FURTHER REMARKS ON THE RAWLSIAN PRINCIPLE OF JUST SAVING

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1. Introduction

In his recent treatise [4], Rawls argues that redistribution should continue to "that point which ... maximizes the expectations of the least advantaged group". Moreover in the intergenerational context he argues that "the appropriate expectation ... is that of the long term prospects of the least favoured extending over future generations."[1]

The following note further explores the savings implications of this 'maxi-min' welfare criterion.

Since a "no (net) saving" rule leaves further generations equally well off, and since saving can only take place at the cost of present consumption, such a criterion might appear to preclude growth either in the short or the long run. Certainly this is the case if present consumption is the only argument in the utility function of potential savers. However the situation is no longer clearcut if the latter also have a personal stake in the future.

One important reason why this should be the case is that generations overlap. Assuming capital is durable, trades between any generation and its immediate descendants become feasible. Phelps and Riley [3] have shown that such trades are in general advantageous to all generations, and that under weak conditions there is long run accumulation of the capital stock.

Rawls' own suggestion is that each generation benefits from an increase in future wealth, in that it cares about its immediate descendants. The implications of this idea have been analyzed in a paper by Arrow [1]. His interpretation is that the utility of generation t is a function of its own consumption (c_t) and the consumption of its children (c_{t+1}).

$$\text{i.e.} \qquad W_t = W(c_t, c_{t+1})$$

To simplify the discussion utility is further assumed to be of the separable form

$$W_t = U(c_t) + \frac{1}{1+\rho} U(c_{t+1}), \ \rho \geq 0 \tag{1.1}$$

Assuming also that the marginal product of capital is a constant γ (> 0), and that the population is stationary, Arrow shows that despite the inclusion of future consumption, net saving after an even number of periods is always zero. Two possibilities are isolated depending upon whether or not the rate of pure time preference (ρ) is larger than the marginal product of capital. The solutions are summarized in figures 1(a) and 1(b), with s_t^* equal to the optimal net saving after t periods.

While economists traditionally assume a non-negative rate of pure time preference, it is far from obvious that such an assumption is appropriate in the context of family transfers. Despite the many desires of the present, and some degree of myopia with respect to the future, the urge on the part of parents to provide greater opportunities for their children is strong. In what follows it is assumed that the latter dominates, that is, children are favored by their parents.

One way of expressing such preference for the future is:

Assumption F. $W(c_t, c_{t+1}) > W(c_{t+1}, c_t)$ if and only if $c_{t+1} > c_t$.

