligned}$$

where the third inequality follows from (40) and (44). Therefore we conclude that

$$\hat{V} = T\hat{V}.$$

Because V^* is the unique fixed point of T (by Proposition 1), $\hat{V} = V^*$. ■

Proof of Proposition 3: Consider an optimal relational contract with the structure described in Theorem 1. If the associated termination function satisfies $Q^*(x,z) = 0$ for all $x \in \mathcal{X}$ with $V^*(x) > \delta \sum_{z\in\mathcal{X}} P_{xz}(0,0)V^*(z)$ and $z \in \mathcal{X}$ then (24) is satisfied, trivially. Suppose not. Fix a state $x \in \mathcal{X}$ with $V^*(x) > \delta \sum_{z\in\mathcal{X}} P_{xz}(0,0)V^*(z)$ and $Q^*(x,z) > 0$ for some $z \in \mathcal{X}$. The termination function $Q^*(x,z)$ satisfies

$$Q^*(x,z) = [V^*(z) - V_m^*(z) - V_b^*(z)]/V^*(z)$$

where (V_m^*, V_b^*) is a solution to the linear program:

$$\max_{V_m, V_b} \sum_{z\in\mathcal{X}} P_{xz} \; [V_m(z) + V_b(z)] \tag{45}$$

$$\begin{aligned} \text{subject to: } & V^*(z) - V_m(z) - V_b(z) \geq 0,\ V_m(z) \geq 0,\ V_b(z) \geq 0 \text{ for every } z \in \mathcal{X} \\ & \sum_{z\in\mathcal{X}} \frac{\partial P_{xz}}{\partial a_m}\delta V_m(z) - \frac{\partial c_m}{\partial a_m} = 0 \\ & \sum_{z\in\mathcal{X}} \frac{\partial P_{xz}}{\partial a_b}\delta V_b(z) - \frac{\partial c_b}{\partial a_b} = 0, \end{aligned}$$

in which, for brevity, we have suppressed the optimal actions: P_{xz} denotes $P_{xz}(a_m^*(x), a_b^*(x))$; c_m denotes $c_m(a_m^*(x), x)$; and c_b denotes $c_b(a_b^*(x), x)$. The two constraints are the first order conditions corresponding to the incentive compatibility constraints in (11). Assumptions (19)-(22) guarantee that these first order conditions are necessary and sufficient. Note that any optimal solution to (45) has $V_m^*(z)$ and $V_b^*(z)$ increasing in z, so that each firm's objective function ((5)-(6)) under the corresponding optimal relational contract is concave.

Let z_n be the largest state $z \in \mathcal{X}$ such that $Q^*(x, z) > 0$. Equivalently, choose n so that $Q^*(x, z_n) > 0$ and $Q^*(x, z_i) = 0$ for all $i > n$. Suppose that $Q^*(x, z_i) < 1$ for some $i < n$. This implies $V_m^*(z_i) > 0$ or $V_b^*(z_i) > 0$. Suppose $V_b^*(z_i) > 0$ and $V_m^*(z_i) = 0$. Then there exists $\Delta > 0$ such that

$$\begin{aligned}
\widetilde{V}_b(z_i) &\equiv V_b^*(z_i) - \Delta \geq 0 \\
\widetilde{V}_b(z_n) &\equiv V_b^*(z_n) + \Delta \frac{\partial P_{xz_i}}{\partial a_b} / \frac{\partial P_{xz_n}}{\partial a_b} \leq V^*(z_n) - V_m^*(z_n)
\end{aligned}$$

and at least one of the following two equalities is satisfied:

$$\begin{aligned}
\widetilde{V}_b(z_i) &= 0 \\
\widetilde{V}_b(z_n) + V_m^*(z_n) &= V^*(z_n).
\end{aligned}$$

