

Price Uncertainty and the Use of Money as  
Standard of Deferred Payment

by

Robert A. Jones\*

University of California, Los Angeles

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## PRICE UNCERTAINTY AND THE USE OF MONEY AS STANDARD OF DEFERRED PAYMENT

Economic agents, when engaging in contracts to make deferred payments, frequently fix the value of such payments in units of money rather than in units of non-money goods. The practise is so common that the use of money as a standard of deferred payment has, along with its use as medium of exchange, been traditionally regarded as one of its major economic functions. But in a world where future prices are uncertain the fixing of deferred payments in units of money and units of goods are clearly not equivalent, and this common use of the medium of exchange as a standard of deferred payment is somewhat remarkable. In this paper we examine the choice of standard of deferred payment by rational individuals, and attempt to provide a theoretical link between the use of a good as medium of exchange and its use as standard of deferred payment. We are asking whether the use of money as standard of deferred payment is more than just a convenient adjunct to its use as medium of exchange.

Consider a hypothetical loan agreement between two individuals. One individual, the lender, gives another individual, the borrower, a quantity of money or goods in the current period in return for a promised deferred payment in some future period. A well-defined loan agreement must specify what object or good will be transferred from the borrower to the lender in the future, and how the quantity of that object or good transferred will be determined. We shall refer to the object or good transferred as the means of payment of the loan. It may, for example, be money, a single good or service, or a standard combination of goods such as equal numbers of apples and oranges. We shall not require the agents to fix in advance the quantity of means of payment to be transferred, but allow them to agree upon rules for determining the quantity

to be transferred once future prices are known. Specifically, we require them to agree upon a fixed bundle of goods to be purchasable with the deferred payment. We refer to this fixed bundle as the standard of deferred payment for the loan. The means of payment and standard of deferred payment need not be the same good or collection of goods, and the considerations relevant to the choice of each are somewhat different.

The standard of deferred payment for the loan, as we have defined it, represents a price index to which the payment is tied. The indexing of loan agreements, or use of a "tabular standard," was suggested as early as 1807 by John Wheatley. Yet despite the advocacy of a tabular standard of value by such distinguished economists as F. Y. Edgeworth, Alfred Marshall, Irving Fisher and more recent proponents as a means of protecting the real value of deferred payments from unanticipated price changes, the medium of exchange remains the principal standard of deferred payment in most economies.<sup>1</sup> Our purpose, however, is not primarily to suggest the virtues of indexing but rather to identify those circumstances under which individuals would rationally choose not to index their debts.

Transaction costs, information costs, and price uncertainty could all be expected to influence the specification of the deferred payment. Suppose that exchanging unwanted goods in the future spot market incurs transaction costs, but that the use of money as medium of exchange is the least costly way of effecting any ultimate trade. Generally, an exchange of some part of the borrower's future endowment for something the lender wishes to consume will have to be made in the future spot market. If money is used as means of payment for the loan, then no additional transactions will be imposed on the borrower and lender beyond those required to effect this exchange in the least costly manner. But if a good which is neither money, nor in the borrower's

endowment, nor in the lender's consumption bundle is specified as means of payment, then two additional transactions are required: namely the acquisition of and disposition of this means of payment good. Hence the existence of transaction costs in spot markets conducive to monetary trade imply that money should also be used as the means of payment for loans.

Suppose that future prices are not known when the loan agreement is made, and that acquiring information in the future about prevailing prices will entail costly search or inquiries. If the good, or vector of goods, used as standard of deferred payment for the loan is identical to that used as means of payment, then the quantity of means of payment transferred will be independent of the future prices prevailing and these information costs need not be incurred. The existence of information costs render the means of payment of the loan an attractive standard of deferred payment. Consequently transaction and information costs together suggest that money would be a desirable standard of deferred payment.

But uncertainty about future prices raises a further consideration whose implications are not so transparent. Until future prices are revealed, the lender does not know what quantities of goods are purchasable with a given payment. Similarly, the borrower does not know how much of his future endowment of goods will be absorbed in making the payment. The standard of deferred payment plays a central role in determining the distribution of consumption possibilities that may face the borrower and lender. The remainder of this

<sup>1</sup> There have been many exceptions, of course, including the issuance of indexed bonds in several countries, escalation clauses in labour contracts, and tying of the future delivery price of long lead-time capital goods to wholesale price indices. Collier (1969) recounts many examples. Such exceptions, however, suggest that the use of non-monetary standards of deferred payment is indeed feasible, and that the prevalence of money as a standard of payment is the result of mutual choice by contracting agents.

paper focuses on this last consideration. How would rational but risk-averse individuals choose to specify their deferred payments when faced with uncertain future prices? An appropriate standard of deferred payment, and that on which we assume individuals will agree, is one which efficiently allocates the risks of future price fluctuations between them. Transaction and information costs will be omitted from the formal analysis. In circumstances where money is a desirable standard from the point of view of risk allocation the other factors can only further enhance its attractiveness as a standard of deferred payment.

## I. Outline of the Model

A loan agreement is defined to be a bilateral exchange of current money for claims to future goods, with the recipient of current money designated the borrower and the other agent the lender. We shall call the vector of claims which the borrower transfers to the lender the standard of deferred payment of the agreement. The special case where a zero quantity of current money is transferred is termed a pure futures contract. For example, a quantity of claims to future money might be exchanged for claims to a particular good. If two individuals for whom no mutually beneficial pure futures contract exists enter into a loan agreement, we term the agreement a pure loan agreement.

We consider a two period pure exchange economy with  $H$  individuals and  $n$  goods, one of which is universally used as a medium of exchange and designated money. Each individual has an initial endowment of current goods and claims to goods in the future period. The different types of exchanges that can be made in the current period can be distinguished by supposing them to take place in distinct markets. Exchanges involving only current goods take place in the current spot market. Exchanges involving only claims to future goods take place in the futures contract market. Exchanges involving current money and future claims take place in the loan market. In the next period all claims to future goods materialize, and there is a further opportunity to trade in the future spot market.

We are specifically interested in the proportions in which future claims are transferred in the loan market. We abstract from trading in the current spot market and take as given the amount of current money to be given by our hypothetical lender to his borrower. This amount may have depended on the costs of borrowing and returns to lending anticipated when each was making

his current spot market trades, but its determination is not important for our purposes.

Although all individuals, including our lender and borrower, anticipate trading in the future spot market the prices that will prevail in that market are currently unknown. Each individual knows his own future endowments and preferences, but no individual knows everyone's endowments and preferences to be able to deduce the future equilibrium outcome. The individuals are assumed to have subjective probability distributions representing their beliefs about possible future prices, but we do not explore how these beliefs are formed.

It adds some complication to consider an economy with  $n$  goods rather than one with just money and a single consumption good, but there are two reasons for doing so. First, differences in consumption preferences between individuals, and properties of preferences such as homotheticity, play important roles in determining the standard of deferred payment. With only one consumption good, however, identical homothetic preferences would be imposed on all individuals and these relationships would be missed. Second, having several consumption goods permits us to consider situations in which the general price level, some weighted average of the money prices of goods, is not uncertain even though the prices of individual non-money goods are uncertain. As will be shown below, it is in these situations where money might plausibly be chosen over other goods as the standard of deferred payment. With only one consumption good such situations would be excluded, and the role of money in allocating the risks of relative price changes in the non-money goods would be lost.

The analysis proceeds as follows: In section II the choice of a portfolio of claims to future goods by an individual facing given prices for those claims is examined. This analysis applies equally to a net purchaser of claims

(lender) or net seller (borrower). Section III assumes that the borrower and lender will negotiate an efficient loan agreement, in the sense that no alternative agreement could further improve the future prospects of both. Since any efficient reallocation of claims between the individuals can be characterized as a competitive equilibrium from some initial redistribution of claims, the outcome of the loan agreement can be characterized using the results from section II. The difference between the lender's pre-agreement and post-agreement allocation of claims is the standard of deferred payment. As might be expected, this difference consists solely of money claims only in rather special circumstances. Finally, in section IV, use is made of money's role as medium of exchange to distinguish it from other commodities. If the velocity of circulation of the medium of exchange is non-random, and some restrictions on preferences are met, it is demonstrated that the medium of exchange is an appropriate standard of deferred payment for pure loan agreements. A possible, though tenuous, link is established between the use of a good as medium of exchange and its use as standard of deferred payment



## II. Choosing a Portfolio of Claims to Future Goods

How might an individual with a given quantity of money  $m$  choose a portfolio of claims  $x = (x_1, \dots, x_n)'$  to future goods when faced with given current money prices  $p^0 = (p_1^0, \dots, p_n^0)'$  for those claims? He is uncertain about the money prices  $p = (p_1, \dots, p_n)'$  that will prevail in the future spot market for these goods, but has a subjective probability distribution on the possible prices he might face with mean  $\bar{p} = E(p)$  and covariance matrix  $\Sigma = E[(p-\bar{p})(p-\bar{p})']$ . The first good is arbitrarily designated to be money so that  $p_1 = \bar{p}_1 = 1$  with certainty. ie: The first row and column of  $\Sigma$  consist of zeroes. Let us assume the individual behaves as if he were maximizing the expected value of some utility function defined on his portfolio of claims  $x$  and the future outcome  $p$ .