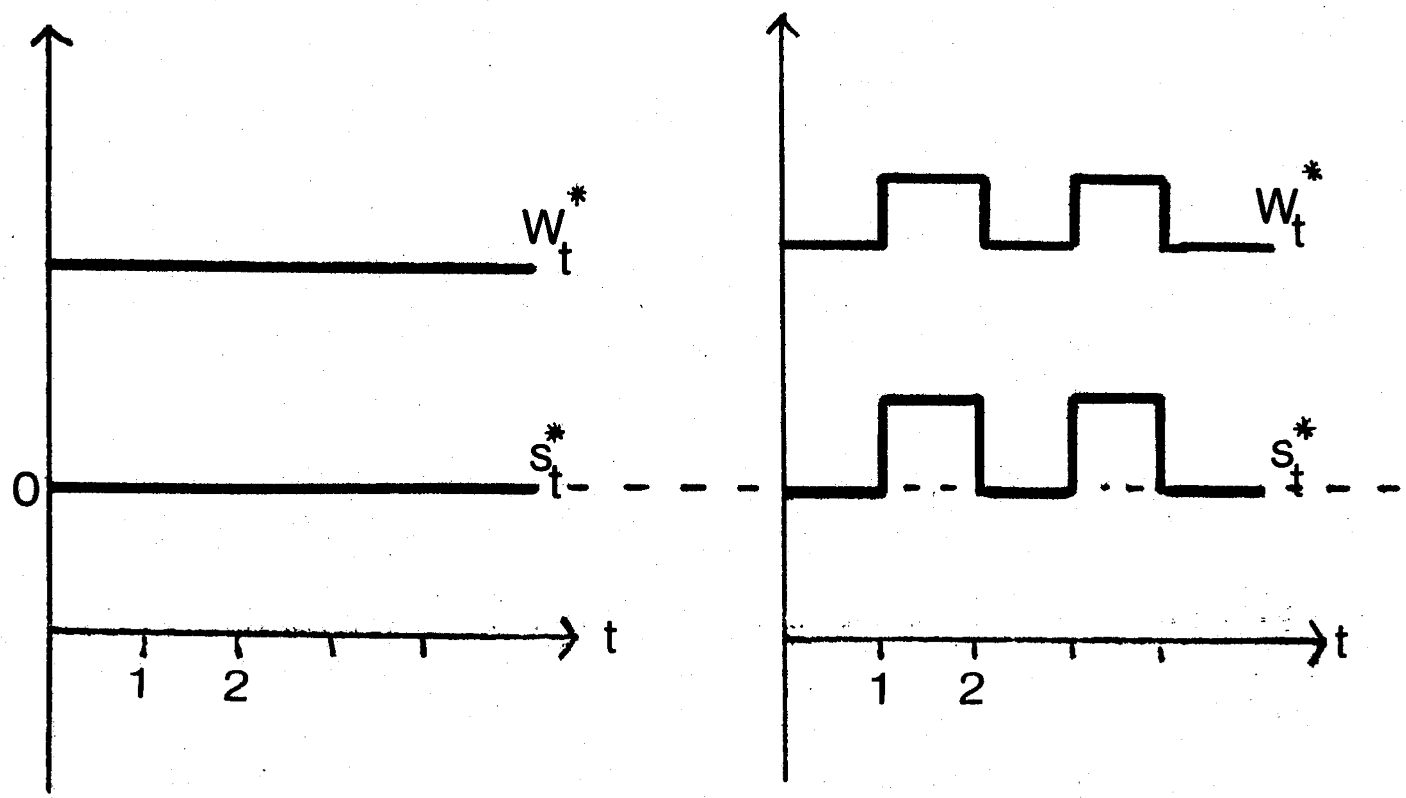
That is, parents prefer to switch their own and their children's consumption bundles if and only if the former are larger. For the separable utility function given in (1.1), assumption F of course requires $\rho < 0$.

In the next two sections it is shown that when this assumption is satisfied the character of the solution changes significantly. First it is no longer the case that some generations have a higher utility than others. Second, and perhaps most surprising, the Rawlsian maxi-min criterion does lead to long run growth in the capital stock. Both results are summarized in Figure 2.

Interesting also, from a technical standpoint, the solution is no longer a 'balanced variation' of the stationary "no savings" economy. Hence the approach

Figure 1

OPTIMAL POLICIES (A LA RAWLS-ARROW)

