

Economics 2102: Homework 3

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Question 1 (Nonlinear Pricing with Three Types)

Consider the nonlinear pricing model with three types, $\theta_3 > \theta_2 > \theta_1$. The utility of agent θ_i is

$$u(\theta_i) = \theta_i q - t$$

Denote the bundle assigned to agent θ_i by (q_i, t_i) . We now have six (IC) constraint and three (IR) constraints. For example, (IC_1^2) says that θ_1 must not want to copy θ_2 , i.e.

$$\theta_1 q_1 - t_1 \geq \theta_1 q_2 - t_2 \tag{IC_1^2}$$

The firm's profit is

$$\sum_{i=1}^3 \pi_i [t_i - c(q_i)]$$

where π_i is the proportion of type θ_i agents and $c(q)$ is increasing and convex.

- (a) Show that (IR_2) and (IR_3) can be ignored.
- (b) Show that $q_3 \geq q_2 \geq q_1$.
- (c) Using (IC_2^1) and (IC_3^2) show that we can ignore (IC_3^1) . Using (IC_2^3) and (IC_1^2) show that we can ignore (IC_1^3) .
- (d) Show that (IR_1) will bind.
- (e) Show that (IC_2^1) will bind.
- (f) Show that (IC_3^2) will bind.
- (g) Assume that $q_3 \geq q_2 \geq q_1$. Show that (IC_1^2) and (IC_2^3) can be ignored.

Question 2 (Downward Sloping Demand I)

Suppose a seller of wine faces two types of customers, θ_1 and θ_2 , where $\theta_2 > \theta_1$. The proportion of type θ_1 agents is $\pi \in [0, 1]$. Let q be the quality of the wine and t the price. Agent θ_i has utility

$$u(\theta_i) = \theta_i q - \frac{1}{2} q^2 - t$$

Let type θ_1 buy contract (q_1, t_1) and type θ_2 buy (q_2, t_2) . The cost of production is zero, $c(q) = 0$, and the seller maximises profit

$$\pi t_1 + (1 - \pi)t_2 \tag{1}$$

- (a) Suppose the seller observes the agent's types. Solve for the first best qualities.
- (b) Now suppose the seller cannot observe which agent is which. Write down the seller's optimisation problem subject to the two (IR) and two (IC) constraints.
- (c) Derive the profit-maximising qualities.

Question 3 (Downward Sloping Demand II)

Suppose a seller of wine faces two types of customers, θ_1 and θ_2 , where $\theta_2 > \theta_1$. The proportion of type θ_1 agents is $\pi \in [0, 1]$. Let q be the quality of the wine and t the price. Agent θ_i has utility

$$u(\theta_i) = \theta_i(q - \frac{1}{2}q^2) - t$$

Let type θ_1 buy contract (q_1, t_1) and type θ_2 buy (q_2, t_2) . The cost of production is zero, $c(q) = 0$, and the seller maximises profit

$$\pi t_1 + (1 - \pi)t_2 \tag{2}$$

- (a) Suppose the seller observes the agent's types. Solve for the first best qualities and prices.
- (b) Now suppose the seller cannot observe which agent is which. Write down the seller's optimisation problem subject to the two (IR) and two (IC) constraints.
- (c) Derive the profit-maximising qualities.

Question 4 (Optimal Taxation)

There are two types of agents, $\theta_H > \theta_L$. Proportion β have productivity θ_L . An agent of type θ who exerts effort e produces output $q = \theta e$. The utility of an agent who produces quantity q with effort e is then

$$u(q - t - g(e))$$

where t is the net tax. Assume $g(e)$ is increasing and strictly convex, and $u(\cdot)$ is strictly concave.

Suppose that output is contractible so that a mechanism consists of a pair $(q(\theta), t(\theta))$. The government's problem is to maximise

$$\beta u \left(q_L - t_L - g \left(\frac{q_L}{\theta_L} \right) \right) + (1 - \beta) u \left(q_H - t_H - g \left(\frac{q_H}{\theta_H} \right) \right)$$

subject to budget balance (BB), $\beta t_L + (1 - \beta) t_H \geq 0$. Notice that there are no (IR) constraints here.