We must specify more precisely how the individual's utility function depends on  $x$  and  $p$  if we are to say anything about the portfolio he might choose. The traditional approach, pioneered by Markowitz and Tobin, assumes a one-to-one correspondence between the future money value  $M = p'x$  of the portfolio and the individual's realized level of utility  $u(M)$ . This would be appropriate if none of the risky (in money terms) assets were the objects of ultimate consumption. However for our purposes it must explicitly be taken into account that the individual's level of future utility  $U(c)$  depends on the vector of goods  $c = (c_1, \dots, c_n)'$  actually consumed. If the future prices turn out to be  $p$ , and if his portfolio has money value  $M$ , he could trade in the future spot market to attain a level of utility

$$V(p, M) = \underset{c}{\text{Max}} \{U(c) \text{ subject to } M \geq p'c \text{ and } c \geq 0\}.$$

With the prices of consumption goods uncertain there is no longer a one-to-one correspondence between  $M$  and achieved utility. Nevertheless, we shall first

solve the portfolio choice problem with the objective of maximizing  $E[u(M)]$ , since the solution when maximizing  $E[V(p,M)]$  can be most conveniently expressed in terms of the former solution, and the effect of price uncertainty on the traditional portfolio choice can be most clearly seen.

The significance of price uncertainty for portfolio choice has long been recognized<sup>2</sup>, but the problem was first stated formally, together with first order conditions for its solution, by Roll (1973). Roll allows very general probability distributions over future prices, but his first order conditions tell us little about the effect of this uncertainty on the portfolio chosen. Fischer (1974) examines the portfolio choice problem in a continuous time framework where prices vary randomly, but continuously, over time. Since an individual faces only infinitesimal price risks from moment to moment, Fischer can partially characterize the portfolio chosen in terms of the infinitesimal means and variances of the anticipated price changes. We also assume that price risks are small, but exploit some simple results from duality theory to relate consumption preferences to the portfolio choice.

## II.1 Solution to the traditional problem: maximizing $E[u(M)]$

Suppose an individual must choose a vector of claims  $a = (a_1, \dots, a_n)'$  subject to the constraint  $a'p^0 = m$  so as to maximize the expected value of a twice-differentiable concave utility function  $u(M)$ .  $M = a'p$  is the future money value of his portfolio. Assume that future prices will lie in some small neighbourhood of their mean  $\bar{p}$  with probability one, so that we may adequately approximate  $u(M) = u(a'p)$  by a second order Taylor Series expansion of  $u(a'p)$  around  $\bar{p}$ , and take its expected value to approximate  $E[u(a'p)]$ .<sup>3</sup>

<sup>2</sup> See Tobin (1958).

$$(1) \quad u(a'p) \approx u(a'\bar{p}) + a'(p-\bar{p})u_M + \frac{1}{2} a'(p-\bar{p})(p-\bar{p})'a u_{MM}$$

$$E[u(a'p)] \approx u(a'\bar{p}) + \frac{1}{2} a'\Sigma a u_{MM}$$

The first and second derivatives of  $u$ ,  $u_M$  and  $u_{MM}$ , are understood to be evaluated at the expected outcome  $\tilde{M} = a'\bar{p}$ . Maximization of the above expression for  $E[u(a'p)]$  subject to the wealth constraint  $a'p^0 = m$  leads to the first order conditions

$$(2) \quad u_M \bar{p} + u_{MM} \Sigma a - \tilde{\lambda} p^0 = 0, \quad m - a'p^0 = 0.$$

The Lagrange multiplier on the wealth constraint,  $\tilde{\lambda}$ , is interpreted as the marginal expected utility of investable funds. Since the first row of  $\Sigma$  consists of zeroes, it is clear that  $\tilde{\lambda} = (\bar{p}_1/p_1^0)u_M$ . Denoting the Arrow-Pratt index of relative risk aversion by  $RRA = -\tilde{M}u_{MM}/u_M$ , and recognizing that  $\bar{p}_1/p_1^0 \equiv 1 + r$  is the "risk-free" (in money terms) return to purchasing dollar claims in the current period, (2) can be rewritten as

$$(3) \quad (RRA)\Sigma(a/\tilde{M}) = \bar{p} - (1+r)p^0, \quad m - a'p^0 = 0.$$

The results of traditional mean-variance portfolio analysis are apparent in equation (3): A solution exists only if every risk-free combination of claims yields return  $r$ . In other words, there cannot be any riskless but profitable arbitrage. Individuals who are infinitely risk-averse ( $RRA \rightarrow \infty$ ) hold only money claims. If the individual's RRA is independent of wealth, then  $a/\tilde{M}$  and hence portfolio shares are independent of the size of the portfolio  $m$ . And finally, if  $a^1$  designates the portfolio chosen when  $RRA = 1$  and  $a^\infty$  the portfolio consisting solely of money claims, then the solution to (3) will satisfy

<sup>3</sup> A formal justification for using such an approximation is given by Samuelson (1970).

$$(4) \quad a/\tilde{M} = (1/RRA)(a^1/\tilde{M}^1) + (1 - 1/RRA)(a^\infty/\tilde{M}^\infty).$$

The significance of (4) is clearer when expressed in terms of portfolio shares at expected future prices, obtained by multiplying each component of (4) by the respective expected future price  $\bar{p}_1$ . Denoting by  $a^*$  the portfolio shares chosen, by  $a^{**}$  the shares chosen if  $RRA = 1$ , and by  $e_1 = (1, 0, \dots, 0)$  the shares when only money claims are held, (4) becomes

$$(5) \quad a^* = (1/RRA)a^{**} + (1 - 1/RRA)e_1.$$

In other words, the individual always chooses a linear combination of the riskless portfolio and an efficient "mutual fund" of risky assets.

## II.2 Utility depends on goods consumed: Maximization of $E[V(p, M)]$

The vector of claims  $x$  maximizing the objective appropriate for our problem,  $E[V(p, M)] = E[V(p, x'p)]$ , is determined in a similar fashion. Expanding  $E[V(p, x'p)]$  in a Taylor Series about the expected outcome  $\bar{p}$  to the second order terms,

$$(6) \quad V(p, x'p) \approx V(\bar{p}, \bar{p}'x) + (p - \bar{p})'V_p + x'(p - \bar{p})V_M + \frac{1}{2}(p - \bar{p})'V_{pp}(p - \bar{p}) \\ + x'(p - \bar{p})(p - \bar{p})'V_{pM} + \frac{1}{2}x'(p - \bar{p})(p - \bar{p})'xV_{MM}$$

All partial derivatives are evaluated at the expected outcome  $(\bar{p}, x'\bar{p})$ . The expected utility achieved is the expected value of (6).

$$(7) \quad E[V(p, M)] \approx V(\bar{p}, x'\bar{p}) + \frac{1}{2}E[(p - \bar{p})'V_{pp}(p - \bar{p})] + x'\Sigma V_{pM} + \frac{1}{2}x'\Sigma x V_{MM}$$

Notice that the first and last terms of (7) correspond to the two terms of (1).

Equation (7) indicates the several ways in which price uncertainty affects expected future utility. The first term represents the utility attainable if the expected prices actually prevail. The second term reflects the effect of price uncertainty on the expected attainable utility if his future

money wealth available for consumption was fixed at  $\bar{M}$ . This term can be either negative or positive depending on whether or not the individual's aversion to risk and the uncertain income effects of price changes dominates the gains from being able to substitute unexpectedly low priced goods for high priced goods when prices are revealed.<sup>4</sup> The third term accounts for the correlation between the money value of the portfolio and the marginal utility of money wealth available for consumption spending. If the portfolio offers a higher than expected money return when prices are such that the marginal utility from consumption spending is high, and vice versa when  $V_M$  is low, then this correlation will be positive and add to expected utility. The last term is negative for risk-averse individuals, indicating the expected utility loss from having uncertain money wealth  $M = p'x$  even if consumption prices were fixed at  $\bar{p}$  with certainty. This term has provided the basis for traditional portfolio choice analysis.