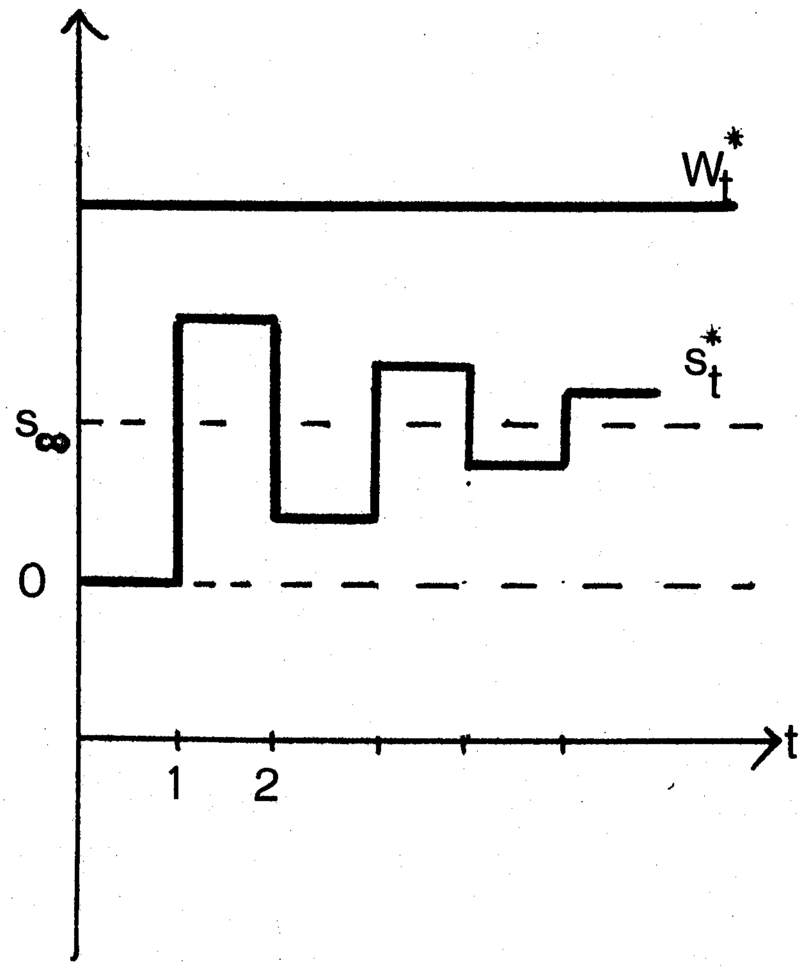


1(a) $\rho > \gamma > 0$

1(b) $\gamma > \rho \geq 0$

Figure 2

JUST SAVING WITH CHILDREN FAVOURED



used by Arrow is not immediately extendible to this case. Instead a dynamic programming formulation is employed, which not only yields the form of the growth path but also readily yields Arrow's solution.

2. A Dynamic Programming Formulation

Individuals of generation t derive utility from their own consumption c_t and that of their children c_{t+1}, according to a differentiable, strictly quasi-concave function $W_t = W(c_t,c_{t+1})$ defined over the positive quadrant. To avoid possible corner solutions it is assumed that $W(0,c) = W(c,0) = -\infty$.

It is further assumed that each unit of capital invested during period t has a net yield γ exceeding zero. Then if k_t is the per-capita stock at the beginning of period t and if population is stationary, we have,

$$k_{t+1} = (1+\gamma)(k_t - c_t), \qquad \gamma > 0 \tag{2.1}$$

Since k_{t+1} must be non-negative, k_t and c_t must satisfy the constraint

$$0 \leq c_t \leq k_t. \tag{2.2}$$

Respecting the preferences of immediate ancestors, the Rawlsian problem in some period τ can be written as:

$$m(k_{\tau-1},c_{\tau-1}) = \max \left[\inf_{\tau \geq \tau-1} \{W(c_t,c_{t+1})\}\right] \tag{2.3}$$

subject to (2.1) and (2.2)

The variables, $c_{\tau-1}$, $k_{\tau-1}$ and hence k_τ, are all predetermined.

Given a finite initial capital stock, the utility of the first generation is certainly bounded from above. Then the infimum is bounded from above, implying the existence of a supremum $m(k_{\tau-1},c_{\tau-1})$. Moreover a unit increase in $k_{\tau-1}$ increase initial resources by one unit. If the latter is invested in perpetuity, it yields an annuity of γ to all generations. Hence m is a strictly increasing function of $k_{\tau-1}$. Making similar arguments for small changes in $k_{\tau-1}$ and $c_{\tau-1}$ it can be shown that m is also a continuous function.[2]

Having just established that $m(\cdot)$ is bounded from above, it follows that the supremum is attained and hence that $m(k_{\tau-1},c_{\tau-1})$ is the maximum.

Applying Bellman's 'Principle of Optimality' [2], decisions after the (τ)-th period must also constitute an optimal policy and we can rewrite (2.3) as the following dynamic programming problem.

$$m(k_{\tau-1},c_{\tau-1}) = \max_{c_\tau} \left(\min[W(c_{\tau-1},c_\tau),\ m(k_\tau,c_\tau)]\right), \tag{2.4}$$

where k_τ is given by (2.1).

Note also that for the first generation, there are no ancestral preferences, hence the first stage of the dynamic programming problem is to obtain the solution for

$$\max_{c_0}\ m(k_0,c_0). \tag{2.5}$$

Before analyzing the nature of the optimal path it is necessary to characterize further the form of the return function $m(k,c)$. This is achieved in the following lemmas.

Lemma 1. The return function $m(k,c)$ is semi-strictly quasi concave.[3]

It is convenient to introduce the notation x^ν to represent a convex combination of two numbers x' and x''

$$\text{i.e.}\quad x^\nu = \nu x' + (1-\nu)x'' \qquad 0 < \nu < 1.$$

Writing the optimal sequences corresponding to two initial states (k_0',c_0') and (k_0'',c_0'') as $\{k_t',x_t'|k_0',c_0'\}$ and $\{k_t'',c_t''|k_0'',c_0''\}$ it is easy to check that the convex combination $\{k_t^\nu,c_t^\nu\}$ is a feasible solution for an initial state (k_0^ν,c_0^ν). (This follows directly from the linearity of the production constraint.)

