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A GENERALIZATION OF
THE PURE THEORY OF PUBLIC GOODS

by

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Introduction

The concept of a pure public good formalized by Samuelson [11] has frequently been criticized for failing to admit the possibility of exclusion or crowding. In most public consumption -- viewing a movie, belonging to a neighborhood, sending children to school or sharing a dwelling -- one shares consumption with some but not all other members of society. Adding another person to a sharing group while maintaining the quality of the service consumed will in general require additional resources. Most goods are public, but few are purely public.

This paper extends the concept of a public good to cover cases in which exclusion or crowding occurs. Like Samuelson [11], I assume that a public good is equally available to every member of a group to which it is provided. However, rather than assuming that everyone shares the public good, I permit the partition of the members of the economy into distinct sharing groups or "jurisdictions". The resources required to provide a public good to the members of a jurisdiction may depend on the size of the jurisdiction. Thus, both exclusion and crowding of public goods are admitted into the analysis.

The theory developed here not only permits the analysis of a wide class of public goods but sheds new light on certain results which have been obtained for the case of pure public goods. In particular, when crowding and exclusion are present, the Lindahl equilibrium need not be in the core and the core itself may be empty. These results stem directly from the fact that, within the context of this generalized model of public goods, the aggregate technology set may not be convex.

In contrast to the usual conclusion in economics, the presence of non-convexity does not prevent further analysis of the allocative process in such an economy. By admitting non-convexity, the theory is able to accommodate "neighborhood" and "local political jurisdiction" as legitimate analytical constructs. For a fixed assignment of individuals to jurisdictions, it is possible to prove the existence and relative Pareto optimality of a Lindahl equilibrium or of what Foley [3] has called a public competitive equilibrium. However, because of the presence of non-convexity, prices can guide the assignment of individuals to jurisdictions only under highly specialized circumstances.

Section 1 reviews the theory of pure public goods in terms of a three-person economy. Section 2 introduces crowding of the public good and the possibility of exclusion into this economy and exhibits cases where the Lindahl allocation lies outside of the core and where the core is empty respectively. Section 3 explores the nature of the non-convexity introduced by crowding and exclusion while Section 4 interprets this non-convexity as a type of indivisibility. After a brief discussion in Section 5 of the type of exclusion assumed in this analysis, Section 6 turns to the general question of the relationship between competitive analysis and the organization of collective consumption -- the pattern of sharing of public goods. Section 7 relates the theory of public goods developed in this paper to Buchanan's theory of clubs [1]. Section 8 illustrates the general failure of prices in guiding the assignment of individuals to jurisdictions and explores one special case in which prices can perform such a function.

1. Pure Public Goods

Consider a three-person economy with two commodities, one private and one purely public. The tastes of the i th individual are represented by

$$u_i = x_i g \quad (i = 1, 2, 3) \quad (1)$$

where x_i is the amount of the private good and g the amount of the public good consumed. The i th individual has an initial endowment, w_i , of the private good; no public goods are initially owned; and public goods cannot be used as inputs to production. The economy faces the resource constraint

$$cg + \sum_{i=1}^3 x_i = \sum_{i=1}^3 w_i \quad (2)$$

where c is a positive constant. The Pareto optimal level of the public good is then¹

$$g = \frac{w_1 + w_2 + w_3}{2c} \quad (3)$$

In their search for an analogue to competitive equilibrium in an economy with public goods, economists have given considerable attention to Lindahl equilibria [4,5,7]. The distinguishing feature of a Lindahl equilibrium is that taxes are levied in such a manner that each individual's "demand" for the public good will be brought into exact equality with the amount supplied. In our three-person economy, the level of public good supplied in the Lindahl equilibrium is given by equation (3). Setting the price of the private good arbitrarily equal to unity, the budget line of the ith individual will be tangent to his indifference curve at the Lindahl allocation provided that he is taxed an amount²

$$t_i = \frac{w_i}{2} \quad (i = 1, 2, 3) \quad (4)$$

Foley [4] has proved under relatively general conditions, satisfied by our three-person economy, that the Lindahl equilibrium exists, that it is Pareto optimal and that it belongs to the core. Foley puts great stress on the third result: no coalition can, using only its own resources, achieve an allocation which makes each of its members better off than in the Lindahl equilibrium. Samuelson [13] has voiced considerable skepticism concerning the relevance of Lindahl equilibria to the political-economic decision-making process. But as Foley observes: "There is some reason to think that the core is a meaningful political concept."³

2. Crowded Public Goods, Lindahl Equilibria and the Core

The case for the relevance of Lindahl equilibria would be stronger if one could prove some analogue of the Scarf-Debreu [15] theorem on the core of a competitive economy. The Lindahl equilibrium would, in the limit, be the only politically relevant allocation. However, a theorem of this type does not seem possible when pure public goods are present. If the resource cost of producing a given level of the public good is independent of the number of individuals who consume it, then blocking is very difficult. In looking for ways to enhance the blocking ability of a coalition, it is natural to consider the possibility that the coalition can, by excluding nonmembers, obtain a given level of the public good at lower cost. When the addition of another consumer increases the resources required to maintain the level of public good consumed by all, we will refer to the public good as "crowded".

We will now introduce crowding into our three-person economy. If all three individuals consume the public good jointly, then we assume, as before, that they face the resource constraint given by equation (2). However, an individual who decides to produce the good g using only his own resources faces the constraint

$$ag_i + x_i = w_i \quad (i = 1, 2, 3) \quad (5)$$

where a is a positive constant.⁴ Any pair of individuals i and j ($i \neq j$) face the constraint

$$bg_{ij} + x_i + x_j = w_i + w_j \quad (6)$$

provided that they share consumption of the public good.⁵

To be in the core, a vector of utilities (u_1, u_2, u_3) must be individually rational; i.e., every individual must obtain at least as much utility as he could using only his own resources. In our three-person economy, (u_1, u_2, u_3) is individually rational if and only if⁶

$$u_1 \geq \frac{(w_1)^2}{4a} \equiv v(\bar{1}); \quad u_2 \geq \frac{(w_2)^2}{4a} \equiv v(\bar{2}); \quad u_3 \geq \frac{(w_3)^2}{4a} \equiv v(\bar{3}) \quad (7)$$

Although a pair of individuals forming a blocking coalition may choose to consume the good g privately rather than sharing it, we will, in order to simplify the exposition, assume that g is a pure public good for coalitions of size less than or equal to two; i.e., $a = b$. In that case, the set of utilities that a pair of individuals can obtain through private consumption of the "public" good is a proper subset of the set of utilities obtainable through joint consumption. Under these circumstances, equation (6) is the appropriate resource constraint for a two-person blocking coalition.