Maximization of  $E[V(p,M)]$  subject to the current wealth constraint  $x'p^0 = m$  leads to the first order conditions

$$(8) \quad V_{M\bar{p}} + \Sigma V_{pM} + \Sigma x V_{MM} - \lambda p^0 = 0, \quad x'p^0 - m = 0.$$

To interpret (8) we make use of the relationships between the derivatives of an individual's indirect utility function  $V(p,M)$  and his consumption demand

<sup>4</sup> The fact that this term can be positive, even if individuals are risk-averse is the basis of the "Vaugh Paradox" that individuals may prefer variable prices to prices stabilized at their arithmetic means  $\bar{p}$ . The literature on this subject is summarized in Hanoch (1974a), and the paradox was resolved by recognizing that stabilization of prices at their arithmetic means is not a feasible policy a government might follow. We are concerned, however, not with a choice among alternative distributions of prices but with individual responses to given price uncertainty. In our problem, the effect of this term on individual behaviour is of an order lower than the other terms, as is seen by its disappearance upon differentiating to get the first order condition (8).

functions  $c(p,M)$ . Specifically, if  $c(p,M)$  is the consumption vector maximizing  $U(c)$  subject to the constraints  $M \geq p'c$ ,  $c \geq 0$ , and  $V(p,M) = U(c(p,M))$  is the level of utility thus attained, then the following relationships hold:<sup>5</sup>

$$\begin{aligned} V_p(p,M) &= -V_M(p,M) c(p,M) \\ V_{pM} &= -V_{MM}c - V_M c_M \end{aligned}$$

From this point on,  $c$  and  $c_M$  refer to the individual's consumption demand function and incremental consumption bundle (expansion path) respectively evaluated at the appropriate  $(p,M)$ . Substituting these relations into (8), defining  $RRA \equiv -MV_{MM}/V_M$  to be the relative risk-aversion index in this context and noting that  $\lambda = (1+r)V_M$  by the same argument as in section II.1, permits the first order conditions to be written as

$$(9) \quad (RRA) \Sigma [(x/\bar{M}) - (c/\bar{M}) + (c_M/RRA)] = \bar{p} - (1+r)p^0, \quad x'p^0 - m = 0$$

The vectors  $c$  and  $c_M$  in (9) are both evaluated at the expected outcome  $(\bar{p}, x'\bar{p})$ .

An explicit solution to (9) is messy, but the relationship between the solution to (9) and that of the traditional portfolio analysis is quite illuminating. Subtracting (3) from (9) tells us that

$$(10) \quad \Sigma \underbrace{[(x/\bar{M}) - (c/\bar{M}) + (c_M/RRA)] - (a/\bar{M})}_{z} = 0.$$

Hence the vector  $z$  must be an eigenvector of  $\Sigma$  associated with a 0 eigenvalue. One such eigenvector is  $e_1 = (1, 0, \dots, 0)$  since the first row of  $\Sigma$  consists of zeroes (ie:  $\Sigma e_1 = 0$ ). Hence one solution for  $x$  in terms of  $\underline{a}$  has the form

$$(11) \quad x/\bar{M} = (c/\bar{M}) - (c_M/RRA) + (a/\bar{M}) + ke_1$$

<sup>5</sup> These relationships between demand functions and indirect utility functions were noted by Hotelling (1932) and first formally derived by Roy (1943). Diewert (1973) admirably surveys the literature on duality theory with current and prospective applications to economics.

where  $k$  is some scalar such that  $x$  satisfies the wealth constraint  $m = x'p^0$ .

The appropriate value for  $k$  is determined by evaluating both sides of (11)

at the future expected prices  $\bar{p}$ , noting that  $x'\bar{p} = \bar{M}$ ,  $c'\bar{p} = \bar{M}$ ,  $c'_M\bar{p} = 1$ ,  $a'\bar{p} = \tilde{M}$  and  $\bar{p}_1 = 1$  :

$$(12) \quad \begin{aligned} \bar{p}'x/\bar{M} &= \bar{p}'c/\bar{M} - \bar{p}'c_M/RRA + \bar{p}'a/\tilde{M} + k\bar{p}'e_1 \\ 1 &= 1 - 1/RRA + 1 + k \\ k &= (1/RRA) - 1 \end{aligned}$$

Consequently one solution for  $x$  is

$$(13) \quad x/\bar{M} = a/\tilde{M} + c/\bar{M} - c_M/RRA - (1 - 1/RRA)e_1.$$

If  $e_1$  is the only eigenvector of  $\Sigma$  associated with a zero eigenvalue, then the portfolios  $a$  and  $x$  are both unique. However if there is some other claim or combination of claims whose future money value is certain, then these claims may be substituted for an equal value of money claims to produce a whole class of portfolios to which the individual is indifferent.

All solutions to the portfolio problem can be constructed in this manner, a point which we make use of in section IV.

The chosen portfolio is represented in share form, at the expected future prices, by multiplying each component of (13) by its respective  $\bar{p}_i$  and substituting equation (5) for  $a^*$ :

$$(14) \quad \begin{aligned} x^* &= a^* + c^* - c_M^*/RRA - (1 - 1/RRA)e_1 \\ &= (1/RRA)a^{**} + c^* - (1/RRA)c_M^* \end{aligned}$$

The vector  $x^*$  represents the portfolio shares chosen;  $a^{**}$  the shares chosen in the traditional analysis when  $RRA = 1$ ;  $c^*$  the respective consumption expenditure shares when the expected outcome prevails; and  $c_M^*$  the proportions in which incremental income is spent on the various goods at prices  $\bar{p}$  and income  $\bar{M}$ .

### II.3 Discussion of portfolio choice with price uncertainty:

The preceding section extended mean-variance portfolio analysis to situations in which consumption goods prices are uncertain. Before proceeding with the problem of choosing a standard of deferred payment, the significance of these results for traditional portfolio choice analysis should be pointed out. For simplicity we confine our attention to utility functions exhibiting constant relative risk aversion for all levels of income and prices. Such utility functions may or may not represent homothetic preferences, meaning that incremental income may or may not be spent in the same proportions as total income (ie:  $c^* = c_M^*$ )<sup>6</sup>

Repeating the portfolios chosen in a form suitable for comparison,

$$\begin{aligned}
 (15) \quad a^* &= (1/RRA)a^{**} + (1 - 1/RRA)e_1 \\
 x^* &= (1/RRA)(a^{**} + c^* - c_M^*) + (1 - 1/RRA)c^* \\
 x^* &= (1/RRA)a^{**} + (1 - 1/RRA)c^* \quad [\text{with homothetic preferences}]
 \end{aligned}$$

where  $a^{**}$  depends only on  $(\Sigma, p^0, \bar{p})$ ,  $c^*$  depends on  $(\bar{p}, \bar{M})$ , and  $c_M^*$  depends on  $(\bar{p}, \bar{M})$ . As expected,  $x^*$  generally differs from  $a^*$ . There are, however, two circumstances in which they coincide. First  $x^* = a^* = a^{**}$  if preferences are homothetic and the utility function is logarithmic in income ( $RRA = 1$ ). Second multiple solutions exist and the sets of optimal  $x^*$  and  $a^*$  coincide if the

<sup>6</sup> Hanoch (1974b) demonstrates that the class of utility functions which are both homothetic and have constant RRA must have the form (up to a linear transformation)  $U(c) = [f(c)]^{1-RRA}/(1 - RRA)$

where  $f(c)$  is any linear homogeneous function of the vector of consumption. The limiting case of  $RRA = 1$  is  $U(c) = \log[f(c)]$ . Interestingly, for that case the marginal utility of money wealth is independent of prices:

$$\partial \log[f(c(p, M))] / \partial M = f'_c c_M / f(c) = f'_c c / M f(c) = 1/M$$

The first equality follows from the chain rule, the second from homotheticity, and the third from Euler's Theorem since  $f$  is linear homogeneous. Thus  $V_{pM}$  is zero when  $RRA = 1$ . The total utility associated with  $M$  does, of course, depend on  $p$ .



prices of goods consumed in positive quantities are not uncertain, in other words if  $c_i^* > 0$  implies that  $\sigma_{ij} = \sigma_{ji} = 0$ . This is not immediately apparent from (15), but the spirit can be captured by thinking of money as being the only consumption good (ie:  $c^* = c_M^* = e_1$ ), and represents the situation that traditional portfolio analysis was intended to portray.

The influences of the degree of risk-aversion on the portfolios chosen are quite similar when preferences are homothetic. The portfolio shares  $a^*$  chosen when only money returns were considered represented a linear combination of the riskless portfolio  $e_1$  and the efficient risky portfolio  $a^{**}$ , with weights depending on RRA. The portfolio shares  $x^*$  chosen when consumption opportunities are considered, and preferences are homothetic, is the same linear combination of the expected consumption bundle  $c^*$  and the portfolio  $a^{**}$ . With many consumption goods there is no riskless portfolio, in the sense that future utility is known with certainty before prices are announced, but holding claims in the proportions  $c^*$  maximize the minimum utility that might be obtained and hence is the least risky portfolio available. Individuals extremely averse to risk thus demand future claims in the same proportions as they anticipate consuming the goods (ie:  $RRA \rightarrow \infty$  implies  $x^* \rightarrow c^*$ ). However individuals who are only mildly risk averse ( $RRA < 1$ ) actually take a "short" position on their expected consumption bundle. As previously noted, homotheticity with  $RRA = 1$  result in demands for claims  $x^* = a^{**}$  which are completely independent of the consumption demands.