But $W(c_t',c_{t+1}') \geq m(k_0',c_0')$ and $W(c_t'',c_{t+1}'') \geq m(k_0'',c_0'')$ for all t. Then from the strict quasi-concavity of W, it follows that whenever $m'' > m'$, there exists a $\delta = \delta(\nu) > 0$ such that:

$$W(c_t^{\nu},\ c_{t+1}^{\nu}) > m' + \delta(\nu)$$

Therefore, $m(k_0^{\nu}, c_o^{\nu}) \geq \inf_t\ [W(c_t^{\nu},\ c_{t+1}^{\nu})] > m'$. Q.E.D.

Corollary 1. m(k,c) is quasi-concave

This follows directly from Lemma 1 since m is also continuous.

Lemma 2. The return function m(k,c) is a strictly quasi-concave function of c.

From the previous lemma we know that m is a semi-strictly quasi-concave function of c. It remains to show that when

$$m(k_0, c_0') = m(k_0, c_0'') \text{ we have}$$

$$m(k_0, c_0^{\nu}) > m(k_0, c_0') \qquad 0 < \nu < 1$$

Without loss of generality assume $c_0' < c_0''$. Then for any μ such that $0 < \mu < \nu$ we have $c_0^{\mu} > c_0^{\nu}$. From Corollary 1 we also have $m(k_0, c_0^{\mu}) \geq m(k_0, c_0')$

Furthermore, since $c_0' \neq c_0''$, the strict quasi-concavity of W implies that

$$U(c_0^{\mu}, c_1^{\mu}) > \min\ [U(c_0', c_1'),\ U(c_o'',\ c_1'')]$$

$$\geq m(k_0, c_0')$$

Then, if $|\nu - \mu|$ is sufficiently small it follows that $U(c_0^{\nu}, c_1^{\mu}) > m(k_0, c_0')$.

The reduction in first period consumption releases $(c_0^{\mu} - c_0^{\nu})$ units of resources which can be invested in perpetuity yielding an annuity $a = \gamma(c_0^{\mu} - c_0^{\nu})$.

But $m(k_0, c_0') \leq U(c_t^{\mu},\ c_{t+1}^{\mu}) < U(c_t^{\mu} + a,\ c_{t+1}^{\mu} + a)$ for all $t > 0$.
We have therefore obtained a feasible sequence $\{c_0^{\nu},\ c_1^{\mu} + a,\ c_2^{\mu} + a,\ \ldots\}$ which yields utility to all generations strictly greater than $m(k_0, c_0')$. Since the

maximized infimum must be at least as large, we have finally,

$$m(k_0,c_0^{\nu}) > m(k_0,c_0') \qquad \text{Q.E.D.}$$

<u>Corollary 2.</u> <u>For any k, m(k,c) achieves a unique maximum at some point c in (0,k).</u>

From our assumptions on W we have $m(k,0) = m(k,k) = -\infty$ with $m(k,c) > -\infty$ on $(0,k)$. Also m is bounded from above. Therefore an interior maximum exists and given strict quasi-concavity it is unique. Q.E.D.

All these results are summarized by the iso-m contours in Figure 3. Since m is an increasing function of k the contours farther to the right correspond to higher values of m. From Corollary 2 it follows immediately that the solution of the first stage of the dynamic programming problem is a unique interior point c_0^*.

We now turn to an examination of the optimal path.

3. Cyclical Growth

From (2.4) the (t)-th stage of the dynamic programming problem is as follows:

$$m(k_{t-1},c_{t-1}) = m_{t-1} = \max_{c_t} \left(\min\,[W(c_{t-1},c_t),\ m(k_t,c_t)]\right). \qquad (3.1)$$

By assumption, $W_{t-1} = W(c_{t-1},c_t)$ is a strictly increasing function of c_t, and from the previous section m_t is a strictly quasi-concave function of c_t with a unique maximum $\tilde{c}_t$ in $(0,k_t)$. Then either the two curves intersect to the right of the maximum (figure 4(a)), or they do not (figure 4(b)). If the former the solution is:

$$m_{t-1} = m_t = U(c_{t-1},c_t) \quad \text{with } c_t > \tilde{c}_t \qquad (3.2)$$