If (u_1, u_2, u_3) is to be in the core, then it must not be possible for any two-person coalition, using only its own resources, to obtain higher utility levels for both of its members. Therefore, (u_1, u_2, u_3) is in the core only if⁷

$$\begin{aligned} u_1 + u_2 &\geq \frac{(w_1 + w_2)^2}{4b} \equiv v(\bar{12}) \\ u_1 + u_3 &\geq \frac{(w_1 + w_3)^2}{4b} \equiv v(\bar{13}) \\ u_2 + u_3 &\geq \frac{(w_2 + w_3)^2}{4b} \equiv v(\bar{23}) \end{aligned} \quad (8)$$

Returning to the case where all three individuals consume the public good jointly,⁸ Pareto optimal utility vectors satisfy the equation

$$u_1 + u_2 + u_3 = \frac{(w_1 + w_2 + w_3)^2}{4c} = v(\overline{123}) \quad (9)$$

The inequalities (7) and (8) and equation (9) describe a type of game that is particularly easy to analyze: a game with transferable utility.⁹ For any coalition S , the number $v(S)$ represents the minimum utility the coalition must receive if it is not to block. $v(S)$, defined over all possible coalitions, is called the characteristic function of the game.¹⁰

This game will have a non-empty core if and only if¹¹

$$v(\overline{12}) + v(\overline{13}) + v(\overline{23}) \leq 2 v(\overline{123}) \quad (10)$$

If (10) is satisfied, the game is said to be balanced.¹² Substituting (8) and (9) into (10) we obtain

$$\frac{c}{b} \leq \frac{k_1 + 2k_2}{k_1 + k_2} \quad (11)$$

as the necessary and sufficient condition for a non-empty core where $k_1 = (w_1)^2 + (w_2)^2 + (w_3)^2$ and $k_2 = w_1w_2 + w_1w_3 + w_2w_3$.

Having completed these preliminaries, we can now demonstrate the two propositions mentioned in the introduction: the Lindahl equilibrium is not necessarily in the core, and the core may be empty.

To establish the first proposition, let $a = b = 1$, $c = 4/3$, $w_1 = w_2 = 2$ and $w_3 = 1$. Substituting these values into (7), (8) and (9)

we obtain the following game:

$$\begin{aligned} v(\bar{1}) &= 1; v(\bar{2}) = 1; v(\bar{3}) = 1/4 \\ v(\bar{12}) &= 4; v(\bar{13}) = 9/4; v(\bar{23}) = 9/4 \\ v(\bar{123}) &= 75/16 \end{aligned} \quad (12)$$

From (11), the game will be balanced if $c \leq \frac{25}{17} \approx 1.47$; since $c = 4/3$, the game has a non-empty core. A utility vector in the core must satisfy the inequality $u_1 + u_2 \geq v(\bar{12}) = 4$, so the third individual can receive at most $v(\bar{123}) - 4 = 11/16$; individual rationality requires that he receive at least $v(\bar{3}) = 1/4$. Thus, for all utility vectors in the core, $1/4 \leq u_3 \leq 11/16$. $(2, 2, 11/16)$ and $(2, 2, 7/16, 1/4)$ are, for example, two utility vectors that are in the core.

We have already observed that, in the Lindahl equilibrium,¹³
 $g = (w_1 + w_2 + w_3)/2c$ and $x_i = w_i/2$; consequently,

$$\tilde{u}_i = \frac{w_i(w_1 + w_2 + w_3)}{4c} \quad (i = 1, 2, 3) \quad (13)$$

For the particular parameter values we have selected, the Lindahl utility vector is $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = (15/8, 15/8, 15/16)$. Since $\tilde{u}_1 + \tilde{u}_2 + \tilde{u}_3 = 75/16$, the Lindahl equilibrium is a Pareto optimum. But individuals 1 and 2, who can obtain $(u_1, u_2) = (2, 2)$ using only their own resources, will block this Lindahl allocation. The Lindahl equilibrium fails to give sufficient recognition to the "power" of individuals 1 and 2 -- they receive too little and individual 3 receives too much -- and, hence, the Lindahl allocation is not in the core. However, individual 3 is willing to allow individuals 1

and 2 to receive more than their "Lindahl share" if they share consumption of the public good with him: there are allocations involving joint consumption by all three individuals that are in the core.¹⁴ We summarize this result in the following proposition:

Proposition 1: If public goods are crowded, then the Lindahl equilibrium may not be in the core.

The second proposition that we wish to establish is that the core can be empty if public goods are crowded.¹⁵ Letting $a = b = 1$ and $w_1 = w_2 = w_3$ the balance condition (11) states that the core will be empty if $c > 3/2$. If $c = 3/2$ is interpreted as "constant returns to scale" in moving from two to three-person sharing groups, then we find that the core is empty when there are diminishing returns to scale. One should not conclude, however, that the core will necessarily be non-empty whenever returns to scale are constant or increasing. For example, when $w_1 = w_2 = 4$, $w_3 = 3$, $a = b = 1$ and $c = 3/2$, the core is empty.¹⁶ In summary, we have demonstrated

Proposition 2: If public goods are crowded, the core may be empty.

3. Non-Convexity

To what can these rather surprising results be attributed? They stem from an assumption, implicit in the preceding section, that the aggregate technology set is non-convex. Making this assumption explicit will be greatly facilitated if we formalize our approach to crowded public goods along the lines of Foley [4].

We assume that the economy has one public good, m private goods and n consumers. An allocation is a vector $(g_1, \dots, g_n; x_1, \dots, x_n)$, with each x_i itself a vector in E^m , such that for all i there is a $(g_i; \bar{x}_i)$ with

$\bar{x}_i < x_i$ belonging to the consumption set of the i th consumer.¹⁷ An allocation is feasible if $(g_1, \dots, g_n; z) \in Y$ where Y is the aggregate technology set, w_i is the i th individual's initial endowment of private goods,¹⁸ and $z = \sum_{i=1}^n (x_i - w_i)$ is the aggregate input of private goods into public good production. Of the assumptions which Foley makes about the aggregate technology set, the one which we will find inappropriate in an economy with crowded public goods is that Y is a convex cone.¹⁹

Definition: A Lindahl equilibrium with respect to $w = (w_1, \dots, w_n)$ is a feasible allocation $(g_1, \dots, g_n; x_1, \dots, x_n)$ and a price system $(p_g^1, \dots, p_g^n; p_x)$ ≥ 0 such that

$$(a) \quad \sum_{i=1}^n p_g^i g_i + p_x z \geq \sum_{i=1}^n p_g^i \bar{g}_i + p_x \bar{z} \quad \text{for all } (\bar{g}_1, \dots, \bar{g}_n; \bar{z}) \in Y;$$

$$(b) \quad \text{if } (\bar{g}_i; \bar{x}_i) >_i (g_i; x_i), \text{ then } p_g^i \bar{g}_i + p_x \bar{x}_i > p_g^i g_i + p_x x_i = p_x w_i.$$

Observe that there is a separate public good price, p_g^i , for each individual. In the Lindahl equilibrium producers maximize profits and consumers maximize utility taking prices as given. Foley's definition of a Lindahl equilibrium is restricted to the special case of a pure public good where $g_1 = \dots = g_n$. However, the concept of a Lindahl equilibrium, as defined above, can be applied to any pattern of sharing.

At this point, it is useful to formalize the notion of alternative patterns of sharing of the public good. Let $N \equiv \{1, \dots, n\}$ denote the index set of consumers in the economy. Any subset of N representing a group of consumers who share consumption of the public good will be called a jurisdiction; the j th jurisdiction will be denoted J_j . Assume that every

consumer belongs to exactly one jurisdiction. A particular assignment of consumers to jurisdictions, $\{J_1, \dots, J_r\} \equiv P_k$, is a partition if $\bigcup_{j=1}^r J_j = N$. The collection of possible partitions will be denoted by $\mathcal{P} = \{P_1, \dots, P_s\}$. We reserve P_1 for the private good partition ($P_1 = \{\{1\}, \dots, \{n\}\}$) and P_s for the pure public good partition ($P_s = \{N\}$). To any partition P_k corresponds an aggregate technology set $Y(P_k)$. In Foley's model of pure public goods, $Y = Y(P_s)$; in the standard model of private goods, $Y = Y(P_1)$.