By construction,

$$\sum_{z \in \mathcal{X} \backslash \{z_i, z_n\}} \frac{\partial P_{xz}}{\partial a_b} \delta V_b^*(z) + \frac{\partial P_{xz_i}}{\partial a_b} \delta \widetilde{V}_b(z_i) + \frac{\partial P_{xz_n}}{\partial a_b} \delta \widetilde{V}_b(z_n) - \frac{\partial c_b}{\partial a_b} = 0,$$

and assumption (22) guarantees that

$$P_{xz_i}\ \widetilde{V}_b(z_i) + P_{xz_n}\ \widetilde{V}_b(z_n) \geq P_{xz_i}\ V_b^*(z_i) + P_{xz_n}\ V_b^*(z_n),$$

so replacing $(V_b^*(z_i), V_b^*(z_n))$ with $(\widetilde{V}_b(z_i), \widetilde{V}_b(z_n))$ leads to a larger objective value in (45). Thus, reducing $V_b^*(z_i)$ to $\widetilde{V}_b(z_i)$ and increasing $V_b^*(z_n)$ to $\widetilde{V}_b(z_n)$ yields a solution to (45) (and hence an alternative optimal relational contract) with a termination function $\widetilde{Q}$ that satisfies at least one of the following two inequalities:

$$\widetilde{Q}(x, z_n) = 0 \tag{46}$$

$$\widetilde{Q}(x, z_i) = 1. \tag{47}$$

Analogous arguments establish (46) or (47) when $V_m^*(z_n) > 0$.

By substituting $\widetilde{Q}$ for Q^* and recursively repeating the above sequence of arguments (at most n times), we obtain an optimal relational contract with the structure described in Theorem 1 and

$$Q^*(x, z_i) = \begin{cases} 1 & \text{for } z_i < z_k \\ 0 & \text{for } z_i > z_k \end{cases}$$

for some threshold state $z_k \in \mathcal{X}$. Iterating the preceding arguments over all states $x \in \mathcal{X}$ with $V^*(x) > \delta \sum_{z \in \mathcal{X}} P_{xz}(0,0) V^*(z)$ establishes (24). ■

Proof of Proposition 4: (a) Because the signal is informative about the manufacturer's action

$$\sum_{s^i \geq \widehat{s}_m(a_m, x)} \frac{\partial}{\partial a_m} P_{x(z,s^i)}(a_m, a_b) > 0.$$

Recall that $\overline{\mathcal{X}} \subset \mathcal{X}$ and $\{\overline{a}_m(x), \overline{a}_b(x) \colon x \in \overline{\mathcal{X}}\}$ denote the first best transaction states and actions, obtained by solving (9). For every state $X_t \in \overline{\mathcal{X}}$, set the court-enforced contract

$$\begin{aligned} p_t(s_{mt}) = {} & c_m(\overline{a}_m(X_t), X_t) - \sum_{s^i \geq \widehat{s}_m(\overline{a}_m(X_t), X_t)} P_{X_t(z,s^i)}(\overline{a}_m(X_t), \overline{a}_b(X_t)) \times \\ & \left[\sum_{s^i \geq \widehat{s}_m(\overline{a}_m(X_t), X_t)} \frac{\partial}{\partial a_m} P_{X_t(z,s^i)}(\overline{a}_m(X_t), \overline{a}_b(X_t))\right]^{-1} \frac{\partial}{\partial a_m} c_m(\overline{a}_m(X_t), X_t) \\ & + \begin{cases} \left[\sum_{s^i \geq \widehat{s}_m(\overline{a}_m(X_t), X_t)} \frac{\partial}{\partial a_m} P_{X_t(z,s^i)}(\overline{a}_m(X_t), \overline{a}_b(X_t))\right]^{-1} \frac{\partial}{\partial a_m} c_m(\overline{a}_m(X_t), X_t) & \text{if } s_{mt} \geq \widehat{s}_m(\overline{a}_m(X_t), X_t) \\ 0 & \text{if } s_{mt} < \widehat{s}_m(\overline{a}_m(X_t), X_t), \end{cases} \end{aligned}$$

and discretionary transfer payment: $d_t = 0$. Assuming that the buyer chooses his first best action $a_{bt} = \overline{a}_b(X_t)$, the manufacturer's first best action $\overline{a}_m(X_t)$ solves the first order condition corresponding to his incentive compatibility constraint (5). Assumptions (20)-(22) imply that the manufacturer's objective is concave, so this is necessary and sufficient. The payment scheme gives the manufacturer zero expected profit in every period, so the manufacturer is willing to transact. The signal s_{mt} is invariant to the buyer's action, so the buyer will indeed choose the first best action

$\overline{a}_b(X_t)$. (b) We provide an explicit example with an independent signal for the manufacturer where the optimal relational contract requires termination. See Figure 2 and the text immediately before it in §5. In this example, the buyer's choice of action influences the ongoing expected discounted profit. ■