Figure 3. Iso-m Contours

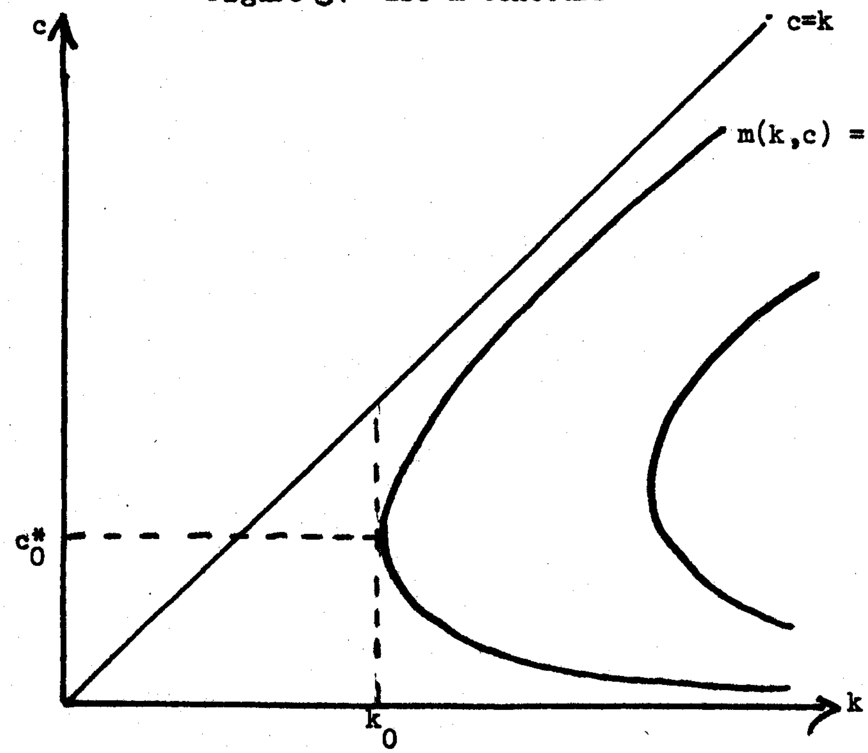
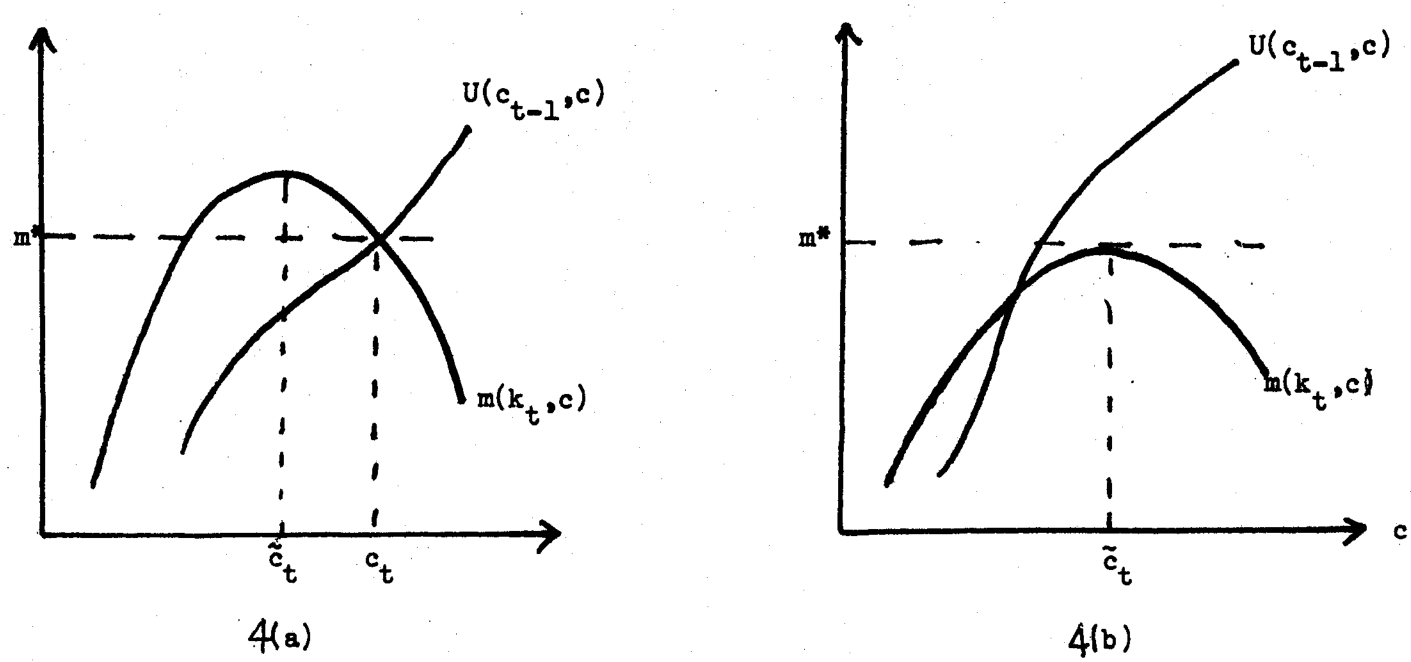