Consider the three-person economy described in section 2. We assumed that the i th individual using only his own resources to produce the good g faced the resource constraint $ag_i + x_i = w_i$ ($i = 1, 2, 3$). Therefore, the production sets for the three consumers are the halflines²⁰ associated with the vectors

$$\begin{aligned} y_1^1 &= (1, 0, 0; -a) \\ y_2^1 &= (0, 1, 0; -a) \\ y_3^1 &= (0, 0, 1; -a) \end{aligned} \tag{14}$$

respectively where the vectors are written in the form $(g_1, g_2, g_3; z)$ with z representing the input, $x_i - w_i$, of private goods into the production of the public good. We also assumed that a pair of individuals \overline{ij} , using their own resources and sharing consumption of the public good, faced the constraint $bg_{ij} + x_i + x_j = w_i + w_j$. The production sets for the pairs $\overline{12}$, $\overline{13}$ and $\overline{23}$ are the halflines associated with the vectors

$$\begin{aligned}
 y_1^2 &= (1, 1, 0; -b) \\
 y_2^2 &= (1, 0, 1; -b) \\
 y_3^2 &= (0, 1, 1; -b)
 \end{aligned} \tag{15}$$

respectively. Finally, we assumed that the constraint facing the three consumers if they consume the public good jointly is $cg + x_1 + x_2 + x_3 = w_1 + w_2 + w_3$; the production set is then the halfline associated with the vector

$$y_1^3 = (1, 1, 1; -c) \tag{16}$$

If we assume that, for a given partition P_k , the economy as a whole has the same technological capacity for producing goods for a component jurisdiction that the jurisdiction has when operating in isolation, then the aggregate technology set $Y(P_k)$ for the partition P_k is a convex cone.²¹ In section 2, the partition P_k was not assumed to be given; the aggregate technology set implicit in that analysis admitted the use of production technologies corresponding to any of the possible partitions. Suppose we assume that the aggregate technology set is the convex cone generated by the halflines associated with the coalition production vectors (14), (15) and (16); i.e.,

$$Y = \sum_{i=1}^3 (y_i^1) + \sum_{i=1}^3 (y_i^2) + (y_1^3) = \text{conv} \left(\bigcup_{k=1}^5 Y(P_k) \right) \tag{17}$$

where (y_i^j) is the halfline associated with y_i^j and conv denotes the convex hull. Since Y is a convex cone and we have adopted Foley's other assumptions

on the aggregate technology set, individual consumption sets and individual preference orderings, the Lindahl equilibrium (as defined above) will exist, it will be Pareto optimal, it will be in the core and, hence, the core will necessarily be non-empty.²² But in section 2 we produced examples where the Lindahl equilibrium was not in the core and where the core was empty. What is the reason for this contradiction?

To find the source of contradiction, consider our three-person economy with $a = b = 1$, $c = 3/2$, $w_1 = w_2 = 2$ and $w_3 = 1$. We will first calculate explicitly the Lindahl equilibrium for this case. Observe that every allocation in Y lies within the convex cone generated by the halflines associated with the production vectors for two-person sharing;²³ i.e., $(y_1^2) + (y_2^2) + (y_3^2)$. This implies that the economy faces the resource constraint

$$(g_1 + g_2 + g_3)/2 + x_1 + x_2 + x_3 = w_1 + w_2 + w_3 \quad (18)$$

Given the constraint (18), part (a) of the definition of a Lindahl equilibrium implies that $p_g^1 = p_g^2 = p_g^3$.²⁴ If $p_x = 1$, then $p_g^i = 1/2$ for $i = 1, 2, 3$; the Lindahl allocation is $(g_1, g_2, g_3; z) = (2, 2, 1; -5/2)$ and the corresponding utility vector is $(u_1, u_2, u_3) = (2, 2, 1/2)$.²⁵ Using the values of the characteristic function given in (12), it is easily verified that this Lindahl equilibrium is in the core.²⁶

However, if we follow the approach described in section 2, we find that the only utility vector in the core is $(2, 2, 1/4)$ corresponding to the partition $\{\{1, 2\}, \{3\}\}$.²⁷ To isolate the source of contradiction, examine the pattern of sharing used to attain the Lindahl allocation:

$$(2, 2, 1; -5/2) = (3/2)y_1^2 + (1/2)y_2^2 + (1/2)y_3^2 \quad (19)$$

This is not a partition, but rather a set of overlapping jurisdictions: individuals 1 and 2 consume 3/2 units of the public good through mutual sharing which each supplements by sharing an additional 1/2 unit with individual 3. But in section 2 we ruled out any pattern of sharing that assigned an individual to more than one jurisdiction; only partitions were regarded as admissible. Although the aggregate technology set admits any production vector that belongs to the union of the partition technology sets, linear combinations of these vectors will be feasible only if the jurisdictions form a partition. As a consequence, the aggregate technology set is non-convex:

$$Y = \bigcup_{P_k \in \mathcal{K}} Y(P_k) \neq \text{conv} \bigcup_{P_k \in \mathcal{K}} Y(P_k).$$

4. Indivisibility

The basis for this non-convexity is, in a fundamental sense, an indivisibility -- an indivisibility not of plants or commodities but of individuals. In assigning consumers to "neighborhoods", for example, we rule out the possibility of fractional assignments; a person cannot be two places at the same time. Referring to our numerical example, individuals 1 and 2 will not, for they cannot, receive two units of the public good by combining 3/2 units from their own two-person neighborhood with 1/2 unit received from two-person neighborhoods formed with individual 3. If 1 and 2 admit individual 3 as a neighbor, then a three-person neighborhood is formed, not a set of three overlapping two-person neighborhoods. But in moving

from a two to a three-person neighborhood, the resources required to maintain the same "neighborhood quality" (g) increase by fifty percent. If individual 3 were as wealthy as 1 or 2, then a three-person neighborhood would be formed and the core allocation would give two units of the public good to everyone. However, individual 3 is not as wealthy as 1 or 2, and the best that a three-person neighborhood has to offer is $5/3$ units of the public good.²⁸ Thus, two separate neighborhoods are formed: a high income neighborhood for 1 and 2 and a low income neighborhood for 3. And that outcome, drawing upon casual observation, seems a reasonable picture of what would in fact happen.

The impossibility of assigning fractions of individuals to jurisdictions is characteristic not only of neighborhoods but local public goods in general. We do not observe consumers belonging to more than one school district: the school services from several districts cannot be added to yield greater benefit. Interjurisdictional spillovers may present an opportunity for fractional membership in some sense, but there is certainly no reason to think that such spillovers will in general lead to convexity of the aggregate technology set. Although further elaboration of the model to incorporate spillovers may be worthwhile, the formulation as it now stands is rich enough to cover a number of interesting cases. Prohibiting overlapping memberships in jurisdictions has one distinct advantage -- the barriers to convexity of the aggregate technology set in the presence of crowded public goods are thrown into bold relief.

Finally, one can view a household, jointly consuming a house or car, as a jurisdiction in our sense. We have, by assumption, ruled out membership in more than one household for any given individual. In assessing

the reasonableness of this assumption, it is important to remember that we have confined our discussion to economies having a single public good. We do not rule out membership in other jurisdictions -- a poker group for the husband, a bridge club for the wife -- if other public goods are involved.

5. Excludability

Implicit in our treatment of public goods is an assumption concerning the ability to exclude which should be made more explicit. Musgrave [9], in particular, puts great stress on non-excludability in consumption as a characteristic of public goods that is quite distinct from "non-rivalness" in consumption (or what I have called the absence of crowding).