Proof of Proposition 5: Let $a_b^*(a_m; x)$ denote an action for the buyer which maximizes total expected discounted profit, given that the manufacturer takes action a_m:

$$a_b^*(a_m; x) \in \arg\max_{a \in A_b(x)} \left\{ -c_b(a, x) + \sum_{z \in \mathcal{X}} P_{xz}(a_m, a)[Y(x, z) + \delta V^*(z)] \right\}.$$

If the optimal relational contract employs action a_m, an upper bound on the value function in state x is given by:

$$V^*(x) \leq -c_m(a_m, x) - c_b(a_b^*(a_m; x), x) + \sum_{z \in \mathcal{X}} P_{xz}(a_m, a_b^*(a_m; x))[Y(x, z) + \delta V^*(z)]. \tag{48}$$

We will show that this upper bound is tight. That is, there exists an optimal relational contract that does not involve termination which implements $(a_m, a_b^*(a_m; x))$.

We begin by establishing that $V^*(X_{t+1})$ is invariant with respect to the buyer's action a_{bt}. By inspection of (10), any two states x_1, $x_2 \in \mathcal{X}$ with $D(x_1) = D(x_2)$ have the same ongoing total expected discounted profit under an optimal relational contract: $V^*(x_1) = V^*(x_2)$. The desired result follows from the assumption that $D(X_{t+1})$ is invariant with respect to the buyer's action a_{bt}. Let $\mathcal{S}$ denote $\{s^1, ..., s^N\}$.

Assumptions (20)-(22) generalized for the signal allow us to substitute a first order condition for the manufacturer's incentive compatibility constraint. Hence (11) evaluated at V^* reduces to

$$\begin{aligned} C(a_m, a_b, V^*, x) \quad = \quad & c_m(a_m, x) + c_b(a_b, x) \\ & + \min_{V_m, V_b} \sum_{s \in \mathcal{S}} \sum_{z \in \mathcal{X}} P_{x(z,s)}(a_m, a_b) Q(x, z, s) \delta V^*(z) \\ & \text{subject to:} \\ & V_m(x, z, s) \geq 0,\ V_b(x, z, s) \geq 0,\ V_m(x, z, s) + V_b(x, z, s) \leq V^*(z) \\ & \qquad \text{for } z \in \mathcal{X},\ s \in \mathcal{S} \\ & \sum_{s \in \mathcal{S}} \sum_{z \in \mathcal{X}} \frac{\partial}{\partial a_m} P_{x(z,s)}(a_m, a_b) \delta V_m(x, z, s) = \frac{\partial}{\partial a_m} c_m(a_m, x) \\ & a_b \in \arg\max_{a \in A_b(x)} \left\{ -c_b(a, x) + \sum_{s \in \mathcal{S}} \sum_{z \in \mathcal{X}} P_{x(z,s)}(a_m, a)[Y(x, z) + \delta V_b(x, z, s)] \right\} \\ & Q(x, z, s) = [V^*(z) - V_m(x, z, s) - V_b(x, z, s)]/V^*(z). \end{aligned} \tag{49}$$

Observe that there exists a feasible solution V_m to constraints

$$V_m(x,z,s) \in [0, V^*(z)] \quad \text{for } z \in \mathcal{X} \text{ and } s \in \mathcal{S} \tag{50}$$

$$\sum_{s\in\mathcal{S}} \sum_{z\in\mathcal{X}} \frac{\partial}{\partial a_m} P_{x(z,s)}(a_m, a_b)\delta V_m(x,z,s) = \frac{\partial}{\partial a_m} c_m(a_m, x) \tag{51}$$

if and only if

$$\sum_{s\geq\widehat{s}_m(a_m,x)} \sum_{z\in\mathcal{X}} \frac{\partial}{\partial a_m} P_{x(z,s)}(a_m, a_b)\delta V^*(z) \geq \frac{\partial}{\partial a_m} c_m(a_m, x). \tag{52}$$

Because $V^*(X_{t+1})$ is invariant with respect to the buyers buyer's action a_{bt}, whether (52) holds depends only on a_m and not on a_b. Thus, $C(a_m, a_b, V^*, x) < \infty$ and the optimal relational contract employs action a_m only if (52) is satisfied. For the remainder of the proof, we restrict attention to the action a_m that is employed in the optimal relational contract. Because (52) is satisfied, there exists a constant $\beta \in [0,1]$ such that

$$V_m(x,z,s) = \begin{cases} \beta V^*(z) & \text{if } s \geq \widehat{s}_m(a_m, x) \\ 0 & \text{if } s < \widehat{s}_m(a_m, x) \end{cases} \tag{53}$$

satisfies constraints (50)-(51). Because $V^*(X_{t+1})$ is invariant with respect to the buyer's action a_{bt},

$$a_b^*(a_m; x) \in \arg\max_{a\in A_b(x)} \left\{ -c_b(a,x) + \sum_{z\in\mathcal{X}} P_{xz}(a_m, a) Y(x,z) \right\}.$$