The qualitative implications of traditional mean-variance analysis hold in our more general context only with appropriate restrictions on individual preferences. Since  $a^{**}$  depends solely on  $(\Sigma, \bar{p}, p^0)$ , conventional analysis implies individuals with identical beliefs about the distribution of future (asset) prices will hold risky assets in identical proportions, which could be supplied by a single "mutual fund" to be combined with money. If there is only one

consumption good, or if goods are consumed in the same proportions by all individuals, then an analogous result holds in our framework. A single "risky" mutual fund  $a^{**}$  and a single "safe" mutual fund  $c^*$  could satisfy all individuals' demands for future claims. But if there is more than one consumption good and individuals' expenditure shares differ, then a different "safe" mutual is required for each investor.<sup>7</sup>

Moreover if preferences are not homothetic, then even these individualized mutual funds do not suffice. With non-homothetic preferences the expenditure shares  $c^*$  generally vary with expected future wealth  $\bar{M}$ . If an individual's RRA were changed, leaving fixed his consumption preferences, then he might choose a portfolio associated with a different  $\bar{M}$  and hence change his expected future expenditure proportions  $c^*$ . He could not achieve an equivalent portfolio as a linear combination of  $a^{**}$  and his previous  $c^*$ . For similar reasons an individual's portfolio shares  $x^*$  are independent of his initial wealth  $m$ , even with constant RRA, only if his preferences are homothetic.

Finally, the traditionally chosen portfolio shares  $a^*$  were independent of the relative prices of assets. Only the relative rates of return mattered. Doubling the current price of some claim, its expected future price, and the standard deviation of its future price left  $a^{**}$ , and hence  $a^*$ , unchanged. But  $c^*$ ,  $c_M^*$  and hence  $x^*$  are generally sensitive to future prices. Only if expenditure shares are independent of future prices -- in other words if all price elasticities of demand are one -- are the portfolio shares  $x^*$  independent of relative prices and initial wealth for constant RRA utility functions.

<sup>7</sup> Fischer(1974) considers the problem of the existence of such mutual funds and arrives at similar conclusions.

The portfolio  $a^{**}$  appears throughout the analysis and deserves further comment. Claims held in the proportions  $a^{**}$  represent an efficient risky portfolio, meaning that no alternative portfolio offers the same expected money rate of return with lower variance of return. Portfolio  $a^{**}$  may consist of more than just risky assets; some money may be included if money is held when  $RRA = 1$ . However it appears somewhat paradoxical that a portfolio which is chosen when only money returns matter (ie:  $u(M) = \log M$ ) would also be chosen when money returns seem least relevant (for example when  $U(c) = \log(\text{oranges})$ , which is trivially homothetic and has  $RRA = 1$ ). In this latter case it is only the expected rate of return and variance of return in terms of oranges which should be relevant to the portfolio decision. For  $a^{**}$  to be optimal it must be true that no alternative portfolio offers the same expected orange rate of return with lower variance of return. In fact,  $a^{**}$  is the only portfolio which is efficient in the above sense for all possible choices of numeraire good.

The expected rate of return and variance of return to  $a^{**}$  is not, however, independent of the numeraire used; it may offer a higher expected rate of return in terms of one good than in terms of another. Fischer (1974) also makes this point, and it forms the basis of the distinction between nominal and "real" rates of return to holding money developed by Boonekamp (1974) and Eden (1974). The following example illustrates the seeming paradox.

Suppose there are only claims to money and one consumption good, oranges, so that "real" return can be unambiguously interpreted to mean expected return in units of oranges. Suppose that the current money prices for claims are 1 for both goods,  $p^0 = (1,1)$ , that the expected future price of oranges is also 1,  $\bar{p} = (1,1)$ , and that the variance of the future money price of oranges is  $\sigma^2$ . Both claims offer the same expected money rate of return, equal to 0, but the

claims on oranges have uncertain future money value. A risk-averse individual whose utility depended solely on money wealth would clearly put all his wealth into money claims; hence  $a^* = a^{**} = e_1$ .

Now let us look at the real returns to the two claims. A unit claim on oranges given the individual one orange in the future with certainty. The expected real rate of return to oranges is 0, and the variance of the real return is 0. A unit claim on money gives the individual  $1/p_2$  oranges when the future (money) price of oranges  $p_2$  is revealed. The expected real rate of return to money claims is thus

$$(16) \quad E[(1/p_2) - 1] = E[1 - (p_2-1) + (p_2-1)^2 - (p_2-1)^3 + \dots] - 1 \approx \sigma^2.$$

Similarly the variance of the real return to money claims is approximately  $\sigma^2$  for small variances. Money claims are clearly the risky asset in real terms, but also offer the highest expected real return. With a RRA = 1, an expected return just equal to the variance of return induces the individual to put his entire portfolio into the risky claims, money.<sup>8</sup> The optimal portfolio shares are again  $x^* = a^{**} = e_1$ , but for a different reason:  $a^{**}$  is a high-risk, high-return portfolio using oranges as numeraire, as opposed to a low-risk, low-return portfolio using money as numeraire.

<sup>8</sup> The usual interpretation of the index of relative risk-aversion (Pratt 1964) is twice the risk-premium per unit variance, with both measured as proportions of total wealth, required to induce an individual to accept a fair bet. But suppose that the individual can choose any convex combination of shares of the fair bet plus risk premium and the expected outcome with certainty, rather than being required to take all or none of the bet. The alternative interpretation suggested here is that the RRA is the risk premium per unit variance required to induce the individual to take all of the bet.

### III. Bilateral Loan Agreements

Turning now to our central problem, what standard of deferred payment should be chosen for a loan agreement? Harry Brown clearly stated the classical objective:<sup>9</sup>

"The intended purpose of a tabular standard for deferred contracts is to shield both borrowers and lenders from losses due to unforeseen changes in the value of the money unit. The ideal is that contracting parties should, in general, be able to count on receiving or paying back with interest an equivalent amount, or purchasing power over an equivalent amount, of certain goods."

Brown went on to point out, however, that if the "certain goods" in which the borrower and lender were interested differed -- for example if their anticipated consumption goods were different -- and if the relative prices of these goods might change over time, then this "ideal" was unobtainable. No tabular standard can guarantee that the same payment will be equivalent to two different collections of goods simultaneously. Some sort of compromise standard is required by which the borrower and lender share the risks associated with future price changes.

It must also be pointed out that the classical "ideal", even if attainable, is not necessarily desirable. We saw in section II that only an extremely risk-averse individual would choose to hold future claims in the same proportions as anticipated consumption. If future spot markets exist the individual is concerned not only with some lower bound on his welfare but also with being able to take best advantage of new information and opportunities as they arise.

The loan agreement takes the following form: The lender gives the borrower a quantity of current money, and the borrower transfers to the lender a vector  $t = (t_1, \dots, t_n)'$  of claims to future goods, including money.<sup>10</sup> This vector  $t$  is the standard of deferred payment for the loan. The borrower has

<sup>9</sup> Brown (1909), p. 714.

some future resources with which to repay the loan and finance future consumption, and the lender may have other sources of future wealth than the proceeds from the loan. Denote by the vectors  $y_L$  and  $y_B$  the claims to future goods held by the lender and borrower respectively prior to negotiating the loan. After negotiations the lender holds claims  $x_L = y_L + t$  and the borrower holds claims  $x_B = y_B - t$ .

We shall not attempt to characterize the actual bargaining process by which the loan agreement is determined. There are  $n+1$  parameters in the contract: the amount of current money received by the borrower, and the quantities of the  $n$  claims transferred to the lender. However we shall assume that the outcome is efficient in the sense that no alternative choice of standard of deferred payment  $t$  can increase the expected future utility of one agent without decreasing that of the other. This approach is also used by Shavell (1975) to investigate Pareto Optimal payment plans in which payments can be contingent on "states of nature" exogenous to the contracting agents.<sup>11</sup> Any strategic aspects of the bargaining process are confined to the distribution of current and future wealth between the borrower and lender. We do not require that this intertemporal distribution of wealth be efficient.

Except where otherwise indicated the following assumptions are maintained for the rest of this section: (i) the lender and borrower have identical

<sup>10</sup> Since we are ignoring any costs of exchanging unwanted goods, this is clearly equivalent to an agreement which specifies payment of a sufficient quantity of money to be able to purchase  $t$ . This latter indexing arrangement is the form one would expect any real-world agreement to take.

<sup>11</sup> Shavell permits payment plans contingent in any differentiable manner on a single dimensional continuum of states of nature, rather than just a linear function of the multidimensional continuum of prices. There is, however, just a single dimensional common argument (one consumption good) to both individuals' utility functions, implying, trivially, identical homothetic consumption preferences.

beliefs  $\bar{p}, \bar{\Sigma}$  about future price possibilities (ii) they have the same constant index of relative risk-aversion and (iii) they have homothetic consumption preferences although their expected expenditure shares  $c_L^*, c_B^*$  may differ.

Homothetic preferences imply that  $c_L^*$  and  $c_B^*$  are independent of the distribution of future wealth between the individuals, and hence are independent of those aspects of the loan agreement we shall not determine.

