

## Economics 2102: Homework 3

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### Question 1 (Nonlinear Pricing with Three Types)

Consider the nonlinear pricing model with three types,  $\theta_3 > \theta_2 > \theta_1$ . The utility of agent  $\theta_i$  is

$$u(\theta_i) = \theta_i q - t$$

Denote the bundle assigned to agent  $\theta_i$  by  $(q_i, t_i)$ . We now have six (IC) constraint and three (IR) constraints. For example,  $(IC_1^2)$  says that  $\theta_1$  must not want to copy  $\theta_2$ , i.e.

$$\theta_1 q_1 - t_1 \geq \theta_1 q_2 - t_2 \tag{IC_1^2}$$

The firm's profit is

$$\sum_{i=1}^3 \pi_i [t_i - c(q_i)]$$

where  $\pi_i$  is the proportion of type  $\theta_i$  agents and  $c(q)$  is increasing and convex.

- (a) Show that  $(IR_2)$  and  $(IR_3)$  can be ignored.
- (b) Show that  $q_3 \geq q_2 \geq q_1$ .
- (c) Using  $(IC_2^1)$  and  $(IC_3^2)$  show that we can ignore  $(IC_3^1)$ . Using  $(IC_2^3)$  and  $(IC_1^2)$  show that we can ignore  $(IC_1^3)$ .
- (d) Show that  $(IR_1)$  will bind.
- (e) Show that  $(IC_2^1)$  will bind.
- (f) Show that  $(IC_3^2)$  will bind.
- (g) Assume that  $q_3 \geq q_2 \geq q_1$ . Show that  $(IC_1^2)$  and  $(IC_2^3)$  can be ignored.

### Question 2 (Downward Sloping Demand I)

Suppose a seller of wine faces two types of customers,  $\theta_1$  and  $\theta_2$ , where  $\theta_2 > \theta_1$ . The proportion of type  $\theta_1$  agents is  $\pi \in [0, 1]$ . Let  $q$  be the quality of the wine and  $t$  the price. Agent  $\theta_i$  has utility

$$u(\theta_i) = \theta_i q - \frac{1}{2} q^2 - t$$

Let type  $\theta_1$  buy contract  $(q_1, t_1)$  and type  $\theta_2$  buy  $(q_2, t_2)$ . The cost of production is zero,  $c(q) = 0$ , and the seller maximises profit

$$\pi t_1 + (1 - \pi)t_2 \tag{1}$$

- (a) Suppose the seller observes the agent's types. Solve for the first best qualities.
- (b) Now suppose the seller cannot observe which agent is which. Write down the seller's optimisation problem subject to the two (IR) and two (IC) constraints.
- (c) Derive the profit-maximising qualities.

### Question 3 (Downward Sloping Demand II)

Suppose a seller of wine faces two types of customers,  $\theta_1$  and  $\theta_2$ , where  $\theta_2 > \theta_1$ . The proportion of type  $\theta_1$  agents is  $\pi \in [0, 1]$ . Let  $q$  be the quality of the wine and  $t$  the price. Agent  $\theta_i$  has utility

$$u(\theta_i) = \theta_i(q - \frac{1}{2}q^2) - t$$

Let type  $\theta_1$  buy contract  $(q_1, t_1)$  and type  $\theta_2$  buy  $(q_2, t_2)$ . The cost of production is zero,  $c(q) = 0$ , and the seller maximises profit

$$\pi t_1 + (1 - \pi)t_2 \tag{2}$$

- (a) Suppose the seller observes the agent's types. Solve for the first best qualities and prices.
- (b) Now suppose the seller cannot observe which agent is which. Write down the seller's optimisation problem subject to the two (IR) and two (IC) constraints.
- (c) Derive the profit-maximising qualities.

### Question 4 (Optimal Taxation)

There are two types of agents,  $\theta_H > \theta_L$ . Proportion  $\beta$  have productivity  $\theta_L$ . The utility of an agent who takes produces quantity  $q$  with effort  $e$  is

$$u(q - t - g(e))$$

where  $t$  is the net tax,  $g(e)$  is increasing and strictly convex, and  $u(\cdot)$  is strictly concave. Assume output is contractible so that a mechanism consists of a pair  $(q(\theta), t(\theta))$ . The government's problem is to maximise

$$\beta u\left(q_L - t_L - g\left(\frac{q_L}{\theta_L}\right)\right) + (1 - \beta)u\left(q_H - t_H - g\left(\frac{q_H}{\theta_H}\right)\right)$$

subject to budget balance (BB),  $\beta t_L + (1 - \beta)t_H \geq 0$ . Notice that there are no (IR) constraints here.