Figure 4. The Maxi-Min Solution



If the latter, the solution becomes

$$m_{t-1} = m_t \leq U(c_{t-1},c_t) \quad \text{with } c_t = \tilde{c}_t \tag{3.3}$$

Combining these results we have immediately;

<u>Theorem</u> 1: <u>The return function $m(k_t,c_t)$ has the same value for all t.</u>

Next, suppose that the solution is m^*. From Theorem 1, $m_0 = m^*$. Also from section 2, the cptimal first period assumption c_0^* maximizes $m(k_0,c_0)$. Therefore $c_0^* = \tilde{c}_0$ as depicted in figure 5.

Furthermore, from (3.2) and (3.3), the optimal consumption c_t^*, is given by the intersection from above, of $m(k_t^*,c)$ and $m = m^*$. Suppose $k_t^* < k_0$. Since m is strictly increasing in k, the profile $m(k_t^*,c)$ lies strictly below $m(k_0,c)$, implying,

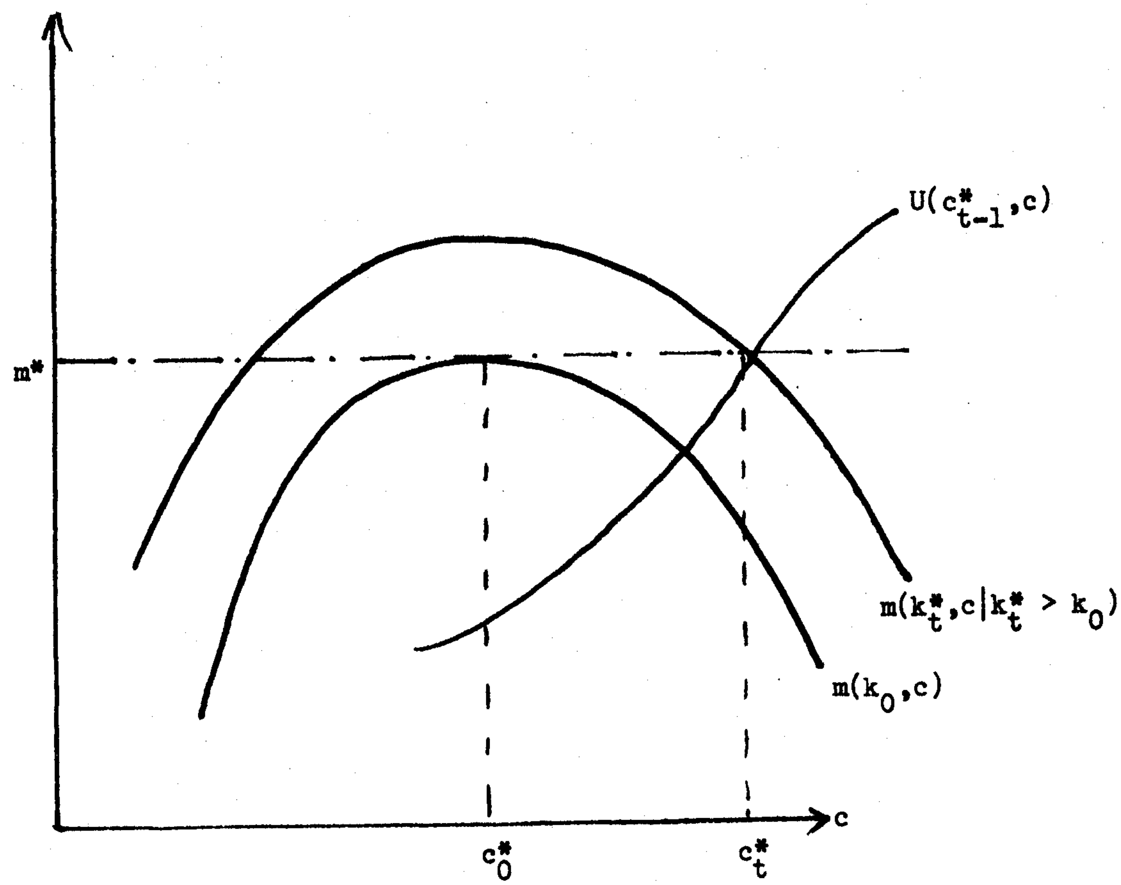
$$m_t \leq \max_c m(k_t^*,c) < m^*$$

But this contradicts Theorem 1, therefore $k_t^* \geq k_0$ for all t. If the equality, the solution of the (t)-th stage of the programming problem is as depicted in figure 4(b) and $c_t^* = c_0^*$.

If the strict inequality, as depicted in Figure 5, the three curves, $m = m^*$. $m = U(c_{t-1}^*,c)$, and $m = m(k_t^*,c)$, all intersect at (c_t^*,m^*). Moreover, the larger the capital stock, the further the m-curve is shifted upwards, and hence the larger the optimal consumption level. All this is summarized in the following theorem.

<u>Theorem</u> 2. <u>Cumulative saving</u> $s_t = k_t^* - k_0$ <u>is either positive or zero for all t. Furthermore,</u> $c_t^*(\geq c_0^*)$ <u>is a strictly increasing function of</u> k_t^* <u>and whenever</u> $c_t^* > c_0^*$ <u>we have</u> $U(c_{t-1}^*, c_t^*) = m^*$.

Figure 5



It follows that $k_1^* = k_0$ if and only if $c_1^* = c_0^*$. But then the second stage of the dynamic programming problem is identical to the first and $k_2^* = k_0$, $c_2^* = c_0^*$, etc. Therefore, if there is no saving in the first period, there is never any saving.

We now characterize cases when first period saving is optimal.