Musgrave concludes that the presence or absence of crowding and the presence or absence of the ability to exclude gives rise to four different types of public good.²⁹ However, of the four possible cases, three have precisely the same core: the two cases of crowded and uncrowded public goods where exclusion is impossible and the uncrowded case with exclusion. If exclusion is impossible, then a blocking coalition S must assume that the members of the complementary coalition \bar{S} will share the benefits of any public good produced by S ; the set of utility vectors that S can obtain using only its own resources, $v(S)$, is the same whether the public good is crowded or not. The set of utility vectors that could be obtained if the members of \bar{S} could be excluded from consumption is irrelevant when exclusion is impossible. If exclusion is possible but the public good is uncrowded, then $v(S)$ is the same as in the two cases just discussed: if exclusion of the members of \bar{S} does not reduce the resource requirements for producing the

public good for S , then whether \bar{S} is excluded or not has no impact on the utility vectors achievable by S using only its own resources.³⁰

The remaining case to be considered, where exclusion is possible and the public good is crowded, is the one most relevant to the analysis presented in this paper.³¹ The core will be smaller than in the three preceding cases. Because exclusion reduces crowding, the blocking possibilities of any coalition S are enhanced. In our earlier discussion, exclusion has been assumed implicitly to be both complete³² and costless, but incorporation of partial or costly exclusion into the analysis does not appear to raise any particular difficulties.

Thus, from the point of view of the analysis of allocations in the core, the four cases we have been discussing reduce to just two: the pure public good case (broadly defined to include the absence of the ability to exclude, the absence of crowding or both) and the crowded public good case with exclusion. Foley's model can be applied directly to the first case; the approach developed in this paper is directed to the second. In our model jurisdictions are permitted to exclude, but there is no exclusion within jurisdictions.³³ Within this analytical framework, the distinction between exclusion and crowding is inessential. Both concepts are incorporated into the definition of the aggregate technology set.

6. Prices and the Organization of Collective Consumption

I have argued that a reasonable theory of crowded public goods must recognize the non-convexity of the aggregate technology set. Only in this way can the theory permit the existence of distinct sharing groups: neighborhoods, local political jurisdictions, clubs and households. But

in relinquishing convexity, have we given up too much? If most consumption involves sharing to some extent -- within the household if nowhere else -- and if non-convexity vitiates competitive equilibrium analysis, then we are forced to conclude that the main body of economic theory has almost no relevance to the way resources are allocated.

The situation we face is remarkably parallel to that addressed by Samuelson as he neared the end of his paper on social indifference curves:

Now I have proved the impossibility of group or community preference curves. But haven't I in a sense proved too much. Who after all is the consumer in the theory of consumer's (not consumers') behavior. Is he a bachelor? A spinster? ... In most of the cultures actually studied by modern economists the fundamental unit on the demand side is clearly the "family", and this consists of a single individual in but a fraction of the total cases.³⁴

Samuelson recognizes explicitly that family decision-making raises issues "exactly of the same logical character"³⁵ as the theory of public expenditure, but his reconciliation between family decision-making and standard demand analysis is limited to the special case where each family member consumes privately and the family as a whole maximizes its "social welfare function." However, it is crucial for our purposes to face squarely the public character of family consumption. Fortunately, a reconciliation between "public" consumption within families and standard competitive analysis is possible -- without the contrivance of a "family welfare function."

Expecting the tools of competitive equilibrium analysis to explain the pattern of sharing that will emerge in an economy is clearly asking too much. General equilibrium analysis has never tried to explain household

formation -- or the formation of firms, for that matter. The organization of the economy, the set of consuming and producing agents, has always been regarded as given. Competitive analysis explains the economic activity (consumption and production) of these agents, not their existence.

Therefore, we might expect that competitive analysis, or some variant of competitive analysis, could be applied to an economy with public goods when the partition, the pattern of sharing, is predetermined. And that, in fact, is precisely the case. If the partition P_k is given, the assumption that $Y(P_k)$ is a convex cone is no less (and no more) reasonable than the corresponding assumption for a pure private good economy. Along with the other assumptions concerning preferences and production used by Foley, the assumption that $Y(P_k)$ is a convex cone is sufficient to guarantee the existence, Pareto optimality and inclusion in the core of the Lindahl equilibrium as we have defined it.³⁶ Competitive equilibrium (for pure private goods) and Foley's definition of the Lindahl equilibrium (for pure public goods) correspond to the special cases where the partition is P_1 or P_s . It is important to note that the Lindahl equilibrium is defined with respect to a given partition. In every case, Pareto optimality and inclusion in the core is proved only relative to a partition: there is no guarantee that the Lindahl equilibrium for some P_k will not be Pareto inferior or blocked by some allocation feasible under a different partition.

As our counter-example in section 2 illustrated, the Lindahl equilibrium is a rigid and not very satisfying concept of political-economic equilibrium. Fortunately, Foley has provided another concept of equilibrium

which is much more flexible and appealing, the public competitive equilibrium.³⁷ Translating his definition into our notation, we have for the pure public good case:

Definition: A pure public competitive equilibrium is a feasible allocation $(g; x_1, \dots, x_n)$, a price system $p = (p_g; p_x)$, and a vector of taxes (t_1, \dots, t_n) with $p_g g = \sum_{i=1}^n t_i$ such that:

$$(a) \quad p \cdot (g; z) \geq p \cdot (\bar{g}; \bar{z}) \quad \text{for all } (\bar{g}; \bar{z}) \in Y(P_g);^{38}$$

$$(b) \quad p_x \cdot x_i = p_x \cdot w_i - t_i \quad \text{and if } (g; \bar{x}_i) \succ_i (g; x_i), \text{ then } p_x \cdot \bar{x}_i > p_x \cdot x_i;$$

(c) there is no vector of public goods and taxes $(\bar{g}; \bar{t}_1, \dots, \bar{t}_n)$ such that for every i there exists \bar{x}_i with $(\bar{g}; \bar{x}_i) \succ_i (g; x_i)$ and $p_x \cdot \bar{x}_i \leq p_x \cdot w_i - \bar{t}_i$. In this definition, producers maximize profits, consumers maximize utility subject to their after-tax budget constraint and there is no alternative level of public sector activity with taxes to pay for it that would make every consumer better off.

In extending this definition to an arbitrary partition P_k , we will also find it convenient to modify part (c) of Foley's version.³⁹

Definition: A public competitive equilibrium under the partition P_k , $P_k = (J_1, \dots, J_r)$, is a feasible allocation $(g_1, \dots, g_r; x_1, \dots, x_n)$, a price system $p = (p_g^1, \dots, p_g^r; p_x)$, and a vector of taxes (t_1, \dots, t_n) with $p_g^j g_j = \sum_{i \in J_j} t_i$ for all $j = 1, \dots, r$ such that:

$$(a) \quad \sum_{j=1}^r p_g^j g_j + p_x \cdot z \geq \sum_{j=1}^r p_g^j \bar{g}_j + p_x \cdot \bar{z} \quad \text{for all } (\bar{g}_1, \dots, \bar{g}_r; \bar{z}) \in Y(P_k);$$

$$(b) \quad p_x \cdot x_i = p_x \cdot w_i - t_i \quad \text{and if } (g_j; \bar{x}_i) \succ_i (g_j; x_i), \text{ then } p_x \cdot \bar{x}_i > p_x \cdot x_i;$$

$$(c) \quad \text{for any } J_j, j = 1, \dots, r, \text{ if } (g_j; \bar{x}_i) \succ_i (g_j; x_i) \text{ for all } i \in J_j,$$

$$\text{then } p_g^{j-} + \sum_{i \in J_j} p_x \cdot \bar{x}_i > p_g^j + \sum_{i \in J_j} p_x \cdot x_i = \sum_{i \in J_j} p_x \cdot w_i.$$

Conditions (a) and (b) have the same interpretation as the corresponding parts of Foley's definition; the budget of each jurisdiction is balanced; and if the members of a jurisdiction unanimously prefer some other allocation to the one they receive, then they cannot afford it at the given prices.⁴⁰ Foley's proofs are easily modified to demonstrate that the public competitive equilibrium for a partition P_k is Pareto optimal⁴¹ and that for any Pareto optimum under the partition P_k there exists a supporting price system taking the form of a public competitive equilibrium.⁴² It should be emphasized again that Pareto optimality is established only relative to a partition.