Therefore, setting

$$\begin{aligned} Q(x,z,s) &= 0 \\ V_b(x,z,s) &= V^*(z) \left[1 - \beta 1_{\{s\geq\widehat{s}_m(a_m,x)\}}\right] \quad \text{for all } s \in \mathcal{S} \text{ and } z \in \mathcal{X} \end{aligned} \tag{54}$$

will induce action $a_b^*(a_m; x)$:

$$a_b^*(a_m; x) \in \arg\max_{a\in A_b(x)} \left\{ -c_b(a,x) + \sum_{s\in\mathcal{S}} \sum_{z\in\mathcal{X}} P_{x(z,s)}(a_m, a)[Y(x,z) + \delta V_b(x,z,s)] \right\}.$$

(Recall that the signal s is also invariant with respect to the buyer's action.) From (V_m, V_b) in (53) and (54), we conclude that there exists a relational contract that does not involve termination which implements $(a_m, a_b^*(a_m; x))$. From (48), this is an optimal relational contract.

Applying these arguments to the special case that $a_m = \overline{a}_m(x)$ and $a_b^*(a_m; x) = \overline{a}_b(x)$ (the first best actions) establishes that the first best is achieved if and only if (27) holds. ■

Proof of Proposition 6: We must verify that conditions (3)-(8) are satisfied when, in every period, the firms transact if and only if $X_t \in \overline{\mathcal{X}}$, and, when they transact, adopt the first best actions and execute the payments (29)-(31). Because $\frac{\partial}{\partial a_m} c_m(\overline{a}_m(x), x)$ and $\min_{z \in \mathcal{X}} \overline{V}(z)$ are non-negative, condition (28) guarantees that for every state $x \in \overline{\mathcal{X}}$

$$d(x) \in \left[0, \min_{z \in \mathcal{X}} \overline{V}(z)\right]. \tag{55}$$

Therefore the manufacturer is willing to execute the discretionary payment and the buyer, assuming that he will capture the ongoing total expected discounted profit $\overline{V}(X_{t+1})$, is also willing to execute the discretionary payment: (7) and (8) are satisfied. The discretionary payment is constructed so that for every state $x \in \mathcal{X}$, the manufacturer's first order condition is satisfied at the first best action:

$$\sum_{s \geq \widehat{s}_m(\overline{a}_m(x), x)} \sum_{z \in \mathcal{X}} \frac{\partial}{\partial a_m} P_{x(z,s)}(\overline{a}_m(x), \overline{a}_b(x)) d(x) - \frac{\partial}{\partial a_m} c_m(\overline{a}_m(x), x) = 0. \tag{56}$$

Assumptions (20)-(22) imply that the manufacturer's objective is concave, so this is necessary and sufficient. It follows immediately from (56) that the first best action is optimal for the manufacturer:

$$\overline{a}_m(x) \in \arg\max_{a_m \in A_m(x)} \left\{ \sum_{s \geq \widehat{s}_m(\overline{a}_m(x))} \sum_{z \in \mathcal{X}} P_{x(z,s)}(a_m, \overline{a}_b(x)) d(x) - c_m(a_m) \right\}, \tag{57}$$

so (5) is satisfied. The court-enforced payment (31) is chosen so that if, in period $t+1$ and every subsequent period that begins in state $x \in \overline{\mathcal{X}}$, the firms transact, adopt the first best actions and execute the transfer payments (29)-(31), then (3) and (4) is satisfied. In particular, the buyer captures the total ongoing expected discounted profit:

$$E[\Pi_{m(t+1)} | X_{t+1}] = 0. \tag{58}$$

$$E[\Pi_{b(t+1)} | X_{t+1}] = \overline{V}(X_{t+1}). \tag{59}$$

Because the signal s_{mt} is invariant with respect to the buyer's action a_{bt}, the discretionary transfer payment d_t is also invariant with respect to a_{bt}. Hence, in period t, the buyer maximizes his own discounted expected profit by adopting the first best action $\overline{a}_b(X_t)$ and executing the discretionary transfer payment: (6) is satisfied.