<u>Theorem</u> 3. <u>If the marginal product of capital exceeds the rate of pure time preference, there is saving in the short run, i.e. $k_1^* > k_0$.</u>

We prove the contrapositive. From the above discussion $k_1^* = k_0$ implies $k_2^* = k_0$. Also from Theorem 2

$$c_t^* = c(k_t^*) \quad \text{with } c'(\cdot) > 0. \tag{3.4}$$

Therefore, $k_1^* = k_0$ also implies $c_2^* = c_0$. We next examine the greatest minimum utility associated with all feasible paths of the form

$$\{k_0, k_1, k_0\} \quad \{c_0, c_1, c_0\}$$

In Arrow's terminology, these are the two period 'balanced variations' of the stationary state. We seek a necessary condition for the optimal first period capital stock k_1^* to be equal to k_0.

From Theorem 2 the additional constraint $c_1 \geq c_0$ must be imposed.

$$\text{i.e. } \max_{c_0, c_1} \left(\min\,[W(c_0,c_1),\, W(c_1,c_0)]\right)$$

s.t. $k_0 = k_2 = (1 + \gamma)(k_1 - c_1) = (1 + \gamma)^2 k_0 - (1 + \gamma)^2 c_0 - (1 + \gamma) c_1$

and $c_1 \geq c_0$.

It is a straightforward matter to check that $c_1^* = c_0$ if and only if

$$-1 + \frac{\partial W}{\partial c_0} \Big/ \frac{\partial W}{\partial c_1}\Bigg|_{c_1 = c_0} \geq \gamma$$

Finally applying (3.4), the optimal first period capital, $k_1^* = k_0$ if and only if this inequality holds. Q.E.D.

We can now prove that the Rawlsian Principle of Just Saving' may imply long run capital accumulation.

<u>Theorem</u> 4. <u>If assumption F is satisfied, total saving</u> $s_t = k_t - k_0$ <u>is always positive and oscillates towards an asymptote</u> $s_\infty > 0$.

The proof is by induction. We know from Theorem 2 that $c_t^* \geq c_0^*$ for all t. Moreover assumption F implies a negative rate of pure time preference, therefore from Theorem 3,

$$k_1^* > k_0 \text{ and } c_1^* > c_0^*. \tag{3.5}$$

Now suppose for some t

$$c_0^* \leq c_{t-1}^* < c_t^*$$

We shall prove that c_{t+1}^* must lie strictly between the consumption levels in the two preceding periods.

