Comprehensive Exam: Mathematical Economics

Question

This question studies optimal auction design with endogenous entry. There are a large number of potential bidders who must pay k in order to enter an auction. After the entry decision, each entering bidder learns their private value θ_i which are distributed independently and identically with positive density $f(\theta)$, distribution function $F(\theta)$ and support $[\underline{\theta}, \overline{\theta}]$. The auctioneer has known valuation θ_0.

Denote the direct mechanism by $\langle N, P_i, t_i \rangle$, which is common knowledge. The auctioneer first allows bidders in the set N to enter. Each entering bidder learns their type θ_i and reports $\tilde{\theta}_i$. If the other bidders report truthfully, bidder i wins the good with probability $P_i(\tilde{\theta}_i, \theta_{-i})$ and pays $t_i(\tilde{\theta}_i, \theta_{-i})$ yielding utility,

$$u_i(\theta_i, \tilde{\theta}_i) = E_{\theta_{-i}} \left[\theta_i P_i(\tilde{\theta}_i, \theta_{-i}) - t_i(\tilde{\theta}_i, \theta_{-i}) \right]$$

where the lowest type gets utility $u_i(\underline{\theta})$.

(a) Show that incentive compatibility (IC) implies that utility obeys an integral equation and a monotonicity constraint.

(b) Write down the ex–ante individual rationality (IR) constraint which ensures that each bidder is happy to pay the entry cost and participate.

(c) Write down the auctioneer's program or maximising revenue, equal to the sum of payments, subject to (IC) and (IR).

(d) Show that the (IR) constraint will bind at the optimum.

(e) Optimal allocation function. Show that the revenue maximising mechanism awards the object to the agent with the highest valuation if that value exceeds θ_0.

(f) Optimal entry policy. Define welfare with n bidders by

$$W(n) := E_\theta \max\{\theta_0, \theta_1, \ldots, \theta_n\}$$

Show that $W(n+1) - W(n)$ decreases in n. Use this to show that the optimal number of bidders, n^*, obeys $W(n^*) - W(n^* - 1) \geq k \geq W(n^* + 1) - W(n^*)$.

(g) Argue that the optimal mechanism can be implemented by a standard auction with reserve price, entry fee and having bidders make their entry decisions sequentially. What are the optimal entry fee and reserve price?