### III.1 Efficient allocations of claims:

If the final allocation of claims between the lender and borrower is efficient, then it can be characterized as a competitive equilibrium from some initial redistribution of claims between them. In other words, there exists some vector of prices  $p^0$ , and initial redistribution of claims  $\tilde{y}_L, \tilde{y}_B$  satisfying  $\tilde{y}_L + \tilde{y}_B = y_L + y_B \equiv Y$ , such that  $x_L$  maximizes  $E[V_L(p, p'x_L)]$  subject to the wealth constraint  $x_L'p^0 = \tilde{y}_L'p^0 = m_L$ ,  $x_B$  maximizes  $E[V_B(p, p'x_B)]$  subject to  $x_B'p^0 = \tilde{y}_B'p^0 = m_B$ , and the market clears (ie:  $x_B + x_L = Y$ ). This characterization of the loan outcome permits us to use the results of section II on portfolio choice in a competitive claims market to determine post-agreement portfolio shares  $x_L^*, x_B^*$  held by the lender and borrower.

From equation (15) the optimal portfolios for the lender and borrower behaving as price-takers must satisfy

$$(17) \quad x_L^* = (1/RRA)a^{**} + (1 - 1/RRA)c_L^*$$

$$x_B^* = (1/RRA)a^{**} + (1 - 1/RRA)c_B^*.$$

Since  $a^{**}$  depends only on expectations and prices  $p^0$  it is common to both agents' portfolios. The market clearing condition is put in expected share form by first writing it as

$$(18) \quad (\bar{M}_L/\bar{M})(x_L/\bar{M}_L) + (\bar{M}_B/\bar{M})(x_B/\bar{M}_B) = Y/\bar{M}$$

in which  $\bar{M}_L \equiv \bar{p}'x_L$ ,  $\bar{M}_B \equiv \bar{p}'x_B$  and  $\bar{M} \equiv \bar{p}'Y$  respectively designate the expected money value of claims held by the lender, borrower and combined. Multiplying each component of (18) by its respective  $\bar{p}_i$ , and denoting by  $\alpha_L \equiv \bar{M}_L/\bar{M}$  and  $\alpha_B \equiv \bar{M}_B/\bar{M}$  the expected future wealth shares of the lender and borrower, yields

$$(19) \quad \alpha_L x_L^* + \alpha_B x_B^* = Y^*.$$

Market clearing requires that their combined portfolio shares must be a weighted average of their individual portfolio shares, with the expected future wealth shares as weights.

Substitution of the asset demand functions of (17) into the market clearing condition (19) allows us to solve for the value of  $a^{**}$  at market clearing prices:

$$(20) \quad (1/RRA)a^{**} = Y^* - (1 - 1/RRA)(\alpha_L c_L^* + \alpha_B c_B^*) \\ = Y^* - (1 - 1/RRA) C^* .$$

$C^* \equiv \alpha_L c_L^* + \alpha_B c_B^*$  designates the combined expected expenditure shares of the lender and borrower. Finally, the post-agreement portfolio shares are obtained by substituting the value of  $(1/RRA)a^{**}$  at equilibrium prices back into (17):

$$(21) \quad x_L^* = Y^* + (1 - 1/RRA)(c_L^* - C^*) \\ x_B^* = Y^* + (1 - 1/RRA)(c_B^* - C^*).$$

Efficient loan agreements leave each agent holding claims in the same proportions, adjusted somewhat for differences in their anticipated expenditure shares according to their attitude towards risk.<sup>12</sup>

<sup>12</sup> The analysis with non-homothetic preferences proceeds in an identical manner yielding, in place of (21),

$$(21)' \quad x_L^* = Y^* + (c_L^* - C^*) - (1/RRA)(c_{LM}^* - C_M^*)$$

in which  $C_M^* \equiv \alpha_L c_{LM}^* + \alpha_B c_{BM}^*$ . The analogous expression holds for  $x_B^*$ .



### III.2 Remarks on efficient allocations of claims:

The most striking aspect of efficient allocations of claims, described by (21), is the lack of any direct dependence of  $x_L^*, x_B^*$  on the distribution of future prices  $\bar{p}, \Sigma$ . The proportions in which claims are finally held, and hence the proportions in which they are transferred from borrower to lender through a loan agreement, depend on endowments  $Y^*$ , the proportions in which goods are expected to be consumed  $c_L^*, c_B^*, C^*$ , and the attitude towards risk RRA. But  $x_L^*, x_B^*$  do not depend on the degree, or particular nature, of the future price uncertainty. The agreement appropriate when price risks are small is also appropriate when they are larger, and the fact that the price of any particular good may or may not be closely correlated with the price of the anticipated consumption bundles has no influence on its use as a standard of deferred payment.<sup>13</sup> The expected future prices  $\bar{p}$  only affect the loan agreement indirectly through their effect on the anticipated expenditure shares  $c_L^*$  and  $c_B^*$ . This does not mean, however, that the lender and borrower are indifferent to price levels or price uncertainty. It only means that those aspects of  $\bar{p}, \Sigma$  which render a particular claim attractive to the lender also render it attractive to the borrower.<sup>14</sup>

It is also apparent from (21) that the classical "ideal" of minimizing uncertainty about future consumption, even when feasible, is voluntarily chosen

<sup>13</sup> These results follow in this model since individuals are free to "index" their loan repayment on any and all goods. If, however, they were constrained to choose  $t$  as a linear combination of some limited set of mutual funds of claims (or published price indices), then the "second-best" solution would involve  $\Sigma$ , which could tell them how closely the various restricted mutual funds are correlated with the unconstrained choice of  $t$ .

<sup>14</sup> These observations carry over to the case where the borrower and lender have different indices of relative risk-aversion, as long as their subjective beliefs about the distribution of future prices are identical.

only when individuals are infinitely risk-averse.  $Y^*$  and  $C^*$  generally will differ since both individuals anticipate trading in future spot markets with other traders who have different endowments. But if  $Y^*$  did equal  $C^*$  then the lender and borrower hold between themselves claims to all the goods they would consume if the expected prices prevailed. In other words the classical "ideal" is feasible in the sense that  $x_L^* = c_L^*$  and  $x_B^* = c_B^*$  is consistent with  $Y^* = \alpha_L x_L^* + \alpha_B x_B^*$ . Each agent could be guaranteed his expected-price-outcome consumption bundle for all price outcomes. Yet this allocation of claims would not generally be Pareto optimal. With  $Y^* = C^*$ , (21) becomes

$$(22) \quad \begin{aligned} x_L^* &= (1/RRA)Y^* + (1 - 1/RRA)c_L^* \\ x_B^* &= (1/RRA)Y^* + (1 - 1/RRA)c_B^* \end{aligned}$$

and  $x_L^* = c_L^*$ ,  $x_B^* = c_B^*$  is chosen only if  $RRA = \infty$ . Otherwise, both individuals accept a higher risk in return for an increase in the expected quantities they might consume.<sup>15</sup>

A simple example, which does not depend on approximations or particulars of the price distribution, can illustrate this last point. Suppose there are

<sup>15</sup> If the two agents had different indices of relative risk-aversion, denoted by  $RRA_L$  and  $RRA_B$  respectively, then, defining  $RRA \equiv \alpha_B RRA_L + \alpha_L RRA_B$ , the appropriate portfolio shares are given by

$$(21)' \quad x_L^* = (RRA_B/RRA)Y^* + \alpha_B [(RRA_L - 1)/RRA]c_L^* + \alpha_L [(1 - RRA_B)/RRA]c_B^*$$

and the analogous expression for  $x_B^*$ . If we let  $RRA_L \rightarrow \infty$  while  $RRA_B$  stays fixed, then  $x_L^* \rightarrow c_L^*$ . The borrower accommodates the infinitely risk-averse lender by permitting him to hold his anticipated consumption bundle. But if we let  $RRA_B \rightarrow 0$  while  $RRA_L$  stays fixed, so that the lender faces a risk-neutral borrower, we find that

$$x_L^* \rightarrow (1/RRA_L)c_B^* + (1 - 1/RRA_L)c_L^* .$$

The result is of interest when contrasted with Shavell's (1975,p.4) result that a payment plan between a risk neutral and risk-averse transactor is efficient only if it leaves the risk-avorter with no uncertainty about his ultimate utility. The seeming conflict is resolved by noting that when there is but one consumption good then  $c_B^* = c_L^*$ , and it appears that  $x_L^* \rightarrow c_L^*$ . However the implications for efficient specification of deferred payments are substantially different.

three goods, and using the first good as numeraire,  $\bar{p}_1 = \bar{p}_2 = \bar{p}_3 = 1$ . Since the first good is numeraire,  $p_1 = 1$ , but let us also assume for computational convenience that  $p_2 + p_3 = 1$  for all price outcomes. In other words the prices of the two non-numeraire goods are perfectly negatively correlated. Let the utility functions for the borrower and lender be  $U_L = \log c_2$  and  $U_B = \log c_3$  respectively, so that the lender only derives utility from consuming good 2 and the borrower from good 3. Assume that they hold between them one unit claim to good 2 and one unit claim to good 3, so that  $x_L = (0, 1, 0)$  and  $x_B = (0, 0, 1)$  is a possible allocation of claims between them. With this allocation each consumes one unit of his good with certainty, no matter what actual  $p_2, p_3$  occur. Hence the expected future utility of each is  $\log(1) = 0$ . Now let us consider the expected future utilities if the claims are allocated  $x_L = (0, \frac{1}{2}, \frac{1}{2})$  and  $x_B = (0, \frac{1}{2}, \frac{1}{2})$ , which is also feasible. Since  $p_2 + p_3 = 1$  with probability one each individual will have one unit of wealth, in terms of the numeraire good, for all price outcomes. The lender will spend all his wealth on good 2, and the borrower on good 3, so the realized utilities will be  $U_L = \log(1/p_2)$  and  $U_B = \log(1/p_3)$ . The expected utilities faced are

$$(23) \quad \begin{aligned} E[U_L] &= E[-\log p_2] > -\log E[p_2] = 0 \\ E[U_B] &= E[-\log p_3] > -\log E[p_3] = 0. \end{aligned}$$

The inequalities follow from  $-\log$  being strictly convex and Jensen's inequality (assuming a positive variance for  $p_2$  and  $p_3$ ). Both individuals thus obtain strictly higher expected utility from the second allocation of claims, though both face more uncertainty about the quantities they will consume.

