# A Note on Holmstrom and Milgrom (1987)

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Suppose $T = 2$. The principal's long term contract consists of choosing wages in the first and second period, $w_1(q_1)$ and $w_2(q_1, q_2)$. The agent only cares about the sum, but it will be useful to separate these to draw a comparison to the results of FHM.

The principal's problem

$$
\begin{aligned}
&\max_{w_1(q_1), w_2(q_1,q_2), a_1, a_2(q_1)} E[q_1 + q_2 - w_1(q_1) - w_2(q_1, q_2)|a_1, a_2(q_1)] \\
&(IR_1) \quad E[u(w_1(q_1) + w_2(q_1, q_2) - g(a_1) - g(a_2))|a_1, a_2(q_1)] \geq u(0) \\
&(IC_1) \quad (a_1, a_2(q_1)) \in \text{argmax } E[u(w_1(q_1) + w_2(q_1, q_2) - g(a_1) - g(a_2))|a_1, a_2(q_1)]
\end{aligned}
$$

Denote the optimal long term contract by $(w_1^*, w_2^*, a_1^*, a_2^*)$. As in FHM, we claim this is sequentially efficient. That is, suppose we are at the start of period 2 and $q_1$ has been revealed. Then let the principal make another contract $(\hat{w}_2(q_2), \hat{a}_2)$ (which implicitly depends on $q_1$). The contract is sequentially efficient if the original contract $(w_2^*(q_1, q_2), a_2^*(q_1))$ achieves the optimum. That is, $(w_2^*(q_1, q_2), a_2^*(q_1))$ is sequentially optimal if it solves

$$
\begin{aligned}
&\max_{\hat{w}_2(q_2), \hat{a}_2} E[q_2 - \hat{w}_2(q_2)|\hat{a}_2] \\
&(IR_2) \quad E[u(w_1^*(q_1) + \hat{w}_2(q_2) - g(a_1^*) - g(\hat{a}_2))|q_1, \hat{a}_2] \geq E[u(w_1^*(q_1) + w_2^*(q_1, q_2) - g(a_1^*) - g(a_2^*))|q_1, a_2^*(q_1)] \\
&(IC_2) \quad \hat{a}_2 \in \text{argmax } E[u(w_1^*(q_1) + \hat{w}_2(q_2) - g(a_1^*) - g(\hat{a}_2))|q_1, \hat{a}_2]
\end{aligned}
$$

**Claim 1.** *With CARA utility the long term contract $(w_1^*, w_2^*, a_1^*, a_2^*)$ is sequentially efficient.*

*Proof.* By contradiction, suppose that after some $q_1'$, $(w_2^*(q_1, q_2), a_2^*(q_2)) \neq (\hat{w}_2(q_2), \hat{a}_2)$. If this is the case we plan to show that the long term contract cannot be optimal. We make two observations. First, by construction, $(IR_2)$ binds under the long term contract. Second, with CARA utility, $(IR_2)$ must bind under $(\hat{w}_2(q_2), \hat{a}_2)$. If $(IR_2)$ does not bind then the principal can always lower the wage to $\hat{\hat{w}}_2(q_2)$ such that $u(\hat{\hat{w}}_2(q_2)) = (1 - \epsilon)u(\hat{w}_2(q_2))$ which will leave the agent's action unchanged.[1]

[1] This is where FHM's decreasing utility frontier comes in. If the utility frontier is increasing then the principal can only become better off by making the agent better off. For a formal proof that condition this holds under CARA utility see their Theorem 4.

Now construct a new contract $(w_1^{**}(q_1), w_2^{**}(q_1,q_2), a_1^{**}, a_2^{**}(q_1))$ such that

$$\begin{aligned}(w_1^{**}(q_1), w_2^{**}(q_1,q_2), a_1^{**}, a_2^{**}(q_1)) &= (w_1^{*}(q_1), w_2^{*}(q_1,q_2), a_2^{*}, a_2^{*}(q_1)) \quad \text{if } q_1 \neq q_1' \\ (w_1^{**}(q_1), w_2^{**}(q_1,q_2), a_1^{**}, a_2^{**}(q_1)) &= (w_1^{*}(q_1), \hat{w}_2(q_2), a_1^{*}, \hat{a}_2) \quad \text{if } q_1 = q_1'\end{aligned}$$

By construction this new long–term contract raises more profit for the principal than the original long–term contract. This contract gives the agent at least as much utility after state $q_1'$, so $(IR_1)$ is satisfied. Finally, we must check that $(a_1^{**}, a_2^{**}(q_1))$ are satisfy $(IC_1)$.

Notice that the $(IR_2)$ constraint binds under the new long–term contract. That is, it yields the agent the same utility as the original contract for every $q_1$. This means the agent will have no incentive to change their period 1 action from that in the original contract. Second, the agent's choice of $a_2$ is incentive compatible by construction. □

The intuition is the same as the optimality principal of dynamic programming: if choice A is beaten by choice B after some state $s$ then we can improve choice A by replacing it with choice B in the sub–tree following $s$ and keeping it unchanged elsewhere.

Now let us apply this result to Holmstrom and Milgrom (1987). Assume that output is binomial $q \in \{0,1\}$ and, for simplicity, that $a \in \{L,H\}$, where the principal wishes to implement $a = H$. Let $p_a$ be the probability of success if action $a$ is taken. A long–term contract consists of four wages $(w_{11}, w_{10}, w_{01}, w_{00})$.

Suppose $q_1 = 1$. Under CARA utility, $(IC_2)$ becomes

$$p_H u(w_{11} - g(H)) + (1-p_H)u(w_{10} - g(H)) \geq p_L u(w_{11} - g(L)) + (1-p_L)u(w_{10} - g(L)) \quad (1)$$

Using sequential efficiency, $(IC_2)$ binds. Similarly, if $q_1 = 0$, $(IC_2)$ becomes

$$p_H u(w_{01} - g(H)) + (1-p_H)u(w_{00} - g(H)) \geq p_L u(w_{01} - g(L)) + (1-p_L)u(w_{00} - g(L)) \quad (2)$$

Thus $w_{11} - w_{10} = w_{01} - w_{00}$. This means the second period increment is independent of the first period output.

Moving to the first period, the (IC) constraint is

$$
\begin{aligned}
&p_H^2 u(w_{11} - g(H)) + p_H(1-p_H)u(w_{10} - g(H)) + (1-p_H)p_H u(w_{01} - g(H)) + (1-p_H)^2 u(w_{00} - g(H)) \\
&\geq p_L p_H u(w_{11} - g(L)) + p_L(1-p_H)u(w_{10} - g(L)) + (1-p_L)p_H u(w_{01} - g(L)) + (1-p_L)(1-p_H)u(w_{00} - g(L))
\end{aligned}
$$

Using the incentive compatibility constraints from the second period (1) and (2), this becomes

$$(p_H - p_L)u(w_{10} - g(L)) \geq (p_H - p_L)u(w_{01} - g(L))$$

Sequential efficiency implies that this will bind and thus $w_{10} = w_{01}$. We thus see that we can write the wage as

$$w(q_1 + q_2) = \alpha + \beta(q_1 + q_2)$$

That is, the wage only depends upon the total output.

A similar backwards induction argument is given by Segal and Tadelis. There is, however, a slight difference. I directly use the sequential efficiency of the optimal policy, while Segal and Tadelis use the fact that (because of sequential efficiency) the long term contract can be implemented by a series of short term contracts of the form $w_t(q_1, \ldots, q_t)$.