Any relational contract which terminates with strictly positive probability on the equilibrium path generates strictly lower expected discounted profit than the first best. Therefore we conclude that every optimal relational contract is non-terminating: $Q^* = 0$. ■

Proof of Proposition 7: Let $\overline{\Pi}(N_m, N_b)$ denote the quantity in (32). First, we show that the optimal solution is symmetric. Note that

$$(\partial/\partial\beta)(\overline{\Pi}(\beta N, (1-\beta)N) = \frac{(-\bar{p}^{\beta N} + \bar{p}^{(1-\beta)N})\delta N[(1-\delta)G + cN]\log(\bar{p})}{[1+\delta(\bar{p}^N - \bar{p}^{\beta N} - \bar{p}^{(1-\beta)N})]^2}.$$

The first order necessary condition holds only if $\beta = 1/2$. Note that

$$(\partial/\partial N)\overline{\Pi}(N, N) = 2\overline{\Lambda}(N)/[1 - \delta\bar{p}^N(2 - \bar{p}^N)]^2,$$

where

$$\overline{\Lambda}(N) = -[1 - \delta\bar{p}^N(2 - \bar{p}^N)]c - \delta\bar{p}^N(1 - \bar{p}^N)[(1-\delta)G + 2cN]\log(\bar{p}).$$

Because $\overline{\Lambda}(0) < 0$, $\overline{\Lambda}(\infty) < 0$, and

$$(\partial/\partial N)\overline{\Lambda}(N) > 0 \Longleftrightarrow N < -\log(2)/\log(\bar{p}),$$

the first order condition

$$\overline{\Lambda}(N) = 0$$

has at most two solutions. Because $\overline{\Pi}(0,0) = 0$, $(\partial/\partial N)\overline{\Pi}(N,N)|_{N=0} < 0$, and the integrated system profit under the optimal number of product designs is strictly positive, (33) follows. Because

$$(\partial/\partial c)\overline{\Lambda}(N)|_{N=\overline{N}} = \delta(1-\delta)\bar{p}^{\overline{N}}(1 - \bar{p}^{\overline{N}})G\log(\bar{p})/c < 0,$$

$\overline{N}$ is decreasing in c. Because

$$(\partial/\partial G)\overline{\Lambda}(N) = -\delta(1-\delta)\bar{p}^N(1 - \bar{p}^N)\log(\bar{p}) > 0,$$

$\overline{N}$ is increasing in G. ■

Proof of Proposition 8: In constructing a relational contract, one only need consider the initial state of 0. For ease of notation, we omit the first argument in the state transition (e.g., $V_m(1)$ denotes $V_m(0,1)$). Further, let V denote $V(0)$, so that $V \leq \delta G$. Consider the optimization problem

$$\begin{aligned} TV &= \max_{V_m(1),\, V_m(0),\, V_b(1),\, V_b(0),\, N_m,\, N_b} \{\delta(P_{0,1}(N_m, N_b)[V_m(1) + V_b(1)] \\ &\qquad + P_{0,0}(N_m, N_b)[V_m(0) + V_b(0)]) - c(N_m + N_b)\} \end{aligned} \tag{60}$$

subject to

$$V_m(1) \geq 0,\ V_m(0) \geq 0,\ V_b(1) \geq 0,\ V_b(0) \geq 0,\ V_m(1) + V_b(1) \leq G,\ V_m(0) + V_b(0) \leq V,$$

$$N_m \in \arg\max_{N \geq 0} \{\delta[P_{0,1}(N, N_b)V_m(1) + P_{0,0}(N, N_b)V_m(0)] - cN\} \tag{61}$$

$$N_b \in \arg\max_{N \geq 0} \{\delta[P_{0,1}(N_m, N)V_b(1) + P_{0,0}(N_m, N)V_b(0)] - cN\}. \tag{62}$$

First we characterize from (61)-(62) the Nash equilibria in the number of designs. We exclude the equilibrium $N_m = N_b = 0$ because a relational contract that calls for these actions will result in zero system profit, and by assumption, the optimal relational contract results in strictly positive profit. The manufacturer's objective function in (61) is concave in N, and the first order condition implies that the manufacturer's best response is

$$N_m = [\log(c) - \log(V_m(1) - V_m(0)) - \log(-\delta(1 - \bar{p}^{N_b})\log(\bar{p}))]/\log(\bar{p}). \tag{63}$$

Similarly, the buyer's best response is

$$N_b = [\log(c) - \log(V_b(1) - V_b(0)) - \log(-\delta(1 - \bar{p}^{N_m})\log(\bar{p}))]/\log(\bar{p}), \tag{64}$$

so (N_m, N_b) is a Nash equilibrium if and only if it satisfies (63)-(64).