### III.3 The standard of deferred payment:

The relationships of (21) are next used to characterize the vector of claims transferred to the lender the loan agreement. That vector is the standard of deferred payment. Let  $\bar{b} \equiv \bar{p}'t$  be the expected future money value of the claims transferred, and  $\beta \equiv \bar{b}/\bar{M}$  be the fraction this represents of the expected value of the combined endowments of the two agents. Denote by  $\alpha_L^0 \equiv \bar{p}'y_L/\bar{M}$  the fraction of their combined expected future wealth held by the lender prior to the loan agreement, so that  $\alpha_L = \alpha_L^0 + \beta$ . The identity  $x_L = y_L + t$  can be written as

$$(24) \quad (\bar{M}_L/\bar{M})(x_L/\bar{M}_L) = [(\bar{M}_L - \bar{b})/\bar{M}][y_L/(\bar{M}_L - \bar{b})] + (\bar{b}/\bar{M})(t/\bar{b})$$

which, upon multiplying each component by its respective  $\bar{p}_1$ , gives

$$(25) \quad \alpha_L x_L^* = \alpha_L^0 y_L^* + \beta t^*.$$

The vector  $t^*$  is the proportions of the expected value of the transfer made up of claims on the respective goods. It shall also be referred to as the standard of deferred payment, though it is in share form. A  $t^*$  equal to  $e_1 = (1, 0, \dots, 0)$  corresponds to money alone being used as the standard of deferred payment.

Solving (25) for  $t^*$  results in

$$(26) \quad t^* = [(\alpha_L^0 + \beta)/\beta]x_L^* - [\alpha_L^0/\beta]y_L^*.$$

Substituting the expression for an efficient  $x_L^*$  from (21), remembering that  $c^* = \alpha_L c_L^* + \alpha_B c_B^* = (\alpha_L^0 + \beta)c_L^* + (1 - \alpha_L^0 - \beta)c_B^*$ , finally yields the following expression for  $t^*$ :

$$(27) \quad t^* = \alpha_L^0 [Y^* - y_L^* + (1 - 1/RRA)(c_L^* - c_L^*)(1 - \alpha_L^0)](1/\beta) + \\ [Y^* + (1 - 1/RRA)(c_L^* - c_B^*)(1 - 2\alpha_L^0)] - [(1 - 1/RRA)(c_L^* - c_B^*)]\beta$$

The above expression is hardly transparent. However it does point out that the

appropriate standard of deferred payment  $t^*$  generally depends not only on the prior endowments, consumption preferences and attitude towards risk of the two agents, but also on the size of the loan as reflected in  $\beta$ .

Our ultimate objective is to isolate those circumstances in which money alone would rationally be chosen as the standard of deferred payment. We first isolate the circumstances in which  $t^*$  is at least independent of the size of the loan.

Proposition 1: The standard of deferred payment chosen by a lender and borrower with homothetic preferences and identical beliefs about the distribution of future prices will be independent of the size of the loan negotiated if and only if<sup>16</sup>

- (a) they both hold claims to future goods in the same proportions prior to negotiating the loan agreement (ie:  $Y^* = y_L^* = y_B^*$ )
- (b) either (i) their utility functions are logarithmic in money income (ie:  $RRA = 1$ )  
or (ii) they would both consume goods in the same proportions at the expected future prices  
(ie:  $C^* = c_L^* = c_B^*$ ).

The proof of the proposition follows immediately from (27) by noting that the coefficients on  $\beta$  and  $1/\beta$  must be identically 0 if  $t^*$  is to be independent of  $\beta$ . Thus  $(1 - 1/RRA)(c_L^* - c_B^*) = 0$ , giving condition (b); and  $Y^* = y_L^*$  then follows from the coefficient on  $1/\beta$  also being 0. Notice that if the conditions of the proposition hold, then it follows that  $t^* = Y^*$  will be the standard of deferred payment.

The choice of standard of deferred payment may be illustrated graphically for a situation with two goods. Figure 1 presents an Edgeworth box with

<sup>16</sup> The "only if" part of the proposition follows only if  $t^*$  is unique, which occurs when  $\Sigma$  has rank  $n-1$ . The sense in which the proposition still holds when  $t^*$  is not unique is developed in section IV with the notion of equivalent vectors of claims. The underlying assumptions that there are no information, transaction or other costs considered by the lender and borrower are also, of course, presumed in the proposition.

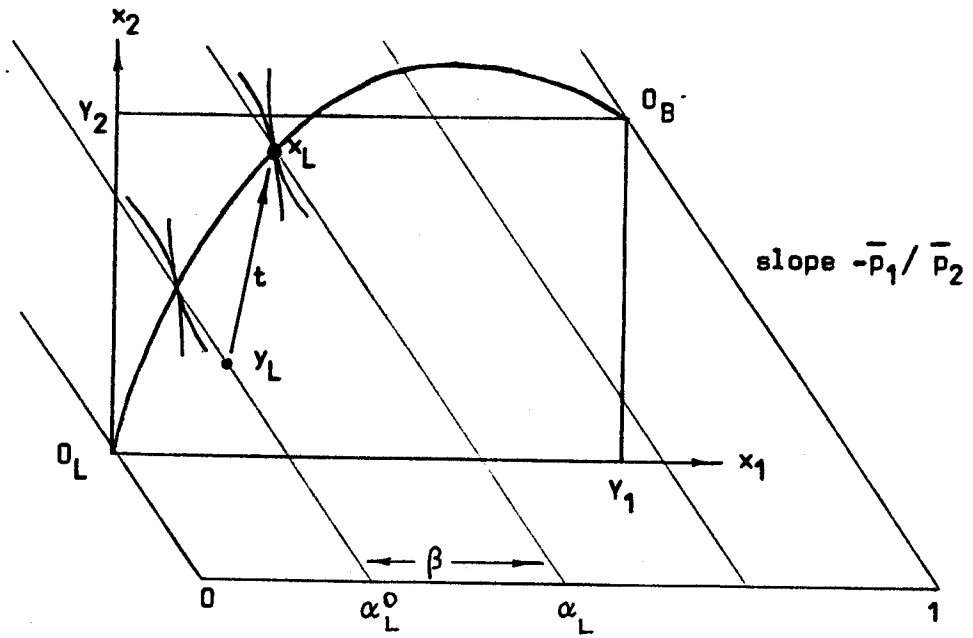


FIGURE 1

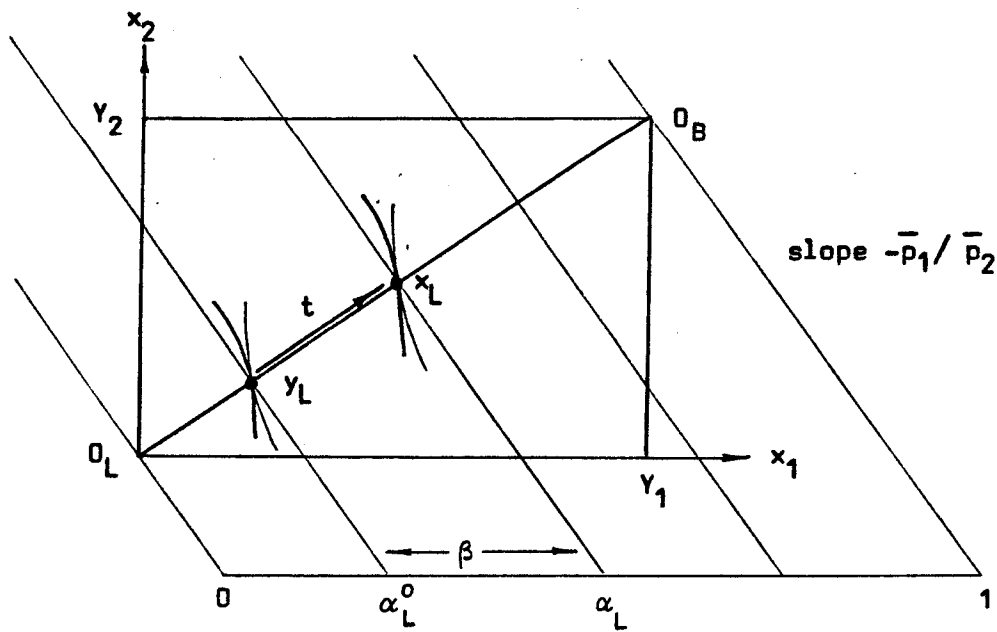


FIGURE 2

dimensions equalling the combined endowments of the borrower and lender  $Y = (Y_1, Y_2)$ , using the zero claims position of the lender as origin. The diagonal lines with slope  $-\bar{p}_1/\bar{p}_2$  represent combinations of claims with the same expected future money value. A point in the  $(x_1, x_2)$  plane represents an allocation of the claims  $Y$  between the lender and borrower, with the projection of that point along the diagonal onto the  $[0,1]$  interval below giving the fraction of the combined expected wealth held by the lender.  $O_L O_B$  is the locus of efficient allocations of claims satisfying (21), which are tangency points of the borrower's and lender's expected utility indifference curves.