Since the Lindahl equilibrium is a public competitive equilibrium, we know that at least one public competitive equilibrium exists for any partition P_k . Proofs of existence for other types of public competitive equilibria could presumably be obtained for alternative theories of how the political decision-making process operates.⁴³

Even when the Lindahl equilibrium is not in the core, some other public competitive equilibrium may result in a core allocation. In the example used to establish Proposition 1, the core imputation $(u_1, u_2, u_3) = (2, 2, \frac{11}{16})$ can be achieved by a public competitive equilibrium with taxes $(t_1, t_2, t_3) = (\frac{14}{15}, \frac{14}{15}, \frac{19}{30})$ and prices $p_x = 1$ and $p_g = \frac{4}{3}$. The core imputation $(u_1, u_2, u_3) = (2, 2, \frac{7}{16}, \frac{1}{4})$ can be achieved by a public competitive equilibrium with taxes $(t_1, t_2, t_3) = (\frac{14}{15}, \frac{21}{30}, \frac{13}{15})$ and prices $p_x = 1$ and $p_g = \frac{4}{3}$. In fact, any imputation in the core for that example can be sustained by a public competitive equilibrium.⁴⁴

Thus, the concept of a public competitive equilibrium offers much greater flexibility than the Lindahl equilibrium in adapting to the blocking power of coalition.

Returning to the example of households as jurisdictions, all that

we have specified about the family decision-making process is that each family member maximizes his utility from goods consumed privately subject to the constraint of his "allowance" and that there is no combination of private and shared goods unanimously preferred to the bundle presently consumed by the family that it can afford. We have demonstrated that if the membership in households is considered exogenous to the model,⁴⁵ then public competitive equilibrium analysis can be applied. We have not reached a full reconciliation with standard competitive analysis, however. Consider the example of automobiles shared by members of a family. The definition of public competitive equilibrium permits a separate automobile price, p_g^i , for each household purchasing a car, but most economists would conclude that these prices will all be equal. It is reasonable to suppose, in this instance, that the resource cost of supplying a car to a family is independent of family size. In that case, condition (a) of the definition of a public competitive equilibrium implies that, for all families purchasing a car, the price must be equal; if prices were unequal, then selling all of the cars to the household having the highest p_g^i would be both feasible and more profitable. In this public competitive equilibrium, the marginal rates of substitution summed over the members of each family is equated to the marginal rate of transformation for the "public" good. The reconciliation of our analysis with standard competitive theory seems complete.

There is an interesting parallel between these results and the analysis of indivisibilities presented some time ago by Koopmans and Beckman [6]. They found that if the profitability of alternative locations for a given plant is independent of the location decisions of the other plants, then a system of location rents exists which supports a non-

fractional assignment of plants to locations. But when profitability of alternative locations depends on the location decisions of the other plants -- in their model, because of transportation costs -- then no price system exists to guide the process of plant location. Prices have a limited role in explaining the organization of economic activity.

Koopmans and Beckmann stressed the importance of indivisibility to location theory in general:

Finally, the theory of the location of economic activities has no chance of explaining such interesting realities as large and small cities without recognizing indivisibilities in the processes of production and human existence.⁴⁶

Neglect of indivisibilities, or non-convexities, explains why the application of general equilibrium analysis to location theory has not been very fruitful. By presupposing the existence of a price system and well-defined demand or supply functions at each location, the location problem is essentially assumed away. Locations are implicitly fixed. It is for this reason that location theorists find it necessary to turn to vague notions of agglomeration or urbanization economies when they try to explain the phenomena they observe. Location theory cannot be brought within the province of general equilibrium analysis solely by adding a subscript to the conventional competitive model.

7. Buchanan's Theory of Clubs

A reader familiar with the work of Buchanan will by now recognize that the treatment of public goods in this paper is closely related to his theory of clubs [1]. What I have defined as a jurisdiction Buchanan calls a club. It is appropriate, at this point, to indicate the relationship between our respective approaches.

In the discussion so far, I have permitted crowding in the production of public goods but excluded the effects of crowding from individual utility functions. Specifically, I have assumed that individuals are indifferent concerning the number of other consumers sharing consumption of the public good. Buchanan, on the other hand, allows crowding to affect utility functions as well as production functions. My reason for avoiding this complication has been solely for purposes of simplifying the exposition. The model is easily extended to incorporate this additional element of crowding.

Introducing a crowding parameter, the number of individuals in a jurisdiction, into utility functions requires no change in the definition of a Lindahl or a public competitive equilibrium for a given partition P_k . Since jurisdictional size is by definition fixed, the effect of this parameter is incorporated into the preference orderings. In analyzing choice among partitions, the presence of crowding in utility functions means that the appropriate preference orderings to consider depends on the partition. Our conclusion that prices have little relevance to the selection of a partition is, if anything, reinforced by the dependence of individual preference orderings on the partition chosen.

Permitting crowding to affect preference orderings does enhance the realism of the model. Turning to an example used before, it is reasonable to expect that, holding wealth constant, larger households will purchase more cars. If a household is wealthy enough and sensitive enough to crowding, it may no longer be the preferred jurisdiction for automobile consumption: husband and wife will have their "own" car and the teenage son will use summer earnings to buy a hot rod.

From our perspective, the theory of clubs emerges as even more general than Buchanan is willing to claim. He asserts that:

The procedure implies that the individual remains indifferent as to which of his neighbors or fellow citizens join him in such arrangements. In other words, no attempt has been made to allow for personal selectivity or discrimination in the models. To incorporate this element, which is no doubt important in many instances, would introduce a wholly new dimension into the analysis, and additional tools to those employed here would be required.⁴⁷

Our analysis demonstrates that introducing preference for particular individuals as members of ones jurisdiction -- on the basis of race, religion or sex -- is not essentially different from the introduction of crowding into the analysis.⁴⁸

There is one area of substantial disagreement between the approaches of Buchanan and myself. He states as a necessary condition for Pareto optimality "... that the marginal rate of substitution 'in consumption' between the size of the group sharing in the use of X_j and the numeraire good, X_r , must be equal to the marginal rate of substitution 'in production'."⁴⁹ Our analysis indicates, however, that determining the "optimal" partition -- whether defined as Pareto optimal or in the core -- requires global rather than marginal comparison of alternative sharing patterns. To any arbitrarily chosen partition there corresponds a price system which supports a public competitive equilibrium. The equilibrium allocation is Pareto optimal with respect to the set of allocations feasible under the given partition. However, under some other partition allocations may exist which are Pareto superior to the public competitive allocation or which permit some coalition to block. The presence of non-convexity means that the price system is no longer a reliable guide to the attainment of allocations that are Pareto optimal and in the core. Decision-making cannot be completely decentralized to the level of individual agents

responding marginally to market prices. The search for the optimal pattern of sharing of public goods requires the global comparison of the allocations achievable under each alternative partition.