Writing m* as the infimum we have,

$$U(c_t^*, c_{t+1}^*) \geq m^* \tag{3.6}$$

also by assumption $c_t^* > c_0^*$, therefore, from Theorem 2, we have,

$$\begin{aligned} m^* &= U(c_{t-1}^*, c_t^*) && (3.7) \\ &> U(c_t^*, c_{t-1}^*) \text{ by assumption F.} \end{aligned}$$

Combining this with (3.6) we must have

$$c_{t-1}^* < c_{t+1}^* .$$

Then $c^*_{t+1} > c^*_0$ and again applying Theorem 2 we have

$$U(c^*_t, c^*_{t+1}) = m^*.$$

Comparing this with (3.7) and noting that we have assumed, $c^*_t > c^*_{t-1}$ it follows that,

$$c^*_{t+1} < c^*_t.$$

An almost identical argument establishes that c^*_{t+2} also lies between the consumption levels in the two periods preceding it. Combining the two results yields

$$c^*_{t-1} < c^*_{t+1} < c^*_{t+2} < c^*_t \tag{3.8}$$

whenever

$$c^*_0 \leq c^*_{t-1} < c^*_t \tag{3.9}$$

From Theorem 2, optimal consumption is a strictly increasing function of the capital stock, therefore we also have

$$k^*_{t-1} < k^*_{t+1} < k^*_{t+2} < k^*_1 \tag{3.10}$$

Finally we note that if (3.9) is true for $t = \tau$, it must be true for $t = \tau+2$ (from (3.8)). Since we have seen that it is true for $t = 1$, (3.8) and hence (3.10) must hold for all odd t.

$$\text{i.e.} \quad k_0 < k^*_2 < \ldots < k^*_3 < k^*_1 \qquad \text{Q.E.D.}$$

For completeness we conclude by summarizing the implications of the alternative assumption that generations favour themselves over their children.

We can express this as:

Assumption S: $W(c_t, c_{t+1}) \geq W(c_{t+1}, c_t)$ if and only if $c_t \geq c_{t+1}$

Clearly the utility functions (1.1) considered by Arrow all have the property S.

From Theorem 2, $c_t^* \geq c_0^*$ and either $c_1^* = c_0^*$, in which case the optimal solution is the stationary solution, or $c_1^* > c_0^*$.

Suppose the latter, and that in addition

$$c_2^* > c_0^* \tag{3.11}$$

Then

$$\begin{aligned} U(c_1^*, c_2^*) &> U(c_1^*, c_0^*) \quad \text{by (3.11)} \\ &\geq U(c_0^*, c_1^*) \quad \text{from Assumption S and } c_1^* > c_0^* \\ &\geq m^* \quad \text{since } m^* \text{ is the infimum} \end{aligned}$$

But from Theorem 2

$$U(c_1^*, c_2^*) > m^* \rightarrow c_2^* = c_0^* \quad \text{contradicting (3.11)}$$

It follows that the optimal second period consumption c_2^* must equal the initial consumption c_0^*, and from Theorem 2, $k_2^* = k_0$.

Therefore, in either case net savings must be zero after two periods. Whether or not savings in the short run is optimal is then determined by examining all feasible two period 'balanced variations'. The proof of Theorem 3 establishes that the critical condition is the relative size of the rate of pure time preference and the marginal product of capital. We therefore have finally:

<u>Theorem</u> 5. <u>If assumption S is satisfied, cumulative saving is zero after two periods. Furthermore, short run (saw-tooth) saving is optimal if and only if the marginal product of capital exceeds the rate of pure time preference</u>.

Footnotes

1. Rawls, op. cit. p 285.

2. For a fuller discussion of the issues involved see Phelps and Riley, op. cit.

3. A function $f(\underline{x})$ is said to be semi-strictly quasi-concave if for any $\underline{x}'$, $\underline{x}''$ such that $f(\underline{x}'') > f(\underline{x}')$ we have $f(\lambda\underline{x}' + (1-\lambda)\underline{x}'') > f(\underline{x}')$, $0 < \lambda < 1$.

4. Having established that $k_t^* > k_0$ for all t, it then follows from Theorem 3 that $U(c_{t-1}^*, c_t^*) = m^*$ for all t, as depicted in figure 2.

References

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