The lender holds claims  $y_L$  and the borrower  $Y - y_L$  prior to negotiating the loan. In agreeing to the deferred payment, the borrower gives up claims  $t$ , leaving the lender with claims  $x_L = y_L + t$ . The loan agreement is efficient only if  $x_L$  lies on the efficiency locus. Notice that the standard of deferred payment, implicit in the slope of the  $t$  vector, generally varies with the size of the loan, reflected in the value of  $\beta$ .

Figure 2 depicts the circumstances in which Proposition 1 holds. Homothetic preferences, identical expectations, identical constant RRA, plus condition (b) imply that the efficiency locus is a straight line through  $O_L$  and  $O_B$ . Condition (a) states that the prior allocation of claims is on this line — no mutually beneficial pure futures contract can be made between the borrower and lender so that any loan will be a pure loan agreement. Efficient transfers of claims entail moving along the straight line  $O_L O_B$  with all claims being transferred in constant proportions. The standard of deferred payment is independent of the size of the loan.

Proposition 1 points to endowments as the major determinant of the standard of deferred payment. If the standard is to be independent of the size of the loan, then it must be proportional to the agents' combined endowments.

Thus if both are endowed solely with claims to apples, apples will be chosen as the standard of loan agreements; if they are endowed solely with claims to labour, labour will be the chosen standard. This occurs despite the possibility that neither may be interested in consuming apples or labour, and the future prices of both goods may be quite uncertain.

Similarly, if both agents were endowed solely with money claims then money alone would be chosen as the standard of deferred payment. But, ignoring for the moment the other requirements of the Proposition, is this likely to be the case in any real situation? Many future endowments of individuals are in the form of money claims: Included are the usual spectrum of assets with returns denominated in money, government transfer payments, leases, and orders for future production accepted at fixed money prices. In addition, labour services are often explicitly or implicitly sold into the future for a predictable money wage, contingent on continued employment;<sup>17</sup> and rents are implicitly fixed in money terms for some time period, contingent on continued tenancy. For many individuals the use of money as the standard for previously entered contracts could plausibly lead them to choose money as the standard for further contracts.

But we cannot explain the almost universal use of money as a standard by extending this argument to the whole economy. A contract increasing the money claims held by one individual decreases those held by another, leaving unchanged the total quantities held throughout the economy. If there are goods other than money available for future consumption then someone holds the claims to these goods. Such individuals, it appears, would not use money alone as the standard of deferred payment.

<sup>17</sup> See Azariadis (1975) for a discussion of implicit wage contracts.



#### IV. General Use of Money as a Standard of Deferred Payment

The previous section arrived at a rather negative conclusion: If there is more than one good in the economy then there generally can be no single good which would universally be chosen as a standard of deferred payment, when the only consideration is the efficient allocation of price risks. For all pairs of individuals to choose the same standard, all must initially hold claims in the same proportions as aggregate future endowments. These proportions could then be used as a universal standard, but involve more than one good.

Must we then conclude that frequent use of money as a standard can only be explained by appealing to factors other than the allocation of risk, such as information and transaction costs, and presume that in the absence of these other factors we would never observe money alone being used as a standard? Not necessarily. It is true that claims proportionate to aggregate endowments would be the only possible universal standard in this framework. But for some probability distributions over future prices claims to just one good can be, in a sense to be defined below, equivalent to a share of aggregate endowments. The line of reasoning we shall follow is to suggest that a claim on the medium of exchange might be equivalent to a claim on an "average" bundle of commodities. If money claims represent "generalized purchasing power" in that sense, then its used as a universal standard can reflect an efficient allocation of price risks.

##### IV.1 Equivalent collections of claims:

We define two vectors of claims  $x$  and  $y$  to be equivalent with respect to the probability distribution of on future prices,  $x \sim y$ , if they will have the same future money value with probability one (ie:  $p'x = p'y$ ). This requires that  $\bar{p}'(x-y) = 0$  and  $\Sigma(x-y) = 0$ , implying that individuals are indifferent

between equivalent portfolios of claims. Two vectors of claims are proportionately equivalent,  $x^* \sim y^*$ , if there exists some scalar  $k \neq 0$  such that  $x \sim ky$ . This equivalence relation divides up the space of possible portfolios into equivalence classes. For most probability distributions there are as many equivalence classes as there are portfolios:  $x \sim y$  implies  $x = y$ , indicating that no two distinct portfolios have the same expected money value and variance of money value. However for some distributions the set of equivalence classes is much smaller than the set of all portfolios, indicating that some prices always move together or in some fixed linear relation to each other.

Only the equivalence class of a vector of claims is relevant for the expected utility maximizing individual's choice of a portfolio, and hence for our entire analysis. In particular, equation (27) for  $t^*$  only defines the equivalence class to which the standard of deferred payment must belong. Similarly Proposition 1 only requires that the borrower and lender initially hold claims in equivalent proportions in place of condition (a),  $Y^* \sim y_L^* \sim y_B^*$ , and that they consume goods in equivalent proportions for condition (b.ii),  $C^* \sim c_L^* \sim c_B^*$ .<sup>18</sup> The Proposition then allows us to conclude that the standard of deferred payment must be equivalent to their combined endowments,  $t^* \sim Y^*$ , which is somewhat weaker than  $t^* = Y^*$ .

Any universally used standard of deferred payment need only be of the same equivalence class as aggregate endowments. Suppose there are  $H$  individuals

<sup>18</sup> The homotheticity of preferences also need only be with respect to equivalence classes of commodities. For example, if the prices of all automobiles move up and down together (belong to the same equivalence class), then we only require individuals to expand their expenditures on automobiles as a class in proportion to increases in their money income. This may involve purchasing more automobiles, more luxurious makes, or any combination thereof.

in the economy, each with an initial endowment of claims  $y_h$ ,  $h = 1, \dots, H$ .

Denote by  $W = (W_1, \dots, W_n) = y_1 + \dots + y_H$  the aggregate endowments of claims, and by  $W^*$  the aggregate endowments in expected share form. A single good, say the  $i^{\text{th}}$ , could be universally used as a standard of deferred payment only if  $e_i = t^* \sim W^*$  in addition to  $y_h^* \sim W^*$  for all  $h$  and the restrictions on preferences being met. From the definition of equivalence,  $e_i \sim W^*$  requires that for some scalar  $k$ ,  $kp'e_i = p'W$  for all price outcomes. In other words

$$(28) \quad \frac{p_1 W_1 + p_2 W_2 + \dots + p_n W_n}{p_i} = k$$

is non-random despite the fact that individual prices may be random. The expression in (28) may be thought of as the general price level in terms of the  $i^{\text{th}}$  good. It is a weighted sum of the prices of each good in units of good  $i$ , using aggregate quantities as weights. Consequently if there is any good in terms of which the general price level is known with certainty then that good could possibly be used as a universal standard of deferred payment.

