

MONEY, CREDIT AND THE TIMING OF TRANSACTIONS

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Recent discussions of the logistics of exchange have directed the attention of economists to numerous weaknesses in the microfoundations of economic theory, including specifically the lack of an acceptable theory of the timing of individual exchange transactions.¹ The present literature on timing phenomena comprises at most a handful of papers, none of which deals formally with anything more complicated than the choice of a trading frequency for a single commodity.² Our purpose in this paper is to investigate a more general class of models in which individual economic agents choose not only the frequencies but also the time phasing of sale and purchase transactions.

The conceptual basis for our argument is provided by an inventory-theoretic model that may be regarded formally as an extension of the classic Baumol-Tobin theory of the transactions demand for money. Our analysis shows, however, that the demands for trade inventories (goods as well as money) depend upon a discontinuous function of the relative frequency of sales and purchases. This novel result forces us to reconsider a variety of issues, e.g., the conditions under which individual traders will choose to hold money in positive amounts, the nature of potential gains from paying competitive interest on money, the economics of hyper-inflation, the effects upon individual demands for money of the availability of trade credit and bonds, and, more generally, the coordination of trading activities in economies characterized by individual diversities in the timing of transactions.

I. THE BASIC MODEL

Our object in Part I is to characterize the stationary equilibrium behavior of a representative trader in an idealized exchange economy where goods can be traded in organized markets only in exchange for units of a pure-stock money commodity.³ In keeping with familiar doctrine, we suppose that on each purchase and sale the trader incurs a set-up cost that is independent of the quantity traded per transaction. As a consequence, the trader will choose to execute trades only at discrete points rather than continuously in time and will hold positive stocks of both goods and money between successive sale and purchase dates. The holding of trade inventories is presumed to impose other costs on the trader, reflecting foregone consumption opportunities and the use of resources to maintain storage facilities. Thus to maximize long run real income the trader must strike a balance between frequent transactions with low holding but high trading costs and infrequent transactions with low trading but high holding costs.

A. Fundamental Assumptions and Ideas

Proceeding to specifics, we begin by considering an individual trader who produces and sells units of just one stock-flow good (S) and who purchases and consumes units of just one other stock-flow good (D). To avoid superfluous notation, we assume that the money prices of both goods are equal to unity, and we use generic symbols S, D and M, respectively, to denote measurable quantities of (S), (D) and the money commodity (M). For similar reasons (also to avoid dealing with feedbacks of trading and holding costs upon the trader's stationary-state level of consumption) we treat all costs as subjective (foregone leisure or consumption) or as

charges that are incurred "outside" the model. We further suppose that the trader's rate of production of (S) is predetermined at the constant level of y units per unit of time, and that the trader holds no precautionary stocks of goods or money (that is to say, a purchase is made only when the trader's stock of (D) has just been exhausted, and a sale is made only when the trader wishes to dispose of the whole of his accumulated stock of (S)). Finally, as conditions for stationary equilibrium, we require that the trader consume (D) at the constant rate y , sell (S) in constant lots of size S at uniform time intervals S/y , and purchase (D) in lots of constant size D at uniform time intervals D/y .

With no loss in generality we may suppose that a sale occurs at date 0 and that the first subsequent (or simultaneous) purchase occurs at date $m \geq 0$. On these assumptions the trader's holdings of (S) and (D) at any date $t \geq 0$ are defined by the time paths:

$$(1) \quad S(t) = yt - [yt/S]S \quad (S(t) \geq 0)$$

and

$$(2) \quad D(t) = D + [y(t-m)/D]D - y(t-m) \quad (D(t) \geq 0)$$

where the symbol $[x]$ denotes the greatest positive integer not greater than the real number x (thus the bracketed expressions in (1) and (2) denote the numbers of sale and purchase transactions that occur between date 0 and date t).

Since the choice of an origin to our time scale is arbitrary we may also suppose that m equals the minimum distance between any contiguous sale and purchase. Obviously m cannot in any circumstances be greater than half the interval between two successive sales or two successive

purchases. In most cases, however, the maximum interval between some purchase and some sale will be considerably shorter. For example, if sales occur every day (24 hours) and purchases occur every seven hours, then at least every seven days a purchase must occur within half an hour of a sale. More generally, it can be shown (see Proposition 7 in the mathematical appendix) that the upper bound to m is given by

$$(3) \quad m \leq G(S,D)/2y = m^*,$$

where $G(S,D)$ -- to which we shall refer subsequently as the divisor function -- denotes the greatest common rational divisor of S and D if S/D is rational and $G=0$ otherwise.⁴ We then define the phase variable θ as

$$\theta \equiv \begin{cases} m/m^* & \text{if } S/D \text{ is rational} \\ 0, & \text{otherwise.} \end{cases} \quad (0 \leq \theta \leq 1)$$

On this definition, the placement of purchases relative to sales over the entire time interval $-\infty < t < \infty$ is determined unambiguously by the trader's choice of a value for θ .

On each sale date the trader receives S units of money and on each purchase date the trader gives up D units of money, so the time path of money holdings is given by:

$$(5) \quad M(t) = M_0 + [yt/S]S - [y(t-\theta m^*)/D + 1]D, \quad (M(t) \geq 0)$$

where M_0 denotes the trader's holdings of money balances after the first sale but before the first purchase.

By hypothesis, marketing costs incurred by the trader are of two types: trading costs associated with the execution of sale and purchase transactions; holding costs associated with the possession of positive average inventories of goods and money. To arrive at a formal model of the trader's decision problem, both types of cost have to be related explicitly to the decision variables S , D and θ .

Dealing first with trading costs, we suppose that these consist of two components, one determined by the sale and purchase frequencies y/S and y/D , the other determined by the time sequencing of sales and purchases. We shall refer to the first of these components as average set-up cost and denote it by C_F . By definition,

$$(6) \quad C_F = ay/D + by/S, \quad (C_F \geq 0)$$

where the constants a and b -- both measured in units of utility -- denote, respectively, fixed costs (foregone leisure) associated with buying (D) and selling (S). The second component we shall call average bunching cost and denote by C_B . The existence of bunching costs is predicated on the assumption that the effort (or other real resources) required to execute any given sale will be greater the more closely that

sale coincides in time with a purchase transaction, and vice versa. We defer detailed discussion of the bunching cost function until later (Section I.C); here we merely observe that the synchronization of sales and purchases at various dates is determined unambiguously by the values of S , D and θ , from which it follows that average bunching cost is definable as a function of the general form

$$(7) \quad C_B = f(S, D, \theta).$$

Average trading cost is therefore given by

$$(8) \quad C_T = C_F + C_B = ay/D + by/S + f(S, D, \theta).$$

Dealing next with holding costs, we assume that these also consist of two components: waiting costs associated with the activity of not consuming average holdings of goods and money; storage costs associated with the physical possession of such inventories. Denoting average holdings of (S) , (D) and (M) , respectively, by \bar{S} , \bar{D} , and \bar{M} , we define the first of these as

$$(9) \quad C_W = \rho(\bar{D} + \bar{S} + \bar{M}),$$

where ρ represents the trader's subjective rate of time discount. The second component, average storage cost, we define similarly as

$$(10) \quad C_S = \alpha\bar{D} + \beta\bar{S} + \gamma\bar{M}$$

where the constants α , β , γ , all expressed as rates per unit of time, denote utility equivalents of real resource costs incurred in holding one unit of (D), (S) or (M) for one unit of time. Combining (9) and (10), we write average holding cost as

$$(11) \quad C_H = \rho(\bar{D} + \bar{S} + \bar{M}) + \alpha\bar{D} + \beta\bar{S} + \gamma\bar{M}.$$

The values of the variables \bar{S} and \bar{D} in (11) are defined by the time paths (1) and (2) as simple functions of D and S, namely:

$$(12) \quad \bar{S} = S/2; \quad \bar{D} = D/2.$$

The value of \bar{M} cannot be determined so directly. In principle, however, \bar{M} is determined by (5) as a function of S, D and θ , a fact that we symbolize by writing

$$(13) \quad \bar{M} = F(S, D, \theta).$$

The properties of the relation (13), which we shall call the finance function, will be established in the section that follows. Here it suffices to note that the function exists, for then it follows from (8), (11), (12), and the assumption of real-income maximization that the trader's decision problem may be expressed formally as

$$(14) \quad \text{Minimize:} \quad (\rho + \alpha)(D/2) + (\rho + \beta)(S/2) + (\rho + \gamma)F(S, D, \theta) \\ \{S, D \geq 0; 1 \geq \theta \geq 0\} \\ + ay/D + by/S + f(S, D, \theta).$$

This formulation is conventional in form, but it generalizes earlier work in four significant respects. First, we allow the trader to choose the frequency of sales as well as the frequency of purchases. The usual procedure in the past has been to regard the frequency of sales (the income period) as predetermined.⁵ Second, we impose no a priori restrictions on the relative trading frequency, S/D . The standard approach in earlier work with similar models has been to suppose that S/D can take on only integer values.⁶ Third, we allow the trader to choose the time phasing of purchases and sales. The procedure followed by previous writers invariably has been to assume that the time paths of $S(t)$ and $D(t)$ start from a date $t = 0$ at which a sale and purchase coincide, implying $\theta = 0$.⁷ The phenomenon of time phasing has thus been ignored. Fourth and finally, we permit the trader's choice of a value for the phase variable θ partially to determine average trading cost through its effect on bunching costs. Earlier theoretical work completely overlooks this possibility.