8. A Tiebout Equilibrium

The two concepts of equilibrium discussed in this paper, the Lindahl equilibrium and the public competitive equilibrium, have been defined relative to the aggregate technology set, $Y(P_k)$, corresponding to a given partition P_k . As a consequence, Pareto optimality and inclusion in the core of the equilibrium allocations are proved only relative to the set of allocations achievable under P_k . There is no guarantee that such allocations will be Pareto optimal or in the core when alternative partitions are regarded as admissible.

However, if we were able to establish the existence of a global Lindahl equilibrium (i.e., a Lindahl equilibrium defined relative to the aggregate technology set $Y = \bigcup_{P_k \in \mathcal{P}} Y(P_k)$), then stronger results would be available.

Theorem 1: If $(g_1, \dots, g_n; x_1, \dots, x_n; p_g^1, \dots, p_g^n; p_x)$ is a global Lindahl equilibrium with respect to w , then it is in the core with respect to w and the aggregate technology set $Y = \bigcup_{P_k \in \mathcal{P}} Y(P_k)$.

PROOF:

Suppose that the coalition S could block $(g_1, \dots, g_n; x_1, \dots, x_n)$ by $(\bar{g}_1, \dots, \bar{g}_n; \bar{x}_1, \dots, \bar{x}_n)$. Since $(\bar{g}_i; \bar{x}_i) \succ_i (g_i; x_i)$ for all $i \in S$, the definition of a Lindahl equilibrium implies that

$$\sum_{i \in S} p_g^i \bar{g}_i + p_x \cdot \sum_{i \in S} \bar{x}_i > \sum_{i \in S} p_g^i g_i + p_x \cdot \sum_{i \in S} x_i = p_x \cdot \sum_{i \in S} w_i \quad (20)$$

or

$$\sum_{i \in S} p_g^i \cdot \bar{g}_i + p_x \cdot \sum_{i \in S} (\bar{x}_i - w_i) > 0 \quad (21).$$

But profit maximization, condition (a) in the definition of a Lindahl equilibrium, requires that

$$\sum_{i=1}^n p_g^i \cdot \bar{g}_i + p_x \cdot \bar{z} \leq 0 \quad \text{for all } (\bar{g}_1, \dots, \bar{g}_n; \bar{z}) \in Y \quad (22).$$

This contradiction establishes the theorem. Letting $S = N$, it follows that the global Lindahl equilibrium is also globally Pareto optimal: i.e., Pareto optimal relative to the set of all allocations $(g_1, \dots, g_n; z) \in Y$.

If a global Lindahl equilibrium exists, then much of our discussion in this paper would be superfluous; prices would guide the choice of partition as well as the choice of an allocation under a given partition. Although non-convexity of the aggregate technology set Y raises some doubts concerning existence of a global Lindahl equilibrium, convexity is not a necessary condition for existence. Is there something in the structure of our problem which will ensure existence of a global Lindahl equilibrium in the general case? Returning to our three-person economy, we can readily establish that this conjecture is false.

In the first place, if the core of the economy is empty, then no global Lindahl equilibrium can exist, and, in the course of demonstrating Proposition 2, we produced such a case. Furthermore, a global Lindahl equilibrium may fail to exist even when the core is not empty. Consider the example discussed in Section 3 where $w_1 = w_2 = 2$, $w_3 = 1$, $a = b = 1$

and $c = 3/2$. The only allocation in the core is $(g_1, g_2, g_3; x_1, x_2, x_3) = (2, 2, 1/2; 1, 1, 1/2)$ corresponding to the partition $\{\{12\}, \{3\}\}$. This allocation is the Lindahl equilibrium relative to that partition with prices $p = (p_g^1, p_g^2, p_g^3; p_x) = (1/2, 1/2, 1; 1)$ where we arbitrarily set $p_x = 1$. Note that

$$p \cdot (g_1, g_2, g_3; z) = p \cdot (2, 2, 1/2; -5/2) = 0 \quad (23).$$

For this equilibrium to be a global Lindahl equilibrium, we require

$$p \cdot (\bar{g}_1, \bar{g}_2, \bar{g}_3; \bar{z}) \leq 0 \quad \text{for all } (\bar{g}_1, \bar{g}_2, \bar{g}_3; \bar{z}) \in Y = \bigcup_{P_k \in \mathcal{P}} Y(P_k) \quad (24)$$

But $(\bar{g}_1, \bar{g}_2, \bar{g}_3; \bar{z}) = (1/2, 2, 2; -5/2) \in Y$, where the allocation is achieved via the partition $\{\{1\}, \{23\}\}$, and

$$p \cdot (1/2, 2, 2; -5/2) = 3/4 > 0 \quad (25).$$

Thus, no global Lindahl equilibrium exists which will sustain the only allocation in the core.

We must conclude that, in general, no price system will exist to guide the choice of membership in jurisdictions, and, for that reason, we have defined our two concepts of equilibrium only relative to a given assignment of individuals to jurisdictions. Nevertheless, we may still ask whether or not under some special circumstances the price system can function to guide the choice of partition. As Samuelson has suggested:

Consider cinemas in a large town. Because of deviations from constant-returns-to-scale ('indivisibilities' if you like), my well-being depends on your being willing to watch movies ... Doubtless, film watching will enter more than one u^i function as a public good variable x_{n+k} . But, if the town is populous and distances are small, free enterprise might well result in optimal replication of cinema theatres, each operating at capacity audiences, with fares set competitively at short and long-run marginal and average costs. This is a case where an exclusion principle can, and should, operate.⁵⁰

Returning to our three-person economy, suppose that we add a fourth individual with the same tastes as the other members and an endowment $w_4 = 1$ unit of the private good so that there are now two individuals of each type. The technology for jurisdictions containing three members or less is assumed to be the same as before. If all four individuals share the public good, then the economy is assumed to face the resource constraint

$$dg + \sum_{i=1}^4 x_i = \sum_{i=1}^4 w_i \quad (26)$$

where g is the level of public good produced and d is a positive constant. In particular, if we let $d = 2$, then we have a pure public good for jurisdictions containing two members or less and "constant returns to scale" for jurisdictions having more than two members. The convex hull of the aggregate technology set, $\text{conv}_{P_k \in \mathcal{P}} \cup Y(P_k)$, for this four-person economy is bounded by the production possibility frontier

$$\left(\sum_{i=1}^4 g_i \right) / 2 + \sum_{i=1}^4 x_i = \sum_{i=1}^4 w_i \quad (27).$$

The global Lindahl equilibrium is described by the price vector

$$(p_g^1, p_g^2, p_g^3, p_g^4; p_x) = (1/2, 1/2, 1/2, 1/2; 1) \quad (28)$$

and the allocation

$$(g_1, g_2, g_3, g_4; z) = (2, 2, 1, 1; -3) \quad (29).$$

The aggregate technology set corresponding to the partition $\{\{12\}, \{34\}\}$, on the other hand, is bounded by the production possibility frontier

$$g_{12} + g_{34} + \sum_{i=1}^4 x_i = \sum_{i=1}^4 w_i \quad (30)$$

where g_{12} and g_{34} are the amounts of the public good provided to the jurisdictions $\{12\}$ and $\{34\}$ respectively. Thus, in contrast to the case of the three-person economy, the global Lindahl equilibrium in this four-person economy is achievable by a partition; in fact, the global Lindahl equilibrium is identical to the Lindahl equilibrium relative to the partition $\{\{12\}, \{34\}\}$.

