#34

A SIMPLE PROOF OF THE ASYMPTOTIC EFFICIENCY OF 3SLS RELATIVE TO 2SLS ESTIMATORS

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1. Introduction.

In this note we provide a simple proof that the 3SLS is asymptotically efficient relative to 2SLS estimator by showing that the appropriate difference of the covariance matrices of their asymptotic distribution is positive semidefinite. The proof is novel and simple in that it utilizes exactly the procedure for showing the efficiency of Aitken relative to ordinary least squares estimators in the context of the general linear model.

Previous proofs of this fact [1], [2], [3], [4] are unduly complicated.

2. Formulation of the Problem.

Utilizing the notation in [1], we deal with the model

$$Y = YB + XC + U \tag{1}$$

where $Y = (y_{.1}, y_{.2}, \ldots, y_{.m})$, $X = (x_{.1}, x_{.2}, \ldots, x_{.G})$ are the matrices (of T observations) on the m current endogenous and G predetermined variables, respectively. It is further assumed that

$$\{u'_{t.} : t = 1, 2, \ldots\}$$

is a sequence of independent identically distributed random variables such that

(2) $$E(u'_{t\cdot}) = 0\,, \qquad Cov(u'_{t\cdot}) = \Sigma$$

Σ being positive definite; i.e., for simplicity we assume that the model contains no identities. It is also assumed that

(3) $$\operatorname*{plim}_{T\to\infty} \frac{X'X}{T} = M$$

is a positive definite matrix.

Imposing the a priori restrictions on the elements of B, C and multiplying on the left by $R^{-1}X'$ we have, for the $i^{\underline{th}}$ structural equation

(4) $$w_{\cdot i} = Q_i\delta_{\cdot i} + r_{\cdot i}\,, \qquad i = 1,2,\ldots,m$$

where R is a nonsingular matrix such that $X'X = RR'$ and

(5) $$w_{\cdot i} = R^{-1}X'y_{\cdot i},\ Q_i = R^{-1}X'Z_i,\ Z_i = (Y_i,X_i),\ \delta_{\cdot i} = (\beta'_{\cdot i},\gamma'_{\cdot i})',$$
$$r_{\cdot i} = R^{-1}X'u_{\cdot i}$$

the $u_{\cdot i}$ being the T element columns of U.

The system may then be written in the compact notation

(6) $$w = Q\delta + r$$

where

(7) $$w = (w'_{\cdot 1},w'_{\cdot 2},\ldots,w'_{\cdot m})',\ Q = diag(Q_1,Q_2,\ldots,Q_m),$$
$$\delta = (\delta'_{\cdot 1},\delta'_{\cdot 2},\ldots,\delta'_{\cdot m})',\ r =(r'_{\cdot 1},r'_{\cdot 2},\ldots,r'_{\cdot m})'\ .$$

It is shown in [1] that

$$(8) \qquad (\tilde{\delta} - \delta)_{2SLS} = (Q'Q)^{-1}Q'r\,, \quad (\hat{\delta} - \delta)_{3SLS} = (Q'\tilde{\Phi}^{-1}Q)^{-1}Q'\tilde{\Phi}^{-1}r$$

where

$$\tilde{\Phi} = \tilde{\Sigma} \otimes I_G$$

and $\tilde{\Sigma}$ is a consistent estimator of Σ.

We now observe that if we write the reduced form as

$$(9) \qquad Y = X\Pi + V$$

and obtain

$$(10) \qquad \tilde{\Pi} = (X'X)^{-1}X'Y$$

then

$$(11) \qquad Q_i'Q_i = Z_i'X(X'X)^{-1}X'Z_i = \tilde{S}_i'X'X\tilde{S}_i$$

where

$$(12) \qquad \tilde{S}_i = (\tilde{\Pi}_i, P_i)$$

such that $\tilde{\Pi}_i = (X'X)^{-1}X'Y_i$ and P_i is a selection matrix such that $X_i = XP_i$. It follows then that

$$(13) \qquad \frac{Q'Q}{T} = \tilde{S}'\left(I_m \otimes \frac{X'X}{T}\right)\tilde{S}, \quad \tilde{S} = \mathrm{diag}(\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_m)$$

and moreover

$$(14) \qquad \frac{Q'r}{\sqrt{T}} = \tilde{S}'\,\frac{1}{\sqrt{T}}(I_m \otimes X')u\,, \quad u = (u'_{\cdot 1}, u'_{\cdot 2}, \ldots, u'_{\cdot m})'$$

Moreover

(15) $$\frac{Q'\tilde{\Phi}^{-1}Q}{T} = \tilde{S}'\left(\tilde{\Sigma}^{-1} \otimes \frac{X'X}{T}\right)\tilde{S}$$

(16) $$\frac{Q'\tilde{\Phi}^{-1}r}{\sqrt{T}} = \tilde{S}'(\tilde{\Sigma}^{-1} \otimes I_G)\frac{1}{\sqrt{T}}(I_m \otimes X')u$$

Since $\tilde{S}_i$ converges in probability to $S_i = (\Pi_i, P_i)$ we see that, asymptotically,

(17) $$\sqrt{T}\,(\tilde{\delta} - \delta)_{2SLS} \sim (S^{*\prime}S^*)^{-1}S'\frac{1}{\sqrt{T}}(I_m \otimes X')u$$

(18) $$\sqrt{T}\,(\hat{\delta} - \delta)_{3SLS} \sim (S^{*\prime}\Phi^{-1}S^*)^{-1}S'\Phi^{-1}\frac{1}{\sqrt{T}}(I_m \otimes X')u$$

An important implication of (17) and (18) is that the arguments establishing the asymptotic distribution of 2SLS are exactly those establishing the asymptotic distribution of 3SLS estimators. In (17) and (18),

$$S^* = \text{diag}(S_1^*, S_2^*, \ldots, S_m^*), \quad S_i^* = \bar{R}S_i, \qquad i = 1,2,\ldots,m$$

and $\bar{R}$ is a nonsingular matrix such that

$$M = \bar{R}'\bar{R}$$

Under the standard assumptions we have that, asymptotically,

(19) $$\frac{1}{\sqrt{T}}(I_m \otimes X')u \sim N(0, \Sigma \otimes \bar{R}'\bar{R})$$

whence it is seen, immediately, that

$$(20) \qquad \sqrt{T}\,(\tilde{\delta} - \delta)_{2SLS} \sim N(0, C_2)\,, \quad \sqrt{T}\,(\hat{\delta} - \delta)_{3SLS} \sim N(0, C_3)$$

where

$$(21) \qquad C_2 = (S^{*\prime}S^*)^{-1}S^{*\prime}\Phi S^*(S^{*\prime}S^*)^{-1}\,, \qquad C_3 = (S^{*\prime}\Phi^{-1}S^*)^{-1}$$

But showing that $C_2 - C_3 \geq 0$ involves exactly the elementary arguments one uses in establishing the efficiency of Aitken relative to ordinary least squares estimators in the context of the general linear model. Precisely, write

$$(22) \qquad (S^{*\prime}S^*)^{-1}S^{*\prime} = (S^{*\prime}\Phi^{-1}S^*)^{-1}S^{*\prime}\Phi^{-1} + F$$

Note that $FS^* = 0$; post multiply by Φ, transpose and post multiply by it the two members of (22) to obtain

$$(23) \qquad (S^{*\prime}S^*)^{-1}S^{*\prime}\Phi S^*(S^{*\prime}S^*)^{-1} = (S^{*\prime}\Phi^{-1}S^*)^{-1} + F\,\Phi\,F'\,, \qquad \text{q.e.d.}$$

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