None of these generalizations is radical; indeed, each is suggested naturally by previous work or by the internal logic of our model. Yet in combination these generalizations yield a model of the timing of transactions with implications that differ fundamentally from those of any model considered in the existing literature. The explanation lies in the as yet unspecified characteristics of the functions $F(S,D,\theta)$ and $f(S,D,\theta)$, for it is only the inclusion of these functions in (14) that clearly distinguishes our formulation of the trader's decision problem from earlier treatments. Our next task, therefore, is to establish the properties of these two functional relationships. We shall then present an explicit solution of the decision problem (14) and go on to explore its implications.

B. The Finance Function

We assumed earlier that the trader holds money only for transactions purposes. From this assumption and our requirements for stationarity, it follows that the value of M_0 at the origin of the money time path (5) must be such that the trader is just able to execute all subsequent purchases and sales without ever violating the feasibility condition $M(t) \geq 0$. We denote this minimally sufficient value of M_0 by \bar{M} and call the corresponding time path of $M(t)$ the required path of money balances. Obviously the average value of the trader's holdings of money along the required path, namely,

$$\bar{M} = \lim_{t \rightarrow \infty} (1/t) \int_{\tau=0}^{\tau=t} M(\tau) d\tau, \quad (M_0 = \bar{M})$$

is determined uniquely by the values of S , D and θ . But this functional relationship is just the finance function $F(S,D,\theta)$. To establish the exact form of the function $F(S,D,\theta)$, therefore, we have only to determine the value of \bar{M} as an explicit function of S , D and θ .

To do this, we temporarily ignore the feasibility requirement $M(t) \geq 0$, and proceed provisionally on the assumption that the trader starts just before the origin with zero money balances. Then this virtual path of money balances is defined by (5) with $M_0 = S$. The virtual path will be as

dashed line in Figure 1(c). It will reach the horizontal axis at some date and, unless S/D is an integer, some portion of the path will lie below the horizontal axis. Let us denote the minimum value of $M(t)$ along the virtual path by $-V$ ($V \geq 0$). The required path of money balances can then be constructed by raising the entire virtual path by the amount V .

P. 9a

INVENTORY TIME PATHS WHEN $S=3$, $D=2$, $y=1$, $\theta=1/2$, $m^*=1/2$

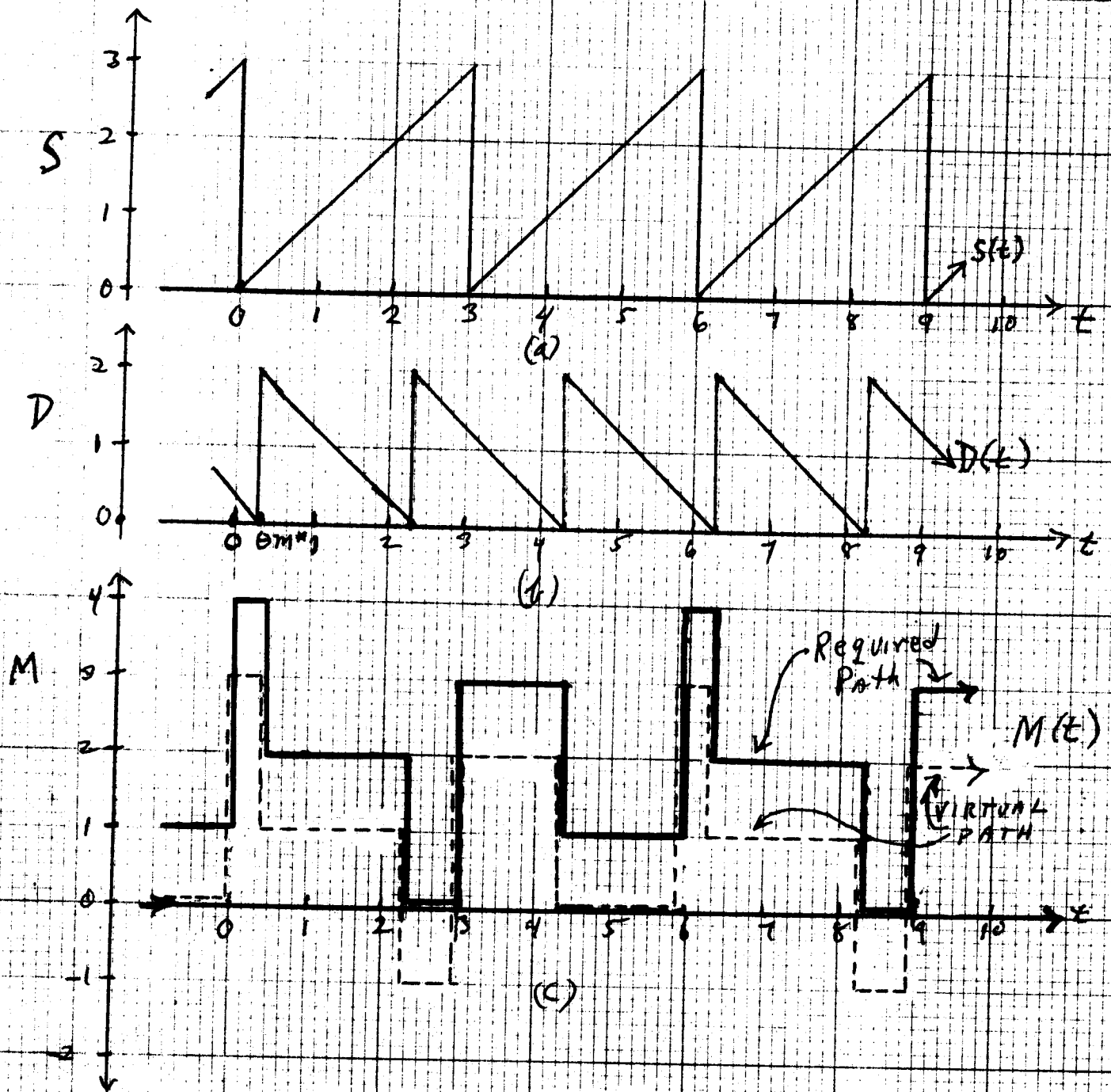


FIGURE 1

Hence the required path is given by (5) with M_0 set equal to $\tilde{M} = S + V$.

To determine $-V$ analytically as a function of S and D , we first observe that the minimum value of $M(t)$ along the virtual path defined by (5) must necessarily occur at some purchase date. Supposing that the first purchase on or after $t = 0$ occurs at time $t_0 = \theta m^*$, it is easy to verify (see the mathematical appendix, proposition 2) that virtual holdings of money balances following the u^{th} purchase of (D) at time t_u are given in general by

$$(15) \quad M(t_u) = S - D - R(uD|S),$$

where $R(uD|S)$ is the remainder when S is divided into uD . Since the minimum of $M(t_u)$ occurs where $S - R(uD|S)$ is minimal, and since this minimal value is $G(S,D)$ (see proposition 3 in the mathematical appendix), it follows that

$$\begin{aligned} -V &= \text{Min}_{\{u\}} M(t_u) = S - D - (S - G(S,D)) \\ &= -D + G(S,D), \end{aligned}$$

which may be rewritten as

$$(16) \quad V = D - G(S,D).^8$$

Immediately after the first sale, but before the first purchase, the trader's inventories will consist of the quantity of money $\tilde{M} = S + D - G(S,D)$ together with the quantity of the consumption good $\theta y m^*$. Also, our stationarity conditions imply that the total value of the trader's inventories must be constant through time. It follows that

$$(17) \quad \bar{M} + \bar{S} + \bar{D} = S + D - G(S,D) + \theta y m^*,$$

from which we obtain:

$$(18) \quad \bar{M} = \bar{S} + \bar{D} - G(S,D) + \theta y m^*$$

as the explicit representation of the finance function $F(S,D,\theta)$.

The characteristics of the finance function (18) obviously depend in a crucial way on those of the divisor function $G(S,D)$. Four properties of the latter function are of particular significance for the present analysis:

- (P.1) $G(\lambda S, \lambda D) = \lambda G(S, D)$ for any $\lambda > 0$; i.e., the function G is homogeneous of degree one in both arguments;
- (P.2) $0 \leq G(S, D) \leq \text{Min}(S, D)$;
- (P.3) $G(S, D) = \begin{cases} 0 & \text{if } S/D \text{ is irrational,} \\ \text{Min}(S, D) & \text{if } S/D \text{ or } D/S \text{ is an integer;} \end{cases}$
- (P.4) $G(S, D)$ has jump discontinuities everywhere that S/D is rational; more precisely, $G(S, D) > 0$ for such points, but if any sequence $\{S^q, D^q\}$ approaches such a point, and $S^q/D^q \neq S/D$ for all q , then $G(S^q, D^q)$ approaches zero.