(a) As a benchmark, solve the government's problem if they knew the agent's types. Which type puts in the most effort?

Now suppose the government cannot observe agent's types. The incentive constraint for type  $L$ , for example, is

$$u\left(q_L - t_L - g\left(\frac{q_L}{\theta_L}\right)\right) \geq u\left(q_H - t_H - g\left(\frac{q_H}{\theta_L}\right)\right)$$

(b) Show that at the optimum (BB) binds.

(c) Show that at the optimum  $u'_L \geq u'_H$ , where  $u'_i$  is the marginal utility of type  $i$ .

(d) Show that  $(IC_H)$  binds.

(e) Using the fact that  $(IC_H)$  binds, show that  $(IC_L)$  can be ignored.

(f) Consider the government's problem of maximising welfare subject to (BB) and  $(IC_H)$ . Show the optimal contract satisfies:

$$1 - \frac{1}{\theta_H}g'\left(\frac{q_H}{\theta_H}\right) = 0 \tag{3}$$

$$1 - \frac{1}{\theta_L}g'\left(\frac{q_L}{\theta_L}\right) = \frac{u'_L - u'_H}{u'_L}(1 - \beta)\left(1 - \frac{1}{\theta_H}g'\left(\frac{q_L}{\theta_H}\right)\right) \tag{4}$$

(g) Show that (4) implies

$$1 - \frac{1}{\theta_L}g'\left(\frac{q_L}{\theta_L}\right) \geq 0 \tag{5}$$

(h) What does (5) imply about the level of work performed by the low type. Provide an intuition for this distortion.

### Question 5 (Costly State Verification)

There is a risk-neutral entrepreneur  $E$  who has a project with privately observed return  $y$  with density  $f(y)$  on  $[0, Y]$ . The project requires investment  $I < E[y]$  from an outside creditor  $C$ .

A contract is defined by a pair  $(s(y), B(y))$  consisting of payment and verification decision. If an agent reports  $y$  they pay  $s(y) \leq y$  and are verified if  $B(y) = 1$  and not verified if  $B(y) = 0$ . If the creditor verifies  $E$  they pay cost  $c(y)$  and get to observe  $E$ 's type.

The game is as follows:

- $E$  chooses  $(s(y), B(y))$  to raise  $I$  from a competitive financial market.
- $y$  is realised.
- $E$  claims project yields  $\hat{y}$ . If  $B(\hat{y}) = 0$  then  $E$  pays  $s(\hat{y})$  and is not verified. If  $B(\hat{y}) = 1$  then  $C$  pays  $c(y)$  and observes  $E$ 's true type. If they are telling the truth they pay  $s(y)$ ; if not, then  $C$  can take everything.
- Payoffs.  $E$  gets  $y - s(y)$ , while  $C$  gets  $s(y) - c(y)B(y) - I$ .

(a) Show that for any incentive compatible mechanism there exists a  $D$  such that  $s(y) = D$  when  $B(y) = 0$  and  $s(y) \leq D$  when  $B(y) = 1$ .

Consider  $E$ 's problem:

$$\begin{aligned} \max_{s(y), B(y)} & E[y - s(y)] \\ \text{s.t.} & s(y) \leq y \quad (MAX) \\ & E[s(y) - c(y)b(y) - I] \geq 0 \quad (IR) \\ & s(y) \leq D \quad \forall y \in B^V \quad (IC1) \\ & s(y) = D \quad \forall y \notin B^V \quad (IC2) \end{aligned}$$

where  $B^V$  is the verification region (where  $B(y) = 1$ ).

(b) Show that constraint (IR) must bind at the optimum.

Now  $E$ 's problem becomes

$$\begin{aligned} \min_{s(y), B(y)} & E[c(y)B(y)] \\ \text{s.t.} & (MAX), (IC1), (IC2) \\ & E[s(y) - c(y)B(y) - I] = 0 \quad (IR) \end{aligned}$$

- (c) Fix  $D$ . Show that any contract  $(s(y), B(y))$  is dominated by one where  $B^V = [0, D]$ .
- (d) Pick a contract  $(s(y), B(y))$  such that  $B^V = [0, D]$ . Show that it is dominated by a contract where  $s(y) = y$  when  $B(y) = 1$ .
- (e) A contract is thus characterised by  $D$ . Which  $D$  maximises  $E$ 's utility? Can you give a financial interpretation to this contract?