Recall that the relational contract must satisfy $V_m(1) + V_b(1) \leq G$. Let us assume that $V_m(1) + V_b(1) = G$. At the conclusion of this proof we will verify that discounted expected profit under the optimal relational contract is strictly increasing in G, so this assumption is without loss of generality in constructing an optimal relational contract. (Intuitively, $V_m(1) + V_b(1) = G$ in the optimal relational contract because destroying value in state 1 weakens the firms' incentives for effort in state 0.) With $V_m(1) + V_b(1) = G$, the optimization problem (60) simplifies to

$$
\begin{aligned}
TV &= \max_{v\in[0,V],\ \alpha\in[0,1],\ N_m,\ N_b} \{\delta[P_{0,1}(N_m, N_b)G + P_{0,0}(N_m, N_b)v] - c(N_m + N_b)\} \qquad (65)\\
&\quad \text{subject to: } g_m(\alpha, N_m, N_b) = 0\\
&\quad \phantom{\text{subject to: }} g_b(\alpha, N_m, N_b) = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
g_m(\alpha, N_m, N_b) &= N_m - [\log(c) - \log(\alpha(G - v)) - \log(-\delta(1 - \bar{p}^{N_b}) \log(\bar{p}))]/\log(\bar{p})\\
g_b(\alpha, N_m, N_b) &= N_b - [\log(c) - \log((1 - \alpha)(G - v)) - \log(-\delta(1 - \bar{p}^{N_m}) \log(\bar{p}))]/\log(\bar{p}).
\end{aligned}
$$

We will now prove that any solution to (65) is symmetric, that is, $\alpha = 1/2$ and $N_m = N_b$. Fix v in the feasible set. For each (α,v), the equality constraints in (65) are the firms' first order conditions in a supermodular game. Because the game is supermodular, it has a largest Nash equilibrium and the expected profit for each firm (and hence the objective in (65)) is maximized at the largest equilibrium (see Milgrom and Roberts 1990, Theorem 7). The total number of designs $(N_m + N_b)$ at the largest Nash equilibrium is maximized by setting $\alpha = 1/2$. To see this, consider the optimization problem

$$
\begin{aligned}
&\max_{\alpha\in[0,1/2],\ N_m,\ N_b} \{N_m + N_b\} \qquad (66)\\
&\text{subject to: } g_m(\alpha, N_m, N_b) = 0\\
&\phantom{\text{subject to: }} g_b(\alpha, N_m, N_b) = 0.
\end{aligned}
$$

Clearly, $\alpha = 0$ cannot be optimal. Suppose in an optimal solution $\alpha < 1/2$; then the Lagrangian multiplier is zero for the constraint $\alpha \leq 1/2$. The Lagrangian is

$$\mathcal{L} = N_m + N_b + \lambda_m g_m(\alpha, N_m, N_b) + \lambda_b g_b(\alpha, N_m, N_b),$$

and any solution must satisfy

$$\lambda_m(\partial/\partial\alpha)g_m(\alpha, N_m, N_b) + \lambda_b(\partial/\partial\alpha)g_b(\alpha, N_m, N_b) = 0 \quad (67)$$

$$1 + \lambda_m(\partial/\partial N_m)g_m(\alpha, N_m, N_b) + \lambda_b(\partial/\partial N_m)g_b(\alpha, N_m, N_b) = 0 \quad (68)$$

$$1 + \lambda_m(\partial/\partial N_b)g_m(\alpha, N_m, N_b) + \lambda_b(\partial/\partial N_b)g_b(\alpha, N_m, N_b) = 0. \quad (69)$$

Note that (67) holds if and only if

$$\frac{\lambda_m}{\lambda_b} = \frac{\alpha}{1-\alpha}; \quad (70)$$

(68) and (69) together imply

$$\frac{\lambda_m}{\lambda_b} = \frac{1-\bar{p}^{N_b}}{1-\bar{p}^{N_m}}. \quad (71)$$

Further, (68) and (69) together imply

$$\frac{1-\bar{p}^{N_b}}{\alpha} = \frac{1-\bar{p}^{N_m}}{1-\alpha}. \quad (72)$$