The first three properties are easy to verify. The fourth is not evident, but a simple proof is given in the mathematical appendix (proposition 4).

As indicated by (P.3) and (P.4), the divisor function is extremely ill-behaved; more precisely, it is an everywhere discontinuous function of the trading ratio S/D . Accordingly, the finance function is also an everywhere discontinuous function of S/D . This is indicated in Figure 2, which illustrates how the value of the finance function varies as D is varied with S held constant and $\theta = 0$.⁹ When $D = S$, we have $\bar{M} = 0$; i.e., perfect synchronization of sales with purchases of equal value implies zero average holdings of money balances. Whenever S/D or D/S is an integer, the values of \bar{M} are indicated by vertical coordinates of the points a, b, c, \dots , and x, y, z, \dots , all of which lie on the broken line defined by $\bar{M} = |\bar{S} - \bar{D}|$. This broken line forms the lower boundary of the finance function. Whenever S/D is irrational, the values of \bar{M} are indicated by vertical coordinates of points that lie on the extended line AB defined by the equation $\bar{M} = \bar{S} + \bar{D}$. This line forms the upper boundary of the function $F(S, D, \theta)$. In all other cases (i.e., whenever S/D is rational but neither S/D nor D/S is an integer), the graph of the finance function lies strictly between these lower and upper boundaries.

As Figure 2 illustrates, F attains a discontinuous local minimum at every value of D for which S/D is rational. This means that for such values of D , the graph of F consists of an infinity of isolated points. The most distinctive of these isolated points are those associated with local minima defined by values of D where S/D or D/S is an integer; i.e., points that lie on the lower boundary of the finance function. Assuming that holding costs on money are positive, the effect of these minima will be to discourage the trader from choosing values of S and D that do not yield integer values for S/D or D/S .

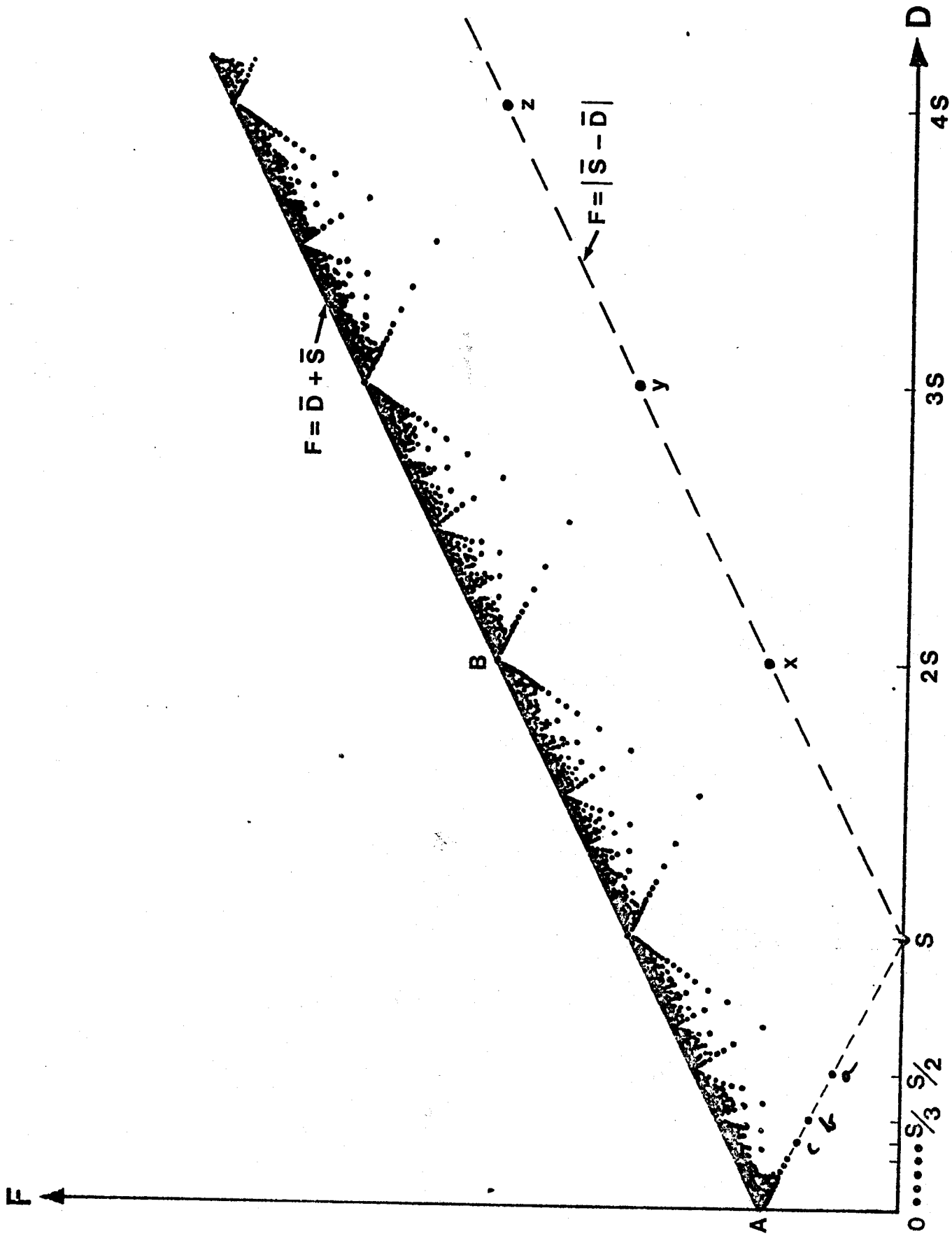


FIGURE 2
THE FINANCE FUNCTION

C. Bunching Costs

By hypothesis, bunching costs are set-up costs that are incurred on closely contiguous trades. Like other trading costs, therefore, bunching costs per unit of time will depend in part upon the absolute frequency of sale and purchase transactions. In addition, however, it is plausible to suppose that the magnitude of bunching costs will be greater for any given pair of trading dates, the smaller is the time interval between the given trades relative to the interval between successive sales or purchases.

To formalize these ideas we begin by noting that each trading date of the least frequently traded good must occur within a time interval $d' = \text{Min } (\bar{S}, \bar{D})/y$ of a trading date of the other good. We define any trading date, t_n , of the least frequently traded good as a bunch point. It follows that to each bunch point there corresponds a trading date t'_n of the other good for which

$$d(n) \equiv |t_n - t'_n| \leq d'.$$

Any pair of trades that satisfies this condition we shall call bunched; any pair of trades that does not we shall call isolated.

As a numerical indicator of the relative closeness of two trades, we next define the measure

$$(19) \quad I(t_i, t_j) \equiv \begin{cases} 1 - d(n)/d' \geq 0 & \text{for } \underline{\text{bunched}} \text{ trades;} \\ 0 & \text{for } \underline{\text{isolated}} \text{ trades.} \end{cases}$$

Using this measure (and noting that all positive values of I occur at bunch points), we define the average square of the $I(t_i, t_j)$'s over the time interval $t \geq 0$ by

$$(20) \quad I^2 = \lim_{N \rightarrow \infty} (1/N) \sum_0^N (1 - d(n)/d')^2.$$

Our basic hypothesis may then be expressed symbolically by writing the bunching cost function $C_B = f(S, D, \theta)$ as

$$(21) \quad C_B = c(y/Y)I^2,$$

where $Y = \text{Max}(S, D)$, y/Y is the frequency of bunch points per unit of time, and c is a given constant representing the fixed cost associated with a single simultaneous purchase and sale.