#### IV.2 Money is the medium of exchange:

There is no a priori reason why the general price level in terms of an arbitrarily chosen good should be certain when aggregate future consumption demands are uncertain. But for the good used as medium of exchange, (28) can be put in a more suggestive form. Since money, good 1, is the medium of exchange, (28) with  $i = 1$  can be written as

$$(29) \quad \frac{p_2 W_2 + p_3 W_3 + \dots + p_n W_n}{p_1 W_1} = (k/W_1) - 1 \equiv v.$$

If the aggregate claims to money currently held represent the future money stock, then (29) is Irving Fisher's (1922) "equation of exchange." The scalar  $v$  is the velocity of circulation of the medium of exchange. The condition that

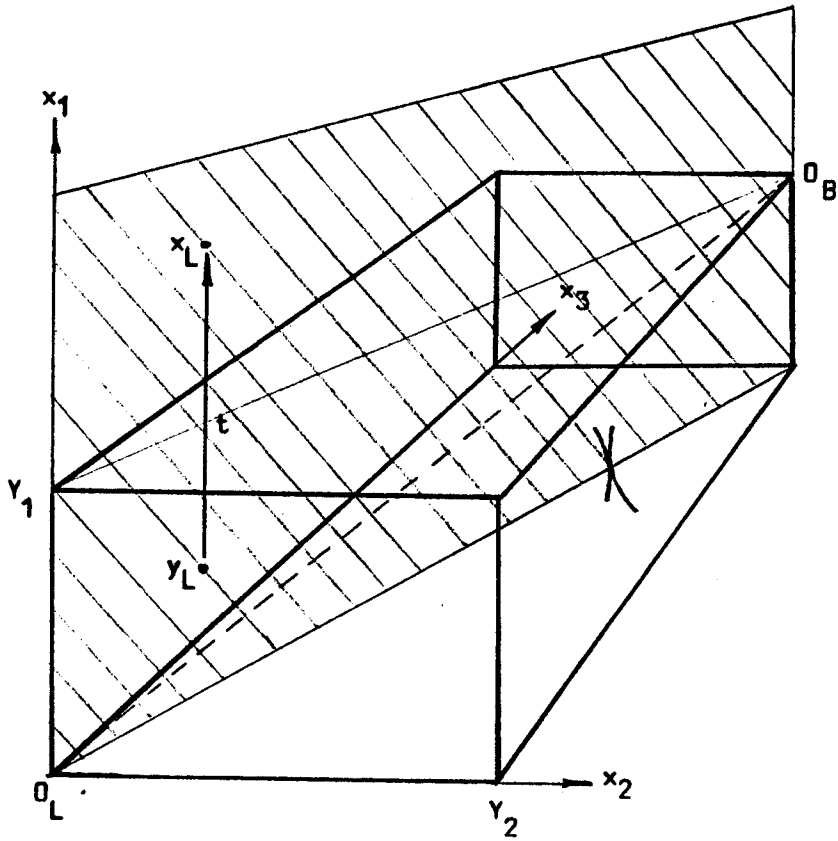


FIGURE 3

vertical plane through  $O_L O_B$ . The diagonal lines on this plane are classes of equivalent portfolios, or lines along which the agents' indifference curves are tangent to each other. Condition (a) states that the initial allocation  $y_L$  lies on this plane -- any loan contemplated is a pure loan agreement. Any efficient reallocation of expected future wealth can thus be achieved by simply transferring money claims from the borrower to lender, moving vertically.

The standard of deferred payment is not unique. Individuals would be indifferent to using  $t^* = e_1$ ,  $t^* = W^*$ , or any combination of these two standards. However standards other than these, such as good 2 or good 3 alone, are not efficient. Moreover individuals are not using money because its use minimizes risk about the quantities of goods they might consume. Neither individual may consume goods in the economy-wide "average" proportions. What the use of money as a standard does accomplish is to eliminate each individual's risk about the share of aggregate wealth he will command when future prices are revealed.

From a historical viewpoint it should be pointed out that Irving Fisher concluded on pragmatic grounds that an index essentially identical to  $W^*$  was the best standard of deferred payment:<sup>19</sup>

"It is clear that no one kind of goods is a fair standard. An index number intended to serve as a standard of deferred payment must have a broad basis.

To cut these Gordian knots, perhaps the best and most practical scheme is that which has been used in the explanation of  $P$  in our equation of exchange, an index number in which every article and service is weighted according to the value of it exchanged at base prices in the year whose level of prices it is desired to find."

Our analysis, based on efficient risk-sharing, would have claims transferred in the proportions  $W^*$ , which differs from Fisher only to the extent that some money claims should be included in the transfer.

<sup>19</sup> Fisher (1922), pp. 213, 217. A more complete discussion of the relationship between Fisher's index and  $W^*$  is contained in Jones (1975).

## V. Conclusion

Our objective was to find some explanation for a commonly observed phenomenon: Individuals frequently choose the medium of exchange as a standard of deferred payment, despite the fact that future prices of the goods they will ultimately consume are often uncertain. In terms of this objective the approach taken was only partially satisfactory. On one hand, if the use of a good as medium of exchange results in the general price level in units of that good being more predictable than in units of other goods, so that claims on that good represent claims on an "average" basket of goods, then use of that good as the standard of deferred payment could reflect efficient sharing of price risks between borrowers and lenders. On the other hand, the conditions on attitudes toward risk, expectations, preferences and the prior allocation of risk required for this to be the case are extremely restrictive, and appear unlikely to be met in real situations.

Several important, and perhaps counter-intuitive, points became apparent in the analysis which warrant repeating. First, the fact that an individual is risk-averse when future consumption prices are uncertain does not necessarily imply he will choose portfolios whose value will be positively correlated with his anticipated consumption bundle. Only the very risk-averse would do so. Second, fixing the "real" value of a loan repayment does not necessarily minimize the risks faced by borrower and lender when their future endowments are not fixed in "real" terms. It only shifts all the risk to the borrower. Finally, choosing an optimal standard of deferred payment is not simply a matter of reducing consumption level uncertainty to the borrower and lender, but involves subtle tradeoffs between risk and return made by both parties.

Many factors other than price uncertainty undoubtedly do influence the choice of standard of deferred payment, and the cost of negotiating complicated

rather than simple standards is not unimportant. Most of these factors, such as the transaction and information costs referred to earlier, could be expected to bias the choice in favour of the medium of exchange. However the difficulties associated with non-monetary standards can always be overcome at some cost. Our results merely give conditions under which achieving the desired allocation of risk does not conflict with the choice dictated by these other considerations. To the extent that the required conditions are not satisfied, some trade-off of the various costs will have to be made. If the costs associated with non-monetary standards are lumpy in nature then our conditions need only be approximately satisfied before money alone would be chosen as the standard of deferred payment.

## Table of Notation

All vectors are column vectors. The vector  $x'$  indicates the transpose of  $x$ . If  $c(p, M)$  is a function of  $p$  and  $M$  then  $c_p, c_M$  indicate the partial derivatives of  $c$  with respect to  $p$  and  $M$ . If  $p$  is a vector, then  $c_p$  is the vector of respective first partial derivatives of  $c$  and  $c_{pp}$  is the matrix of second order partial derivatives.

## Section II:

$m$	wealth constraint on current money value of the portfolio
$x, a$	vectors of claims on future goods
$p^0$	vector of current money prices for future claims on goods
$p$	random vector of future money prices of goods
$\bar{p}$	expected future money prices $E[p]$ ( $p_1 \equiv \bar{p}_1 \equiv 1$ )
$\Sigma$	covariance matrix of future money prices ( $\sigma_{i1} = \sigma_{1i} = 0$ )
$M$	future money value $p'x$ of portfolio of claims $x$
$\bar{M}$	expected future money value of portfolio $x$
$\tilde{M}$	expected future money value of portfolio $a$
$u(M)$	utility function defined on future money wealth alone
$U(c)$	utility function defined on future consumption vector $c$
$V(p, M)$	indirect utility function $\text{Max}_c \{U(c) \text{ s.t. } M \geq p'c \text{ and } c \geq 0\}$
$e_i$	vector with 1 in the $i^{\text{th}}$ position and 0 elsewhere
$r$	the risk-free money interest rate $(1/p_1^0) - 1$
RRA	index of relative risk aversion $-Mu_{MM}/u_M$ or $-MV_{MM}/V_M$
$c(p, M)$	vector of consumption demands at $p, M$ (abbreviated $c$ )
*	$x^*$ is the vector of quantities $x$ in share form at expected future prices, obtained by multiplying the elements of $x$ by the corresponding elements of $\bar{p}$ then normalizing so that the elements of $x^*$ sum to 1.
$a^{**}$	the value of $a^*$ chosen when RRA = 1 ie: when maximizing $E[\log(p'x)]$



## Section III:

$t$	vector of claims transferred from borrower to lender
$y_L, y_B$	claims held by lender and borrower prior to loan agreement
$Y$	$y_L + y_B$
$x_L, x_B$	claims held by lender and borrower after loan agreement
$c_L, c_B$	consumption demand functions of lender and borrower evaluated at the expected price outcome
$C$	$c_L + c_B$
$c_{LM}, c_{BM}$	incremental consumption bundles $\partial c_L / \partial M_L, \partial c_B / \partial M_B$ at the expected price outcome
$\bar{M}_L, \bar{M}_B$	expected future money wealths of the lender and borrower $\bar{p}'x_L, \bar{p}'x_B$
$\bar{M}$	$\bar{M}_L + \bar{M}_B$
$\alpha_L, \alpha_B$	expected future wealth shares of the lender and borrower $\bar{M}_L / \bar{M}, \bar{M}_B / \bar{M}$
$\bar{b}$	expected future value of claims transferred $\bar{p}'t$
$\beta$	share of combined expected wealth transferred $\bar{b} / \bar{M}$
$\alpha_L^0$	share of combined expected wealth held by lender prior to loan agreement $\bar{p}'y_L / \bar{M}$

## Section IV:

$y_h$	initial endowment of claims of individual $h = 1, \dots, H$
$W$	aggregate endowment of claims to future goods $y_1 + \dots + y_H$
$v$	velocity of circulation of the medium of exchange (money) defined by $vW_1 = p_2 W_2 + \dots + p_n W_n$

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