In any equilibrium with $\alpha < 1/2$, $N_m < N_b$, and, hence,

$$\frac{1-\bar{p}^{N_b}}{\alpha} > \frac{1-\bar{p}^{N_m}}{1-\alpha},$$

which contradicts (72). Hence, we conclude that the solution to (66) has $\alpha = 1/2$. At $\alpha = 1/2$, the game is symmetric so the largest Nash equilibrium is symmetric (see the last corollary to Theorem 5 in Milgrom and Roberts 1990). Let $N(v)$ denote the number of designs per firm in the largest Nash equilibrium at $\alpha = 1/2$. Then $N(v)$ is the largest positive solution to

$$N = [\log(c) - \log((G-v)/2) - \log(-\delta(1-\bar{p}^N)\log(\bar{p}))]/\log(\bar{p}), \quad (73)$$

if a positive solution exists; otherwise $N(v) = 0$. It is straightforward to establish that

$$N(v) = \begin{cases} 0 & \text{for } c > -\delta(G-v)\log(\bar{p})/8 \\ \log\left(\left[1-\sqrt{1+8c/[\delta(G-v)\log(\bar{p})]}\right]/2\right)/\log(\bar{p}) & \text{for } c \le -\delta(G-v)\log(\bar{p})/8. \end{cases}$$

Finally, consider the optimization problem

$$\max_{v\in[0,V],\ N_m,\ N_b} \quad \{\delta[P_{0,1}(N_m,N_b)G + P_{0,0}(N_m,N_b)v] - c(N_m+N_b)\} \quad \text{subject to: } N_m + N_b \leq 2N(v). \tag{74}$$

This is a relaxation of the optimization problem (65), because any solution to the equality constraints in (65) has $N_m + N_b \leq 2N(v)$. We next show that in the region of interest, $V \leq \delta G$, any solution to (74) satisfies $N_m = N_b = N(v)$. This is immediate if $c > -\delta(G-v)\log(\bar{p})/8$. To see this when $c \leq -\delta(G-v)\log(\bar{p})/8$, let $\pi(N_m, N_b)$ denote the objective function in (65). It is straightforward to verify that for $\beta \in [0,1]$,

$$\pi(2\beta N, 2(1-\beta)N) \leq \pi(N,N). \tag{75}$$

Further, because $\pi(N,N)$ is convex on $N \in [0, -\log(2)/\log(\bar{p})]$ and concave on $N \in (-\log(2)/\log(\bar{p}), \infty)$, $(\partial/\partial N)\pi(N,N)|_{N=0} = -2c < 0$, $(\partial/\partial N)\pi(N,N)|_{N=N(v)} = 2c > 0$, and $\pi(0,0) < \pi(N(v),N(v))$,

$$\pi(N,N) \leq \pi(N(v),N(v)), \tag{76}$$

for all $N \leq N(v)$. (75) and (76) together imply that any solution to (74) satisfies $N_m = N_b = N(v)$. This completes the proof that every solution to (65) has $\alpha = 1/2$ and $N_m = N_b = N(v)$.

For the region of interest, $V \leq \delta G$, we have simplified the optimization problem (65) to

$$TV = \max_{v\in[0,V]} F(v),$$

where

$$F(v) = \delta[P_{0,1}(N(v),N(v))G + P_{0,0}(N(v),N(v))v] - 2cN(v).$$

Because

$$(\partial/\partial v)F(v) = \frac{\delta\bar{p}^{N(v)}}{1-2\bar{p}^{N(v)}}\left(\frac{2c}{\delta(G-v)\log(\bar{p})} + \sqrt{1+\frac{8c}{\delta(G-v)\log(\bar{p})]}}\right),$$

$F(v)$ is maximized at

$$v^* = \left[G + \frac{(4+2\sqrt{5})c}{\delta\log(\bar{p})}\right]^+,$$

is strictly increasing on $v \in [0, v^*]$ and is strictly decreasing on $v \in [v^*, \infty)$.

