Economics 2102: Homework 2

3 November, 2004

Question 1 (Hidden savings and CARA utility)

There are two periods. In period 1 the agent (privately) chooses to consume c. In period 2 they choose effect $a \in \{L, H\}$ at monetary cost $\{0, g\}$ respectively. Output is binomial, $q \in \{0, 1\}$, where the probability that $q = 1$ given action $a \in \{L, H\}$ is p_a and $p_H \geq p_L$. The principal chooses wages (w_1, w_0).

The two–period (IC) constraint says that

$$u(c_H) + p_H u(w_1 - c_H - g(H)) + (1 - p_H)u(w_0 - c_H - g(H)) \qquad (1)$$
$$\geq u(c_L) + p_L u(w_1 - c_L - g(L)) + (1 - p_L)u(w_0 - c_L - g(L))$$

where c_a is the optimal consumption when the agent plans to choose a.

Show that under CARA utility, $u(c) = -\exp(-rc)$, we have $c_H = c_L$ when the (IC) constraint binds. Why is this important?

Question 2 (Normal learning model)

Suppose that $z_t = \theta + \epsilon_t$, where $\theta \sim N(m_0, 1/h_0)$ and $\epsilon_t \sim N(0, 1/h_\epsilon)$ are IID. Show that

$$E[\theta|z_1] = \frac{h_0 m_0 + h_\epsilon z_1}{h_0 + h_\epsilon}$$

and that

$$E[\theta|z_1, \ldots, z_t] = \frac{h_0 m_0 + h_\epsilon \sum_{s \leq t} z_s}{h_0 + t h_\epsilon}$$

Question 3 (Short–term and long–term contracts)

Suppose there are three periods, $t \in \{1, 2, 3\}$. Each period a principal and an agent must share a good; let $x_t \in \mathbb{R}$ be the share obtained by the agent. The principal gets $\sum_t \pi_t(x_t)$ and the agent gets $\sum_t u_t(x_t)$, where $\pi_t(x_t)$ is decreasing in x_t and $u_t(x_t)$ is increasing in x_t. The agents's outside option is a share of the assets $(\underline{x}_1, \underline{x}_2, \underline{x}_3)$.

(a) Suppose the principal can write a long term contract. Write down the program of maximising profit subject to individual rationality.

(b) Now suppose the principal offered a spot contract each period. Using backwards induction derive the optimal sequence of spot contracts. Explain why this may differ from the long–term contract.

(c) Suppose the principal offers two–period contracts. In the first period they offer $({}_1x_1,\, {}_1x_2)$. If it is rejected the agent gets $\underline{x}_1$. At the start of the second period the a new contract $({}_2x_2,\, {}_2x_3)$ may be proposed by the principal. If this is rejected the agent gets ${}_1x_2$ if they accepted the first contract or $\underline{x}_2$ otherwise. In the third period the a spot contract is offered to the agent. If this is rejected the agent gets ${}_2x_3$ if they accepted the second contract or $\underline{x}_3$ otherwise. Show that if $\lim_{x\to-\infty} u_t(x) = -\infty$ and $\lim_{x\to\infty} u_t(x) = \infty$ then this can implement the optimal long term contract.

(d) Provide an example where the two–period contracts cannot implement the long–term contract.

Question 4 (Teamwork)

A firm employs two workers $i \in \{1, 2\}$. The agents simultaneously choose actions $a_i\{L, H\}$ at cost $\{0, c\}$ respectively. Their actions induce verifiable signals $x_i \in \{0, 1\}$. With probability σ there is a common shock and $x_i = 1$. With probability $1 - \sigma$, $x_i = 1$ with probability p_a, where $p_H \geq p_L$. Assume that the principal wishes to induce high effort.

The contract consists of four wages $(w_{11}, w_{10}, w_{01}, w_{00})$, where w_{10} is $i's$ wage if i succeeds and j fails. Agents are risk neutral but have limited liability, so that the wage must be nonnegative. If i picks action k and j picks l then i's utility is

$$u(k,l) = [\sigma+(1-\sigma)p_k p_l]w_{11}+(1-\sigma)p_k(1-p_l)w_{10}+(1-\sigma)(1-p_k)p_l w_{01}+(1-\sigma)(1-p_k)(1-p_l)w_{00}$$

(a) The principal minimises expected wages subject incentive compatibility. Write down this program. Derive the optimal contract $(w_{11}, w_{10}, w_{01}, w_{00})$. In particular, show that it exhibits an extreme form of relative performance evaluation (where i doing well hurts j).

Now consider the repeated version of the same game. Each period the principal offers the same contract $(w_{11}, w_{10}, w_{01}, w_{00})$. Everyone has discount factor δ.

Assume agents can observe each others actions (but cannot directly report them to the principal). A necessary condition for (H, H) to be implementable is

$$u(H,H) - c \geq (1-\delta)u(L,H) + \delta \min\{u(L,L), u(L,H)\} \tag{IC}$$

(b) Consider the principal's problem of minimising expected wages subject to (IC). Show the optimal program is of the form $(w_{11}, 0, 0, 0)$ when $\delta \geq \hat{\delta} := \frac{\sigma}{(1-\sigma)p_H p_L}$, where you should assume that $\hat{\delta} < 1$.

[I suggest you have a bash at the problem in your own way first. However, the following steps may prove useful. First, break the problem into two cases depending upon $u(L, L)$ and $u(L, H)$. Observe that one these cases is easy. For the other case consider which wage is positive: there are only four alternatives. Then compare across the two cases.]

(c) Under the optimal contract in (b) show the following trigger strategy is subgame perfect: play H until someone plays L and then play L thereafter.

(d) Compare the contracts in (a) and (b). Intuitively what's going on? Should I curve the final in the light of this result?