In the mathematical appendix (proposition 6) we show that formula (20) may be rewritten as:

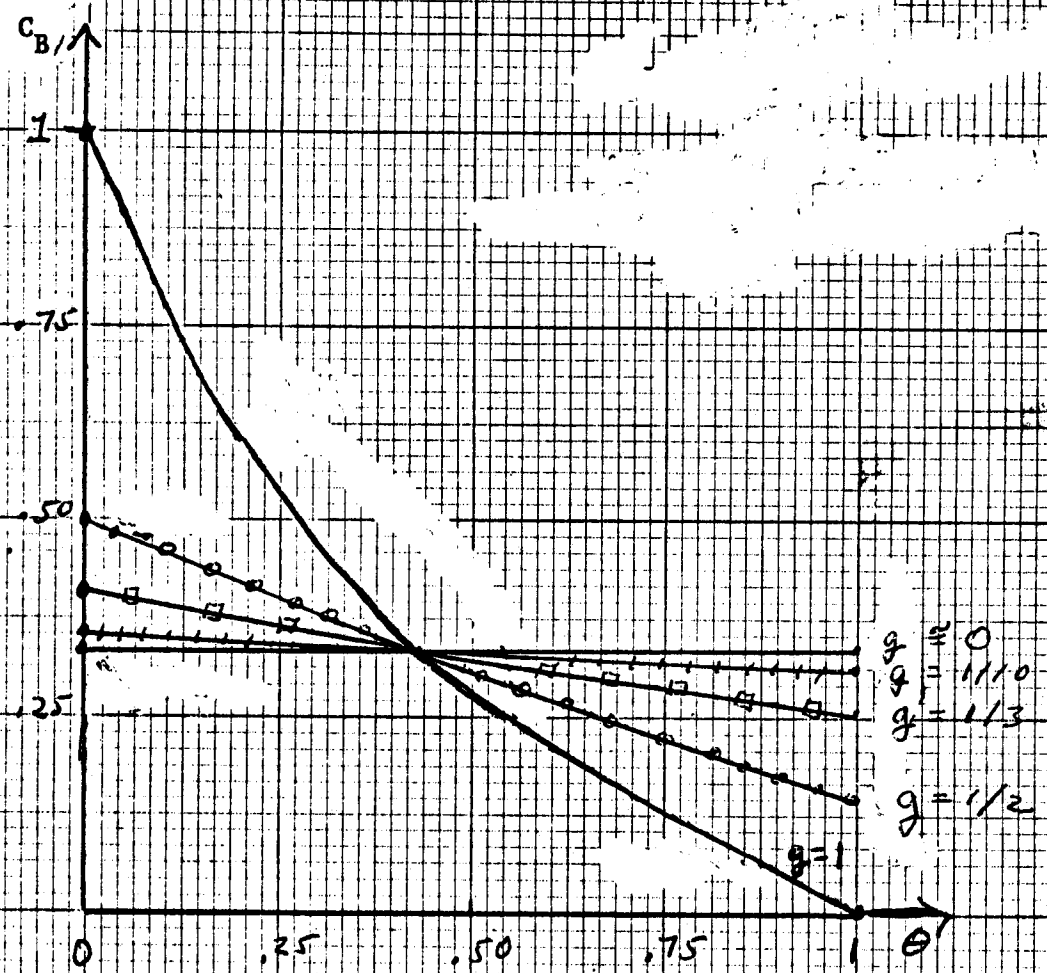
$$I^2 = (1/3)(1-g^2) + g^2(1-\theta)^2,$$

where $g = G(S, D)/\text{Min}(S, D)$. It follows that for all values of S , D and θ the bunching cost function may be expressed as:

$$(22) \quad C_B = (cy/Y)\{(1/3)(1-g^2) + g^2(1-\theta)^2\}.$$

The graph of (22) for the particular case $c=y=S=1$, $D > 1$, is illustrated in Figure 3. For any given value of θ , the bunching cost function--like

14-A



GRAPH OF THE BUNCHING COST FUNCTION C_B

FIGURE 3

the finance function $F(S,D,\theta)$ is an everywhere discontinuous function of the relative trading frequency S/D . This may be seen most easily by noting that the only possible values of $g = G(S,D)/\text{Min}(S,D)$ are given by the terms in the monotonically decreasing sequence

$g^{(n)} = \{1/n\} = \{1, 1/2, 1/3, 1/4, \dots\}$. For relatively small values of g (i.e., $g \leq 1/10$), $C_B \approx 1/3$ regardless of the value of θ . For moderate values of g (i.e., $1/10 \leq g \leq 1/2$), C_B declines moderately as θ increases, but C_B is always strictly positive for all $c > 0$. However, in the case $g = 1$ (i.e., S an exact multiple or divisor of D), C_B approaches zero as θ approaches unity, for in this case (and only in this case) the time phasing of sales and purchases may be so chosen that all trades are strictly isolated.

D. Solution of the Model

Because the finance and bunching cost functions are everywhere discontinuous, standard calculus techniques cannot be used directly to establish the existence or properties of solutions to (14). This problem can be effectively circumvented here, however, by working with graphical methods. For simplicity of exposition, we begin by considering the special case where bunching costs are identically zero ($c = 0$) so that the finance function is the only source of discontinuity in (14).

Case (1): Zero Bunching Costs. Let the curves ab , cd , etc. in Figure 4 represent iso-trading-cost (ITC) loci corresponding to alternative constant values of the trading cost function

$$C_T = ay/D + by/S$$

(i.e., equation (8), above, with $f(S,D,\theta) \equiv 0$). These loci are continuous,

P. 15A

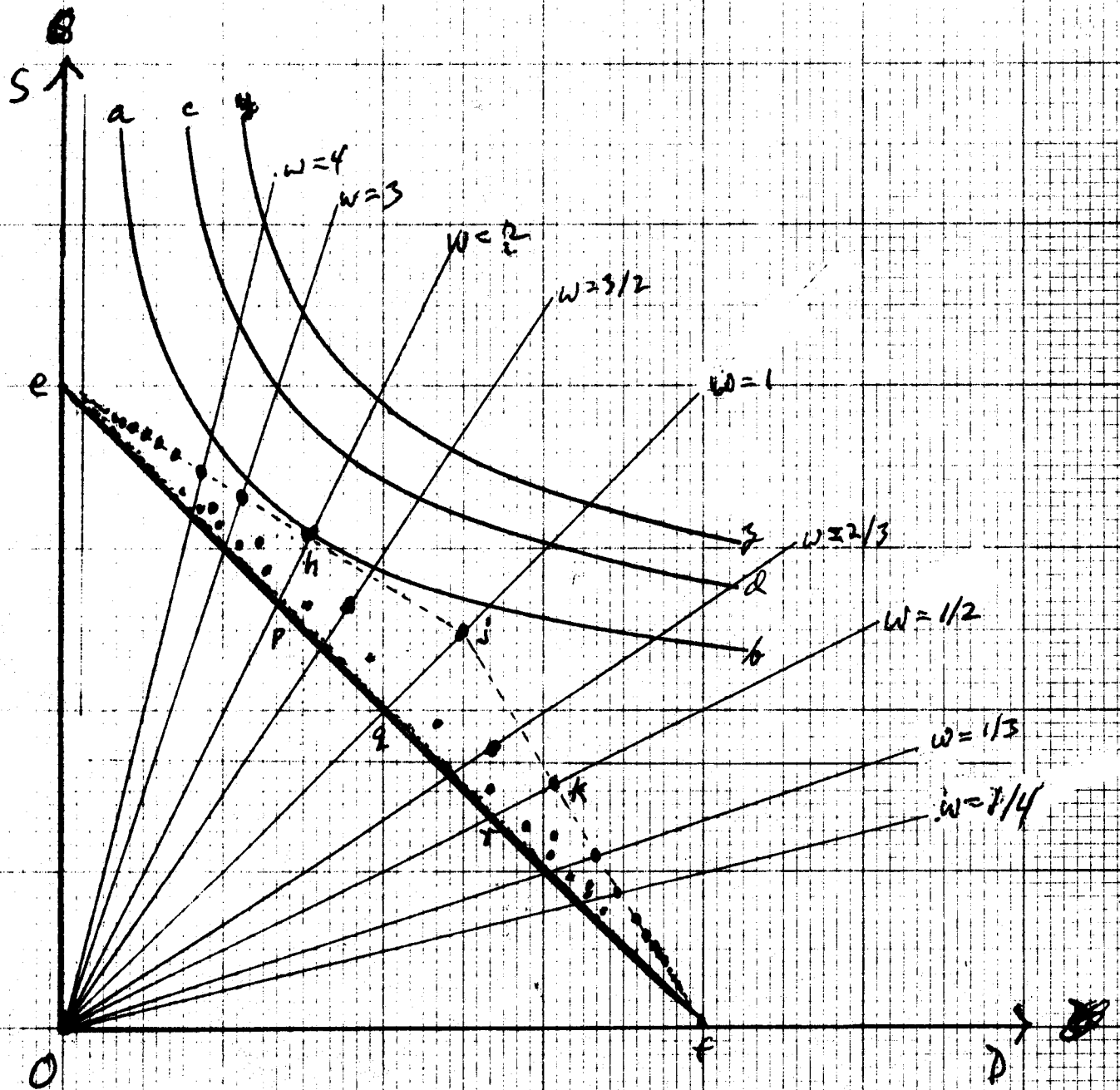


FIGURE 4

downward sloping and convex to the origin; and "higher" loci (larger values of S and D) correspond to lower values of total trading cost. Let the points on or within the triangle *efj* represent an arbitrarily chosen iso-holding-cost (IHC) set corresponding to some given value of the holding cost function

$$C_H = \rho(\bar{D} + \bar{S} + \bar{M}) + \alpha\bar{D} + \beta\bar{S} + \gamma\bar{M}$$

(i.e., equation (11), above). The points at which S/D or D/S is an integer lie on the upper boundary of the IHC set, indicated by the lines *ej* and *jf*, the slopes of which are, respectively, $(\gamma - \alpha)/(2\rho + \beta + \gamma)$ and $(2\rho + \alpha + \gamma)/(\gamma - \beta)$. The points at which S/D is irrational lie on the lower boundary of the IHC set, indicated by the line *ef*, the slope of which is $-(2\rho + \alpha + \gamma)/(2\rho + \beta + \gamma)$. All other points lie strictly inside the triangle *efj*. Because the finance function is discontinuous, the IHC set consists entirely of isolated points except along its lower boundary.