If $F(v^*) > v^*$, then $TF(v^*) = F(v^*)$; thus, $V^* = F(v^*)$ and the optimal relational contract requires $Q^*(0) = 1 - v^*/F(v^*)$ and $N^* = N(v^*)$. If $F(v^*) \leq v^*$, then $V^* = F(\hat{v})$, where $\hat{v} = \max\{v : F(v) = v\}$. Further, the optimal relational contract requires $Q^*(0) = 0$ and $N^* = N_2(\hat{v})$. To see this, observe that $F(v^*) \leq v^*$ implies that for any v, $TF(v) = F(F(v))$. Let $F(\tilde{v}) = \max\{F(v) : F(F(v)) = F(v)\}$; note $\hat{v} = F(\tilde{v})$. Clearly, $TF(\hat{v}) = F(\hat{v})$, so $V^* = F(\hat{v})$. Because

$$\begin{aligned} F(v^*) &> v^* \Leftrightarrow G < \overline{G} \\ v^* &= 0 \Leftrightarrow G \leq \underline{G}, \end{aligned}$$

(34)-(37) follow. If $G \geq \overline{G}$, then the optimal contract must satisfy $v = \hat{v}$, which holds only if

$$\Lambda^*(N) = 0,$$

where

$$\Lambda^*(N) = -2[1 - \delta \bar{p}^N (2 - \bar{p}^N)]c - \delta \bar{p}^N (1 - \bar{p}^N)[(1-\delta)G + 2cN] \log(\bar{p}).$$

Because $N(v) > -\log(2)/\log(\bar{p})$, because $(\partial/\partial N)\Lambda^*(N) < 0$ on $N > -\log(2)/\log(\bar{p})$, and because the profit under the optimal relational contract is strictly positive, (39) follows.

We have assumed that the optimal relational contract satisfies $V_m(1) + V_b(1) = G$. We now verify that this assumption is without loss of generality in that the discounted expected profit under the optimal relational contract, V^*, is strictly increasing in G. If $G < \overline{G}$, then the optimal profit is given in closed form, $V^* = F(v^*)$, and it is straightforward to verify that $(\partial/\partial G)F(v^*) > 0$. If $G \geq \overline{G}$, then $F(v^*) \leq v^*$ and $V^* = \hat{v}$. Thus, $(\partial/\partial G)F(v) > 0$ implies $(\partial/\partial G)\hat{v} > 0$.

That $Q^*(0,0)$ is decreasing in G and p and increasing in c is immediate from (34), (36) and (38). Because N^* is continuous in G, to establish that N^* is increasing in G and decreasing in c it is sufficient to establish this result for N^* as given in (34), (36) and (38). The result is immediate for (34) and (36). For (38), the result follows by argument analogous to that in the proof of Proposition 7. ■

Proof of Proposition 9: From Proposition 8, $N^* > -\log(2)/\log(\bar{p})$. Further, $(\partial/\partial N)\overline{\Lambda}(N) < 0$ for $N > -\log(2)/\log(\bar{p})$, where

$$\overline{\Lambda}(N) = -[1 - \delta \bar{p}^N (2 - \bar{p}^N)]c - \delta \bar{p}^N (1 - \bar{p}^N)[(1-\delta)G + 2cN] \log(\bar{p}).$$

Therefore, it is sufficient to show that

$$\overline{\Lambda}(N^*) > 0 \tag{77}$$

From Proposition 8, $G \leq \underline{G}$ implies (35), so

$$\begin{aligned} (\partial/\partial G)\overline{\Lambda}(N^*) &= -4c^2 N^*/G^2 < 0 \\ \overline{\Lambda}(N^*)|_{G=\underline{G}} &= \{1 - \delta[5 - \sqrt{5} + (4\sqrt{5} - 8)\log((3 - \sqrt{5})/2)]/2\} > 0, \end{aligned} \tag{78}$$

which together imply (77). From Proposition 8, $G \in \left(\underline{G}, \overline{G}\right)$ implies (37), so

$$(\partial/\partial G)\overline{\Lambda}(N^*) = -(3 - \sqrt{5})(\sqrt{5} - 1)\delta(1 - \delta)\log(\bar{p})/4 > 0.$$

This, together with (78) and the fact that $\overline{\Lambda}(N^*)$ is continuous in G, implies that for $G \in \left(\underline{G}, \overline{G}\right)$, (77). From Proposition 8, if $G \geq \overline{G}$, then (39). Because $\Lambda^*(N) < \overline{\Lambda}(N)$, where

$$\Lambda^*(N) = -2[1 - \delta\bar{p}^N(2 - \bar{p}^N)]c - \delta\bar{p}^N(1 - \bar{p}^N)[(1 - \delta)G + 2cN]\log(\bar{p}),$$

$\Lambda^*(N^*) = 0$ implies (77). ■