Applying Theorem 1, this global Lindahl equilibrium is globally Pareto optimal and in the core with respect to w and the set of allocations in $\bigcup_{P_k \in \mathcal{G}} Y(P_k)$. Furthermore, if the economy is replicated in an appropriate fashion, the only allocations in the core are global Lindahl equilibria. Specifically, if we add one individual of each type to the economy (i.e., let $w_5 = 2$ and $w_6 = 1$) and continue to assume constant returns to scale for jurisdictions containing more than two members, then the global Lindahl equilibrium which gives $(g_i; x_i) = (2; 1)$ and $(g_i; x_i) = (1; 1/2)$ to individuals with endowment $w_i = 2$ and $w_i = 1$ respectively is the only allocation in the core.

As will be demonstrated in a subsequent paper, these results can be generalized: if the "efficient" size of jurisdictions is small relative to total population, then a global Lindahl equilibrium exists which guides the choice of partition as well as the allocation under a given partition. When the economy is replicated in an appropriate fashion, the only allocations in the core for all replications are global Lindahl equilibria. Under these circumstances, the conjecture of Tiebout [18] is valid: the process of "voting with one's feet" is equivalent to a competitive market process. For this reason, I will refer to a global Lindahl equilibrium of this type as a Tiebout equilibrium.

As Samuelson has emphasized, a Lindahl equilibrium has, in general, "no relevance to motivated market behavior."⁵¹ Under the particular circumstances which give rise to a Tiebout equilibrium, Samuelson's objections to the relevance of Lindahl prices no longer apply. But a Tiebout equilibrium is a very special type of Lindahl equilibrium: as illustrated by our four-person example, all Lindahl prices will be equal in a Tiebout equilibrium.⁵² When the Lindahl equilibrium is relevant to motivated market behavior, it mimics the standard competitive price system very closely. There is no reason to believe that Lindahl prices will be relevant to market (or non-market) behavior in any other case.

One feature of Tiebout equilibria deserves special comment: as illustrated by our four-person example, households form jurisdictions which are homogeneous with respect to household type. This tendency toward specialization of jurisdictions is a reflection of the non-convexity present in the model, and evidence that such specialization in fact occurs would provide support for the analysis developed in this paper. If we interpret

the characteristics of a neighborhood as a public good, then it seems reasonable to suppose that a market process of this type is involved in the formation of urban residential neighborhoods. The tendency for households to cluster into neighborhoods or "social areas" with others of similar wealth, occupation and ethnic background is a familiar theme in the literature of urban sociology. It is tempting to conclude that a similar process is also at work in producing stratification of suburbs by income class, but that conclusion needs careful qualification.

In the first place, our analysis has been confined to the case of a single public good, but suburban governments typically provide several public services. The natural extension of the analysis would assume the existence of a separate system of jurisdictions for each public service. But this multiplicity of jurisdictions is undoubtedly precluded by the presence of transactions costs, a possibility that seems particularly likely in view of the number of local governmental services that are tied to the consumer's place of residence. The issue raised here is, in a fundamental sense, another aspect of the "indivisibility" of people stressed in this paper: it may be impossible, or at least inefficient, for a person to live in a separate jurisdiction for each public service consumed. To the extent that the survival of relatively large multiple-purpose jurisdictions represents an efficient response to transactions costs, the tendency of suburbs to stratify may be attenuated.

It is also important to recognize that blocking by a coalition, while feasible in economic terms, may not be politically feasible. For example, a coalition desiring to form a new governmental jurisdiction may be unable to muster sufficient political support for changes in existing

jurisdictional boundaries. Political constraints on blocking, while operative in the case of neighborhoods (e.g., when open housing legislation prohibits the use of race as a basis for exclusion), seem particularly significant in the case of local government. To the extent that existing jurisdictional boundaries are, for political reasons, resistant to change, stratification of suburbs by income class and tastes will be less sharp.

In spite of these qualifications, there is some evidence, as I have suggested elsewhere [2], that suburbs do tend to stratify by income class. Furthermore, as the growth of metropolitan areas brings more jurisdictions within effective commuting range, we can expect such tendencies to be reinforced.

9. Conclusion

We have demonstrated that the concept of a public good, a good available equally to every member of a jurisdiction, can be extended to cases in which exclusion or crowding occurs. The techniques of equilibrium analysis can be applied to any given assignment of individuals to jurisdictions including, of course, the special cases of purely private and purely public consumption. However, these techniques fail, in general, to answer the question of how the assignment of individuals to jurisdictions will be determined.

Nevertheless, the theoretical framework developed in this paper provides a precise formulation of the problem to be solved. We are presented with a class of n -person games which, while rich enough to cover a number of interesting cases, seems more amenable to analysis than the more general case of consumption externalities. Within this framework we can give precise meaning to the notion of a neighborhood or a local

political jurisdiction. Progress in understanding models of this type is, in particular, essential to an urban economic analysis capable of dealing with the phenomena of slums, ghettos, suburbs and cities themselves. Understanding how and why such entities come to exist seems at least as important as explaining their behavior once formed.

FOOTNOTES

1. Equating the sum of marginal rates of substitution to marginal cost, $(x_1 + x_2 + x_3)/g = c$. Using equation (2), this relation can be translated into equation (3).

2. To derive the Lindahl taxes, maximize $u_i = x_i g$ subject to the budget constraint $p_g^i g + p_x x_i = p_x w_i$ where p_g^i is the Lindahl price charged to the i th individual and p_x is the price of the private good. From the necessary conditions for a maximum, we obtain $p_g^i = p_x x_i$. Substituting into the budget constraint yields $x_i = w_i/2$ and $p_g^i = (p_x w_i)/2$. Letting $p_x \equiv 1$ and $t_i \equiv p_g^i$ gives the desired result.

3. Foley [4], p. 71.

4. If the public good is uncrowded (i.e., if it is a pure public good) then $a = c$.

5. If they consume the good g privately, then the resource constraint would be $a(g_i + g_j) + x_i + x_j = w_i + w_j$. To avoid an unnecessary complication, we will rule this possibility out by assumption.

6. These expressions, as well as those of (8) and (9) below, are derived by determining the Pareto optimal "utility frontier" for each coalition using its resources alone and the resource constraint appropriate to its size.

7. See footnote 6. In this case, the "utility frontiers" happen to be linear, but that will not be true in general even for three-person two commodity economies of the sort we are discussing. The counter-examples of this section can be obtained in the more general case, but the exposition would be unnecessarily complicated.

8. We should consider other possible "partitions": e.g., 1 and 2 consuming jointly and 3 privately. But we rule out these other partitions by assuming that all individually rational utility vectors achievable under such a partition form a proper subset of the utility vectors determined by $u_1 + u_2 + u_3 \leq v(\overline{123})$. One can show that this condition will be satisfied if and only if the game as represented by (7), (8) and (9) is superadditive. In all of the examples to be presented, this superadditivity condition will be satisfied.

9. As noted previously, utility is transferable only because the utility frontiers happen to be linear in this particular example.

10. $v(S)$ is also the best that S can do assuming that \bar{S} does its worst -- the usual definition of the characteristic function.