Using this diagram, we can illustrate the solution to (14) in two stages. A necessary condition for any D and S to solve (14) is that total trading costs be minimal for any given level of total holding cost. Thus in searching for a solution we may restrict attention to points such as *h* in Figure 4 that lie on the highest possible ITC curve that intersects the given IHC set. If bunching costs are identically zero (as we are presently assuming), the trading cost and holding cost functions will both be homogeneous in the variables S and D; hence the set of all points that minimize trading cost for given levels of holding cost will lie on a common ray through the origin. The slope of this ray, $w = S/D$, is therefore equal to the relative trading frequency

of the solution to (14); i.e., if $h = (S', D')$, then $w' = S'/D' = \hat{S}/\hat{D} = \hat{w}$, where \hat{S} and \hat{D} denote solution values of S and D . So the first stage in the solution of (14) is to determine \hat{w} by finding a point that minimizes total trading cost on some arbitrarily given IHC set.

Having determined \hat{w} we may treat the equation $\hat{w} = S/D$ as a constraint and use it to rewrite (14) as

$$(23) \quad \text{Min}_{\{D \geq 0; 1 \geq \theta \geq 0\}} (\rho + \alpha)(D/2) + (\rho + \beta)(\hat{w}D/2) + (\rho + \gamma)\{G(1, \hat{w})(\theta - 2) + (1 + \hat{w})\}(D/2) + ay/D + by/\hat{w}D.$$

The minimand in this second-stage problem is a polynomial in D and θ ; accordingly, standard calculus methods can be used to solve for optimal values of D and θ . By direct inspection, the optimal value of θ is just $\theta = 0$; i.e., in the absence of bunching costs the trader will choose θ so that for some date t a sale and a purchase coincide. To obtain \hat{D} we differentiate (23) with respect to D and solve as usual. This yields the formulae

$$(24) \quad \hat{D} = \frac{2y(a + b/\hat{w})}{\alpha + \beta\hat{w} + (2\rho + \gamma)(1 + \hat{w}) - 2(\rho + \gamma)G(1, \hat{w})},$$

$$(25) \quad \hat{S} = \hat{w}\hat{D},$$

and

$$(26) \quad \hat{M} = \hat{S}/2 + \hat{D}/2 - G(\hat{S}, \hat{D}).$$

Regrettably, the formulae (24)-(26) do not permit us to calculate numerical solution values for \hat{D} , \hat{S} and \hat{M} corresponding to given values of the parameters $y, a, b, \alpha, \beta, \gamma$, and ρ . The difficulty lies in the first stage problem -- the determination of a solution value for w . Of course in any particular problem we could attempt to find a numerical value for \hat{w} by trial and error, but except in contrived special cases such a

procedure is unlikely to yield anything but frustration.

A more promising approach is to establish limits on admissible solution values of \hat{w} by imposing artful and (stringly speaking) invalid restrictions on the finance function (18). Suppose, for example, that we follow earlier literature and approximate the finance function by fitting straight lines through the points satisfying integer constraints. On this assumption the finance function (18) takes the form $\bar{M} = |\bar{S} - \bar{D}|$, so that the graph of F consists of all points on the upper boundary of the IHC set in Figure 4; i.e., the decision problem takes the form

$$(27) \quad \text{Min}_{\{D, S > 0; 1 > \theta > 0\}} (\rho + \alpha)(D/2) + (\rho + \beta)(S/2) + (\rho + \gamma)(|\bar{S} - \bar{D}| + y\theta m^*) + a(y/D) + b(y/D),$$

the minimand of which is differentiable almost everywhere.

The solution to (27) can be found in two stages. First, the slope of the ITC curves is compared with the slopes of the two branches of the upper boundary of the IHC set on the ray $w=1$ to determine whether $\hat{w} \begin{matrix} > \\ = \\ < \end{matrix} 1$.

From this stage we get:

$$\left\{ \begin{array}{l} S/D < 1 \text{ if } a/b < (\alpha - \gamma)/(2\rho + \beta + \gamma); \\ S/D > 1 \text{ if } a/b > (2\rho + \alpha + \gamma)/(\beta - \gamma); \\ S/D = 1 \text{ otherwise.} \end{array} \right.$$

In the second stage, conventional calculus techniques can be used to derive the formulae:

$$(28) \quad \hat{D} = \begin{cases} \sqrt{ay/2(\alpha - \gamma)} & \text{if } S/D > 1 \\ \sqrt{ay/2(2\rho + \alpha + \gamma)} & \text{if } S/D < 1 \\ \sqrt{(a+b)y/2(2\rho + \alpha + \beta)} & \text{if } S/D = 1 \end{cases} \quad \hat{S} = \begin{cases} \sqrt{by/2(2\rho + \beta + \gamma)} & \text{if } S/D > 1 \\ \sqrt{by/2(\beta - \gamma)} & \text{if } S/D < 1 \\ \sqrt{(a+b)y/2(2\rho + \alpha + \beta)} & \text{if } S/D = 1. \end{cases}$$

As may be easily verified, the value of \hat{w} determined by these equations

approximates the "true" value of \hat{w} in the general solution to (23) to the nearest integer value of \hat{w} or its reciprocal. In virtually every case this approximation will lead quickly (two trials!) to an exact numerical solution; for the "true" solution to (23) usually will correspond to a point on the upper boundary of some IHC set, and all such points define integer values of w or $1/w$. But the rule suggested by this line of reasoning is not universal, as the following example demonstrates. Suppose that $y = 1$, $a = 5/11$, $b = 1$, $\rho = .01$, $\alpha = \beta = .2$, $\gamma = 0$. Pick the IHC set with holding cost equal to 1.0. Then the ITC curve with trading cost equal to $(3.36)/11$ touches two adjacent points on the upper boundary of the given IHC set for which $w = 1$ and $w = 2$; but the same ITC curve passes to the left of a point $w = 3/2$ in the given IHC set, so the optimal value of w cannot be an integer or its reciprocal.

Case (ii): Positive Bunching Costs. When the bunching cost coefficient c is positive the trading cost function (8) takes the form

$$(29) \quad C_T = ay/D + by/S + (cy/Y) \left\{ (1/3)(1-g^2) + g^2(1-\theta)^2 \right\}.$$

Since (29) is an everywhere discontinuous function of $w = S/D$, its graph--as illustrated in Figure 5--is represented generally by a set of isolated points rather than a continuous curve. If $\theta = 0$, for example, the upper boundary of the ITC set is defined by points on the dashed curve $a'b'$ in Figure 5 that correspond to integer values of w or $1/w$; ¹⁰ the lower boundary is defined by points on the solid curve ab that corresponds to irrational values of w ; and all other points lie within the area $abb'a'$ (all but a few of these points are relatively closer to the lower boundary ab than to the upper boundary $a'b'$). Alternatively, if $\theta = 1$, the lower boundary of the ITC set consists of "integer" points