(a) First, suppose the government can observe agents' types. Solve for the first-best contract. Which type puts in the most effort?

Now suppose the government cannot observe agent's types. The incentive constraint for type L , for example, is

$$u \left(q_L - t_L - g \left(\frac{q_L}{\theta_L} \right) \right) \geq u \left(q_H - t_H - g \left(\frac{q_H}{\theta_L} \right) \right)$$

(b) Show that at the optimum (BB) binds.

(c) Show that at the optimum $u'_L \geq u'_H$, where u'_i is the marginal utility of type i .

(d) Show that at the optimum (IC_H) binds.

(e) Consider the government's relaxed problem of maximising welfare subject to (BB) and (IC_H), ignoring (IC_L). Show the optimal contract satisfies:

$$1 - \frac{1}{\theta_H} g' \left(\frac{q_H}{\theta_H} \right) = 0 \tag{3}$$

$$1 - \frac{1}{\theta_L} g' \left(\frac{q_L}{\theta_L} \right) = \frac{u'_L - u'_H}{u'_L} (1 - \beta) \left(1 - \frac{1}{\theta_H} g' \left(\frac{q_L}{\theta_H} \right) \right) \tag{4}$$

(f) Show that (4) implies

$$1 - \frac{1}{\theta_L} g' \left(\frac{q_L}{\theta_L} \right) \geq 0 \tag{5}$$

(g) Using equations (3) and (5) show that $q_H \geq q_L$. Use this and the fact that (IC_H) binds, to show that (IC_L) holds.

(h) What does (5) imply about the level of work performed by the low type. Provide an intuition for this distortion.

Question 5 (Costly State Verification)

There is a risk-neutral entrepreneur E who has a project with privately observed return y with density $f(y)$ on $[0, Y]$. The project requires investment $I < E[y]$ from an outside creditor C .

A contract is defined by a pair $(s(y), B(y))$ consisting of payment and verification decision. If an agent reports y they pay $s(y) \leq y$ and are verified if $B(y) = 1$ and not verified if $B(y) = 0$. If the creditor verifies E they pay cost $c(y)$ and get to observe E 's type.

The game is as follows:

- E chooses $(s(y), B(y))$ to raise I from a competitive financial market.
- y is realised.
- E claims project yields \hat{y} . If $B(\hat{y}) = 0$ then E pays $s(\hat{y})$ and is not verified. If $B(\hat{y}) = 1$ then C pays $c(y)$ and observes E 's true type. If they are telling the truth they pay $s(y)$; if not, then C can take everything.
- Payoffs. E gets $y - s(y)$, while C gets $s(y) - c(y)B(y) - I$.

(a) Show that a contract is incentive compatible if and only if exists a D such that $s(y) = D$ when $B(y) = 0$ and $s(y) \leq D$ when $B(y) = 1$.

Consider E 's problem:

$$\begin{aligned} \max_{s(y), B(y)} & E[y - s(y)] \\ \text{s.t.} & s(y) \leq y \quad (MAX) \\ & E[s(y) - c(y)b(y) - I] \geq 0 \quad (IR) \\ & s(y) \leq D \quad \forall y \in B^V \quad (IC1) \\ & s(y) = D \quad \forall y \notin B^V \quad (IC2) \end{aligned}$$

where B^V is the verification region (where $B(y) = 1$).

(b) Show that constraint (IR) must bind at the optimum.

Now E 's problem becomes

$$\begin{aligned} \min_{s(y), B(y)} & E[c(y)B(y)] \\ \text{s.t.} & (MAX), (IC1), (IC2) \\ & E[s(y) - c(y)B(y) - I] = 0 \quad (IR) \end{aligned}$$

- (c) Fix D . Show that any contract $(s(y), B(y))$ is dominated by one where $B^V = [0, D]$.
- (d) Pick a contract $(s(y), B(y))$ such that $B^V = [0, D]$. Show that it is dominated by a contract where $s(y) = y$ when $B(y) = 1$.
- (e) A contract is thus characterised by D . Which D maximises E 's utility? Can you give a financial interpretation to this contract?