11. See Shapley [16]. Strictly speaking, this condition will be sufficient only if the game is superadditive. This additional condition is satisfied in all of the examples to be discussed.

12. For a discussion of balance in an economic context, see Scarf [14].

13. See equation (3) and footnote (2).

14. In fact, all allocations in the core involve sharing among all three individuals in this example.

15. This proposition runs counter to a theorem of Shapley and Shubik [17] that when external economies are present but external diseconomies are not then the core is always non-empty, provided that utility and production functions satisfy the sufficient conditions for the existence of a competitive equilibrium. However, as we will see in the following section, the aggregate technology set will be non-convex in this example.

16. The reader can verify that this game is superadditive so that (9) is the appropriate characteristic function for $v(\overline{123})$.

17. The assumptions on the individual consumption sets and preference orderings introduced by Foley, which are quite standard, will be adopted in this study. Each consumer chooses a point (g_i, x_i) in a consumption set X_i on which there is defined a complete and transitive ordering $>_i$. The individual consumption sets are closed and convex and have an interior in the private good subspace. If $(g_i, x_i) \in X_i$, then there is a point $(g_i, \hat{x}_i) \in X_i$ with $\hat{x}_i < w_i$. The aggregate consumption set, $X = \sum_{i=1}^n X_i$, has a lower bound for \leq . The individual preference orderings satisfy assumptions of continuity, convexity and monotonicity.

18. No public goods are initially owned.

19. We will adopt the other assumptions on Y introduced by Foley: Y is closed, $0 \in Y$ (inaction is possible), no production other than inaction is possible without input of some private good, it is possible to produce the public good and the public good is unnecessary as a productive input.

20. The halfline (b) associated with the vector b is the set $\{x | x = \lambda b, \lambda \geq 0\}$.

21. Specifically, $Y(P_1) = \{y_i | y_i = \alpha y_1^1 + \beta y_2^1 + \gamma y_3^1, \alpha, \beta, \gamma \geq 0\}$ for the partition $P_1 = \{\{1\}, \{2\}, \{3\}\}$ which implies the constraint

$$a(g_1 + g_2 + g_3) + x_1 + x_2 + x_3 = w_1 + w_2 + w_3;$$

$$Y(P_2) = \{y_i | y_i = \alpha y_1^2 + \beta y_3^1, \alpha, \beta \geq 0\}$$

for the partition $P_2 = \{\{12\}, \{3\}\}$ which implies the constraint

$$b(g_{12}) + a g_3 + x_1 + x_2 + x_3 = w_1 + w_2 + w_3;$$

$$Y(P_3) = \{y_i | y_i = \alpha y_2^2 + \beta y_2^1, \alpha, \beta \geq 0\}$$

for the partition $P_3 = \{\{13\}, \{2\}\}$ which implies the constraint

$$b(g_{13}) + a g_1 + x_1 + x_2 + x_3 = w_1 + w_2 + w_3;$$

$$Y(P_4) = \{y_i | y_i = \alpha y_3^2 + \beta y_1^1, \alpha, \beta \geq 0\}$$

for the partition $P_4 = \{\{23\}, \{1\}\}$ which implies the constraint

$$b(g_{23}) + a g_1 + x_1 + x_2 + x_3 = w_1 + w_2 + w_3; \text{ and}$$

$$Y(P_5) = \{y_i | y_i = \alpha y_1^3, \alpha \geq 0\}$$

for the partition $P_5 = \{\{123\}\}$ which implies the constraint

$$c g + x_1 + x_2 + x_3 = w_1 + w_2 + w_3.$$

22. The proofs in Foley [4] are easily extended; to conserve on the length of this article, they are left to the reader.

23. Provided that at least two of the three individuals consume positive amounts of the public good. The assumptions in our example rule out corner solutions where some individuals consume none of the public good. Observe that when $g_1 = g_2 = g_3$, the constraint (18) is the same as the constraint faced when the technology vector for three-person sharing, y_1^3 , is used.

24. Assuming that every individual consumes a positive amount of the public good.

25. $p_g^i = \text{marginal cost} = \frac{1}{2}$. From the demand side, $p_g^i g_i = p_x^i x_i = x_i = w_i/2$ if $p_x = 1$ (see footnote 2) so $(g_1, g_2, g_3) = (2, 2, 1)$ and $z = -(w_1 + w_2 + w_3)/2 = -5/2$.

26. The value for $v(\overline{123})$ is no longer correct since c now equals $3/2$.

27. If consumption of the public good is shared by all three individuals, then $v(\overline{123}) = 25/6$. Using this value in place of the value used for $v(\overline{123})$ in (12), we find that the game described is no longer superadditive. As

discussed in footnote 8, this non-superadditivity implies that equation (9) no longer represents the entire set of individually rational and Pareto optimal allocations for the coalition of the whole. In particular, the sole core allocation is obtained via the partition $\{\{1,2\}, \{3\}\}$ for which the Pareto optimal "utility frontier" is given by

$$(u_1 + u_2)^{1/2} + (u_3)^{1/2} = 5/2.$$

28. Use equation (3).
29. Musgrave [9], pp. 126-129.
30. However, in this case the ability to exclude may still be relevant to the bargaining process. The characteristic functions, as we have defined them, assume that the complementary coalition will "do its worst": \bar{S} is assumed to provide no public good, or, if it does, to exclude S . This approach essentially rules out the "free rider" problem. Introducing "more realistic" assumptions about the complementary coalition's behavior would increase the number of allocations that can be blocked and decrease the size of core. For a related discussion, see Rosenthal [10].
31. It is instructive to work through all four cases in terms of the example presented in sections 2 and 3.
32. In other words, S can exclude \bar{S} from all benefits of the public good produced by S .
33. The absence of internal exclusion is essentially the defining characteristic of a jurisdiction.
34. Samuelson [12], pp. 8-9.
35. Op. cit., p. 10.
36. Modification of Walley's proofs, an elementary exercise, is left to the reader.

37. Although mentioned in Foley [4], a much more complete discussion of public competitive equilibria is contained in [3].

38. Recall that we defined $z \equiv \sum_{i=1}^n (x_i - w_i)$.

39. In the pure public goods case, the two definitions are equivalent.

40. Do not confuse these prices (where there is one price for each jurisdiction) with the Lindahl prices (where there is one price for each individual). The Lindahl equilibrium is a special case of the public competitive equilibrium.

41. See the proof in Foley [3], pp. 60-61.

42. See the proof in Foley [4], pp. 68-69. As usual lump sum transfers may be required.

43. Foley, for example, proves the existence of a public competitive equilibrium in the pure public goods case when taxes take the form of a proportional income tax (Foley [3], pp. 71-72). However, the tax in the example used to establish Proposition 1 is a proportional income tax in the sense of Foley, so we also have a demonstration that proportional income taxes need not be in the core when public goods are crowded.

44. In every case the price of the public good, p_g , equals the marginal cost of producing the public good.

45. For example, we may assume that household membership is determined by social and cultural factors outside the scope of economic analysis.

46. Koopmans and Beckmann [6], p. 53.

47. Buchanan [1], p. 13, footnote 1.

48. Of course, in particular extensions of this analysis the assumption that preferences depend only upon crowding and the wealth and tastes of the other members may be a useful simplification.

49. Buchanan [1], p. 5.
50. Samuelson [13], p. 110.
51. Samuelson [13], p. 106.
52. A proof of this assertion for the general case will be presented in a subsequent paper.

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