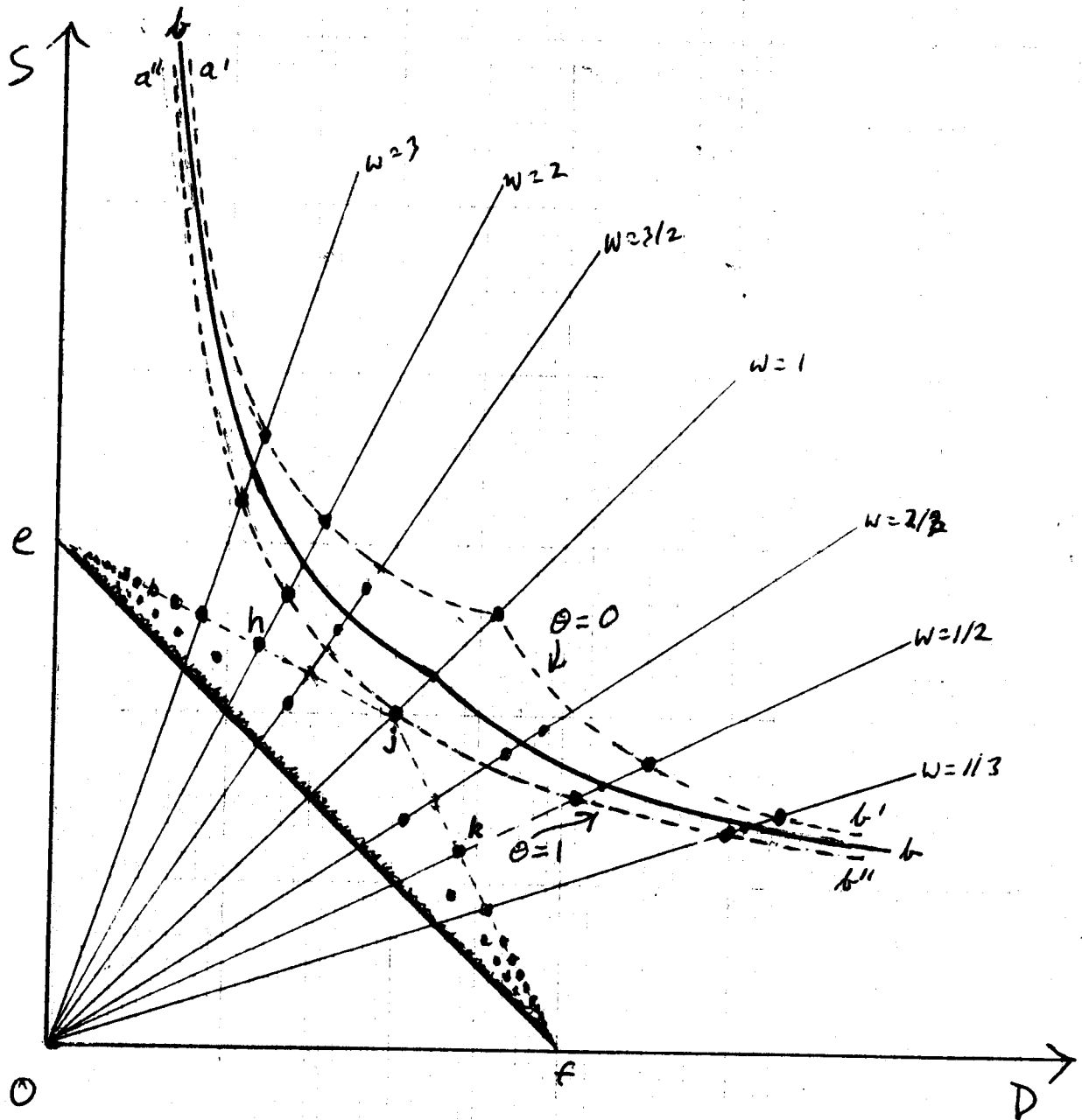


FIGURE 5

on the dashed line a"b" while the upper boundary is defined by the solid line ab. As indicated in Figure 3, above, there exists one value of θ (namely, $\theta = 1 - \sqrt{1/3}$) for which all points in the ITC set lie on the solid line ab; in this special case, therefore, the graph of (29) is continuous even when $c > 0$.

The procedure used to solve for \hat{w} in case (i) does not apply to case (ii) because the marketing costs are not homogeneous in θ . Nevertheless the nature of the diagram in Figure 5 makes it clear that once again there is a strong presumption that \hat{w} will satisfy an integer constraint. For all parameter variations that leave this integer constraint unchanged we may proceed much as in case (i) by rewriting (14) as:

$$(30) \quad \text{Min}_{\{D \geq 0, 1 \geq \theta \geq 0\}} (\rho + \alpha)(D/2) + (\rho + \beta)(\hat{w} D/2) + (\rho + \gamma)\{G(1, \hat{w})(\theta - 2) + (1 + \hat{w})\}(D/2) + ay/D + by/\hat{w}D + cy/\hat{v}D\{\frac{1}{3}(1 - g^2) + g^2(1 - \theta)^2\}$$

where $\hat{v} = \max\{1, \hat{w}\}$. The first order conditions for a minimum are:

$$(31) \quad \hat{v}(\rho + \gamma)(D/2)^2 - cyg(1 - \theta) \geq 0 \quad (\text{with equality if } \theta > 0)$$

and

$$(32) \quad D^2\{\alpha + \beta\hat{w} + (2\rho + \gamma)(1 + \hat{w}) + (\rho + \gamma)g(\theta - 2)\} - 2y\{a + b/\hat{w} + (c/\hat{v})(\frac{1}{3}(1 - g^2) + g^2(1 - \theta)^2)\} = 0,$$

which (in principle) may be solved simultaneously for θ and \hat{D} . As may be seen by direct inspection, however, the solution for \hat{D} will not be of the familiar square-root variety because the elimination of θ from (31) and (32) leaves D defined as a quartic rather than a quadratic equation.

II. IMPLICATIONS AND EXTENSIONS

In subsequent pages we discuss some implications of our basic model and consider various extensions. Our analysis is in no sense exhaustive; its purpose is less to elucidate logical properties of our basic model than to illustrate the wide range of economic insights that may be gained from even the simplest formal theory of the timing of transactions.

A. Comparative Statics: General Observations

It cannot be emphasized too strongly that the discontinuities in our basic model arise not from strained assumptions about the discreteness of time or the atomistic character of commodity and money units but rather from the fact that trades involve stocks rather than flows so that small changes in the relative timing of transactions can produce large jumps in average finance requirements and in average bunching costs. These jumps would be less obvious if our model dealt with non-stationary processes so that between-trade time intervals were not necessarily uniform; as a matter of logic, however, jumps analogous to those implied by our model must occur in any ongoing economy where trades take place at discrete points rather than continuously in time. Appearances to the contrary notwithstanding, therefore, the comparative statics implications of our model are of more than purely academic interest.

The most important comparative statics conclusion to be drawn from our model is negative, namely, the consequences of parameter changes upon equilibrium values of S, D, and \bar{M} are generally ambiguous. The source of these ambiguities lies mainly in the different effects of parameter changes upon relative and absolute transactions frequencies. As is clear from

earlier graphical analysis (Figures 4 and 5), small changes in parameters may leave the relative frequency $\hat{w} = \hat{S}/\hat{D}$ unchanged because one or both of the touching frontiers of the IHC and ITC sets consist of isolated points. In such cases the only effect will be to change the absolute frequencies y/\hat{D} and y/\hat{S} , which are determined in the second-stage maximization problem. It is worth remarking that the standard assumption of regarding the absolute frequency of sales (the income period) as predetermined hides this sometimes crucial distinction by making a change in the absolute frequency of purchases equivalent to a change in the frequency of purchases relative to sales.

To illustrate the preceding remarks, let us consider a change in γ , the storage-cost coefficient on money holdings. If we employed the usual approximation giving rise to the

square-root formulae (28), (above, p.18), we would infer from these that when $S > D$ an increase in γ would lead to an increase in \hat{D} . If the change in γ did not affect the relative frequency \hat{w} , however, we would infer from the "correct" formula (24) that \hat{D} would decrease. To get the result implied by the usual integer constraint, we should require the rise in γ to produce (i) a large enough increase in D (decline in w) in the first-stage decision problem to offset (ii) the decrease in D at the second stage. The standard formulae (28) presuppose that the relative effect (i) always dominates the absolute effect (ii), but this cannot always be so because sometimes a change in γ will produce no change in \hat{w} .

In simpler models when integer constraints are taken into account the individual demand functions will have discontinuous steps in them, but the smoothing properties of aggregation may be invoked to argue that the aggregate function is smooth and behaves qualitatively like the usual square-root formula.¹¹ The present model shows that this is generally not valid. For example, when γ increases some traders will increase D and others will decrease D ; the aggregate effect will depend crucially upon the form of the distribution of traders between the two categories. The form of the distribution is not so crucial in simpler models because they do not allow for the possibility of traders moving in different directions, just for some not moving at all.

This is not to say that we cannot derive comparative static results from the present approach. On the contrary, the approach allows us to isolate those results that are robust enough to survive this degree of generality. For example, Samuelson's well-known technique (1947, pp. 46-52) of manipulating the inequalities associated with an extremum allow us to conclude that an increase in either of the trading cost coefficients a or b , will lead to an increase in the average holding of the associated commodity, \bar{D} or \bar{S} , that an increase in the storage cost coefficient, α , β , or γ , will lead to a decrease in the average holding of \bar{D} , \bar{S} , or \bar{M} , and that an increase in the rate of time preference, ρ , will lead to a decrease in the total size of inventories, $\bar{D} + \bar{S} + \bar{M}$. Note, however, that when ρ increases any single inventory holding may increase.

One other familiar result that holds in the general case (provided that $\theta = 0$) is that the income-elasticity of the demand for each of the inventories has a value of one half. This follows from the modified

square-root formulae (24)-(26) and the fact that, because of the homogeneity of trading costs and holding costs in y , a change in y will not affect the value of \hat{w} determined in the first stage decision problem.

B. Positive Money Holdings

Though a trader might be willing to hold positive money balances simply to avoid bunching costs, it appears that such balances would not be held for any other reason unless money were less costly to hold than the most frequently traded good. Otherwise the trader would synchronize purchases perfectly with sales at a frequency equal to that of the most frequently traded good and would hold no money at all, for there would be no advantage to holding money rather than goods as a store of purchasing power or consumption. This may be seen most easily from our analysis by supposing that the solution point illustrated in Figure 4 yields a value of \hat{w} greater than unity. Then this point must lie to the left of the ray with $w = 1$. But if $\gamma \geq \alpha$, the segment of the upper boundary of the IHC set to the left of the ray with $w = 1$ will be horizontal or even upward sloping, which implies that total marketing costs could be reduced by moving the solution point to the ray with $w = 1$.

These considerations have some bearing on the familiar question, Why do people choose to hold money when all other assets have a higher net return?¹² Our answer is that they won't -- at least not in a stationary state without bunching costs. More generally, it appears that the usefulness of money as a means of payment is limited by its costliness to store, which

may help to explain why representative monies have tended to displace commodity and commodity-backed monies in modern times.

The same considerations also shed light on the holding of money balances during hyperinflations. The coefficient γ in our model may be interpreted as the sum of physical storage costs plus the expected rate of inflation. The model then implies that when inflation reaches some critical point people will hold no money at all except for brief intervals between transactions which, according to (31),

would become smaller as the expected rate of inflation increased. This prediction of our model accords well with behavior observed during actual hyperinflations. During even the most severe hyperinflations, however, people appear to be extremely reluctant to forego trade in organized markets that require them to use conventional media of exchange.¹³ That is to say, money continues to circulate with finite velocity even when it is clearly the most costly of all goods to store. This observation, combined with our theoretical analysis, casts serious doubt on the conventional assumption that bunching costs may be ignored. The evidence suggests, on the contrary, that bunching costs are substantial at least for "nearly simultaneous" sale and purchase transactions.¹⁴

C. Competitive Interest on Money

It has been shown by many authors that social optimality requires the payment of competitive interest on money.¹⁵ Optimality also requires that the money commodity be as inexpensive as possible to produce.¹⁶ Generally speaking, it is taken for granted that the money commodity is almost costless to store relative to the cost of storing a typical non-money commodity of equal money value.

We can analyze the effect of paying interest on money by interpreting the coefficient γ as the money storage-cost coefficient minus the rate of return paid on money. Suppose that it is possible to find a money commodity that is literally costless to store. Then the optimal situation is to have $\gamma = -\rho$, in which case the trader's decision problem (cf. (30), above) may be written as:

$$(33) \quad \text{Min}_{\{D, S \geq 0; 1 \geq \theta \geq 0\}} (\alpha + \rho)(D/2) + (\beta + \rho)(S/2) + a(y/D) + b(y/S) \\ + c(y/Y)\{(1/3)(1-g^2) + g^2(1-\theta)^2\}.$$

If $c > 0$, the minimand in this expression is continuous only in θ , so we proceed by first differentiating partially with respect to this variable. This yields the requirement

$$(34) \quad \theta = 1.$$

In this case the IHC sets are straight lines of slope $-(\alpha + \rho)/(\beta + \rho)$ and the lower boundary of the ITC set consists of all intersections of the integer rays with the loci defined by

$$(35) \quad ay/D + b(y/S) = \text{constant}.$$

Thus when c is relatively large the optimal solution will lie along an integer ray, although this will not hold for small values of c . When $c = 0$ the same IHC sets will exist and the ITC sets will consist entirely of

curves described by (35). In this case standard calculus techniques can be used to derive the optimal quantities:

$$(36) \quad \bar{D}^* = \sqrt{ay/2(\alpha+\rho)} \quad , \quad \bar{S}^* = \sqrt{by/2(\beta+\rho)} \quad .$$

This solution is the one that would be obtained if the trader ignored the interaction of trading frequencies and minimized total holding and trading costs of each non-money good separately. Thus the gain from the payment of competitive interest on money (when $c = 0$) derives from the freedom such a policy gives the trader to choose trading frequencies without regard to cash constraints.

For reasons indicated earlier, it is hard to say what the qualitative effects of optimal money policy might be on specific variables. In particular, our example in section II.A shows that a reduction in γ will have an ambiguous effect on D even if we know a priori that $S > D$. However, two conclusions hold under quite general circumstances. The first is that a reduction in γ will increase \bar{M} (cf. p. 23 above). The second result is that total trading cost must fall. This follows from (32) together with the fact that a fall in γ necessarily reduces total marketing cost ($C_H + C_F + C_B$). For from (32) we infer that $C_H = C_F + C_B$; so we conclude that total trading costs ($C_F + C_B$) must decrease.

D. Money Substitutes

Our analysis up to this point has been conducted on the assumption that the trader's financial inventories consist only of cash. We now relax this assumption by extending our model to include two types of money substitutes, one an earning asset (B) called "bonds" that can be used in place of money as a temporary store of value, the other a loan instrument (L) called "trade credit" that can be used instead of cash as an immediate means of payment. Our object is to discover if either of these generalizations significantly alters any of our earlier conclusions.

(i) Temporary Bond Holdings. Suppose that the trader is able to buy and sell (but not to issue as debt) units of an asset (B) that yields a non-pecuniary income of i units per unit of time. Suppose further that i is no greater than ρ , the subjective rate of time discount, so that the trader has no incentive to hold bonds rather than money except as an alternative store of value. On these assumptions, the trader's total and average holdings of financial assets (money and bonds) will be given, respectively, by

$$(37) \quad A(t) = [1 + yt/S]S - [y(t - m^*)/D]D - G(S,D),$$

and

$$(38) \quad \bar{A} = F(S,D,\theta) = \bar{S} + \bar{D} - G(S,D) + y\theta m^*$$

(cf. equations (5), (16) and (18)). Then for reasons given earlier, the trader will always choose values of S and D for which the relative frequency S/D is rational.¹⁷ In general, therefore, there will be a time interval of length T (namely, the least common multiple of S/y and D/y) and a sequence of dates $\{t', t'+T, t'+2T, \dots, t'+T, \dots\}$ at

which $A(t) = 0$. Since inventory holdings will be exactly the same at each of these dates, the trader's optimal pattern of portfolio management over any given half-open time interval $[t+jT, t'+(j+1)T)$ will be independent of j . In the discussion that follows, therefore, we lose no generality by confining attention to the trader's decision problem over the single basic interval $[t', t'+T)$.

Temporarily ignoring bunching costs on contiguous trades of goods and bonds, and assuming that a set-up cost of k is incurred on every bond transaction, we may suppose that the trader's decision problem over the basic interval $[t', t'+T)$ is to choose non-negative values of S , D and θ , a non-negative integer n representing the total number of bond transactions, a sequence of transaction dates $\{\tau_1, \tau_2, \dots, \tau_n\}$ within the interval, and associated quantities $\{B_1, B_2, \dots, B_n\}$ so as to:

$$\begin{aligned}
 & \text{Min } (\alpha + \rho)(D/2) + (\beta + \rho)(S/2) + (\rho + \gamma)F(S, D, \theta) - (1 + \gamma)\bar{B} \\
 & \quad + k \cdot n + a(y/D) + b(y/S) + f(S, D, \theta), \\
 & \text{Subject to:} \\
 & \left. \begin{aligned}
 B(t) & \equiv \sum_{\tau \leq t} B_r \geq 0 \\
 M(t) & \equiv A(t) - B(t) \geq 0
 \end{aligned} \right\} t \in [t', t'+T) \\
 & \bar{B} \equiv (1/T) \int_0^T B(t) dt \\
 & \eta = n/T \\
 & T = SD/yG(S, D) \equiv \Delta(D/y, S/y)
 \end{aligned}
 \tag{39}$$

where \bar{B} represents the trader's average bond holdings and η is the frequency of bond transactions per unit of time.

This problem will have a solution with $\bar{B} \geq 0$ only if $\theta > 0$ or $S \neq D$; for if $\theta = 0$ and $S = D$, then $\bar{B} \leq F(S, D, \theta) = 0$. Even if $S = D$, however, \bar{B} may be positive provided that $\theta > 0$, for then $\bar{B} \leq F(S, D, \theta) = y\theta m^*$. More generally, the existence of a solution with \bar{B} positive depends sensitively on the absolute magnitude of S and D (since these indirectly determine the absolute value of \bar{B}) and on the set-up cost of bond transactions, k . It is worth remarking, more particularly, that the likelihood of solutions with bond holdings positive does not depend in any way on sales occurring more frequently than purchases; this follows directly from the symmetry of the finance function in the variables S and D .¹⁸

Assuming that (39) has a solution with \bar{B} positive, we infer from the structure of the decision problem that bonds will be purchased only on dates when (S) is sold and will be sold only on dates when (D) is purchased. Moreover, bond sales will occur only when the trader's money balances otherwise would be too small to finance a scheduled purchase of goods. Thus if bunching costs are incurred on each bond transaction (and to suppose otherwise would be incongruous in a model where we assume that such costs are incurred on simultaneous goods transactions), the trader will have an incentive to delay scheduled bond purchases and to advance scheduled bond sales up to the point where the marginal reduction in bond bunching cost is just offset by the marginal loss of interest income associated with shortened bond-holding periods. We shall not attempt to formalize these observations here; suffice it to say that explicit recognition of bunched bond transactions in the statement of (39) would often convert what would otherwise be a solution of (39) with $\bar{B} > 0$ into a solution with $\bar{B} = 0$.

Obviously the solution to (39) often will be relatively insensitive to small changes in parameters because of the discontinuities in the functions $F(S,D,\theta)$ and $f(S,D,\theta)$. As in earlier discussions, therefore, we cannot expect to obtain unambiguous comparative statics results on the basis of a priori considerations. A case in point is the effect of a change in the bond interest rate, i , on average holdings of money balances, \bar{M} . One would expect the sign of this partial derivative to be negative,¹⁹ but that need not be so. If i increases, this might have no effect either on the frequency w or the number of bond transactions during the basic interval, n . But, by using Samuelson's technique of manipulating inequalities it is easily seen that \bar{B} must increase. If we assume that $\theta = 0$ before and after the change, then \bar{B} , \bar{D} , \bar{S} and \bar{M} must all increase in proportion.

If the introduction of bond holdings does not make earlier comparative statics results less ambiguous, neither does it force us to revise earlier conclusions that were unambiguous. For example, it can be shown (albeit with some difficulty) that the income elasticity of all inventory demands is equal to one half unless bunching costs on goods are positive (cf. II.A, p. 23).

Moreover a necessary condition for positive money holdings in the absence of bunching costs on goods is, again, that the storage-cost coefficient on money balances be less than the corresponding coefficient on the most frequently traded of the two non-financial commodities. For example, if $S > D$ and $\bar{M} > 0$, then there must be some purchase date at which money for the purchase was available for some interval of time before the purchase date, in which case the trader clearly could have reduced storage costs by making the purchase at the beginning of that interval rather than the end. Finally, it can be shown that payment of competitive interest on money that is costless to store would still eliminate the discontinuity in the IHC sets. No bonds would ever be held in this situation, for to do so would be to incur needless trading costs; so the behavior of the trader would be precisely as described in the analysis of the preceding section. Thus we conclude that the addition of earning assets to our model serves little purpose other than to provide assurance that the complications to which such a procedure leads are essentially gratuitous.

(ii) Trade Credit. The introduction of trade credit into our model provides rather more interesting results. Suppose that the trader has access to trade credit or bank overdraft facilities that permit him to defer cash payment for goods by having the non-negotiable option at each purchase date of incurring a debt (to the seller of goods in one case, to a banker in the other) up to a predetermined limit, L . Suppose that interest is charged on used credit at the rate s . Let \bar{C} denote the average amount of unused credit and let $L - \bar{C}$ represent the average amount

of used credit. The finance function $F(S,D,\theta)$ will in this case give the sum of $\bar{M} + L - \bar{C}$. As long as $s \leq \rho$ it will be optimal for the trader to arrange for as much as possible of his financing to be done by credit. If L satisfies

$$(40) \quad L \geq D + S - 2G(S,D) + y\theta m^*,$$

then holdings of money may be avoided altogether, for the trader can run so large an average debt without exceeding the credit limit (40) that all sales of (S) are accompanied by a running down of debt rather than an accumulation of money balances.

On these assumptions the trader's total holding cost will be given in general by

$$(41) \quad \rho(\bar{D} + \bar{S} + \bar{M} + \bar{C}) + \alpha\bar{D} + \beta\bar{S} + \gamma\bar{M} + s[L - \bar{C}].$$

The term $\rho\bar{C}$ is included in waiting costs because not to use possible credit lines involves the trader in the same abstention cost as for other commodities. In the case where (40) holds, of course, we have $\bar{M} = 0$. If in addition we have $s = \rho$, the holding cost function (41) reduces to

$$(42) \quad (\rho + \alpha)\bar{D} + (\rho + \beta)\bar{S} + sL,$$

which is, except for the irrelevant constant term sL , identical to the function derived earlier for the case where competitive interest is paid on money (Section II.C, p.25). In other words, competitive trade credit or bank overdraft arrangements provide two alternative routes to monetary optimality equivalent in effect to paying competitive interest on money. This result is interesting because it helps to rationalize--at least on an individual level--the presence in all advanced economies of a wide and (apparently) still expanding array of specialized credit facilities.

E. The Coordination of Individual Trading Activities

Though much of our analysis of individual trading behavior appears to have significant import also for market behavior, we shall limit our discussion of such matters here to a few general observations. To go beyond this would be ill advised, for the existing literature does not contain a satisfactory theoretical account of the overall working of an economy in which the resource costs of trading activity depend in an essential way on the frequency of exchange transactions.²⁰ In the absence of such a theory there is reason to believe that a too hasty generalization of results derived from individual experiments will deal either superficially or not at all with what appears to be a fundamental externality.²¹

This externality arises from the fact that, in an economy with set-up costs of trading, individuals will trade only at isolated points in time; hence the set of transaction dates that are feasible for one trader cannot in general be specified independently of choices made by other traders. As Perlman(1971, p. 235) has put it, there exists in such an economy not only a problem of double coincidence of wants but also a problem of double coincidence of timing.

The timing-externality problem cannot be avoided by supposing (à la Debreu-Arrow) that all trades are the result of prearranged contractual obligations that specify exactly the dates at which trades are to be executed, for this procedure begs the question of how traders coordinate the timing of contract negotiations. The Neo-Walrasian "auctioneer" is not an answer to but rather an evasion of this question. The only plausible and logically satisfactory solution is to posit the existence of specialist

traders who, unlike the primary traders of the present paper, agree to do business continuously at dates chosen by the primary traders with whom they deal.²²

Such specialist traders--shopkeepers, wholesalers, brokers, agents, marketing managers of manufacturing concerns--play a central role in every developed economy. Their usefulness arises from their willingness in normal circumstances to quote buying or selling prices (or both) at which they are ready to trade in large quantities at dates that suit their customers. By so doing they not only solve the double coincidence of timing problem but also allow primary traders to plan and execute trades in accordance with budget constraints that do not require contingent allowance for possible non-price rationing.²³

To perform effectively, of course, each specialist trader has to hold substantial inventories of all commodities in which he deals. The magnitude of these holdings depends partly on the timing and magnitude of sales and purchases by primary traders, partly on each specialist's own demands for inventories; but specialists' demands for inventories depend on past and prospective trading volume which, in turn, depend (with lags) on buying and selling prices posted by specialists.²⁴ Thus any deterministic study of inventory behavior that focusses on primary traders--or for that matter, on specialist traders--is at best a partial equilibrium analysis. Precisely how the activities of primary traders are interrelated with the price-adjustment and inventory-management activities of specialist traders, however, is much too large an issue to be considered here. Though we claim that the specialist-trader approach provides, in principle, a solution to the timing-externality problem, therefore, we leave the practical justification of this claim to another occasion.

MATHEMATICAL APPENDIX

This appendix contains the proofs of nine propositions referred to in the text. We owe special thanks to Georges Monette, Joel Fried, R. A. Jones and John Riley for helpful comments and suggestions at various stages in its preparation.

Notation:

(N.1) $I =$ the set of non-negative integers, $\{0,1,2,\dots\}$; $R =$ the set of real numbers; $R^* =$ the set of rational numbers.

(N.2) $[x]$ denotes the largest integer no greater than the real number x .

(N.3) $G(S,D) = \begin{cases} \text{the greatest common divisor of } S \text{ and } D \text{ iff } S/D \in R^* \\ 0, \text{ iff } S/D \text{ is irrational.} \end{cases}$

(N.4) $R(nx|y)$, $n \in I$, $x,y \in R$, denotes the remainder when y is divided into nx (i.e., $R(nx|y) = nx - ky$, where $n,k \in I$).

(N.5) $J(t) = yt - [yt/S]S$ ($t \geq 0$), the time path of stocks of (S).

(N.6) $\beta(t) = ym - yt + [y(t-m)/D + 1]D$ ($t \geq 0$), the time path of stocks of (D).

(N.7) $M(t) = S - [y(t-m)/D + 1]D + [yt/S]S$, the virtual time path of stocks of (M).

(N.8) $d(n) = |t_n - t'_n| \leq \text{Min}(S,D)/2y$: the time interval between a sale at t_n and a purchase at t'_n if $S \geq D$; the time interval between a purchase at t_n and a sale at t'_n if $S < D$.

Proposition 1: Given any values of D and S such that $S/D \in R^*$, the function $R(nD/S)$ is periodic in n, repeating itself at intervals of length $T = S/G(D,S)$. Within any such interval the function assumes the value $m G(S,D)$ exactly once for each $m \in \{0,1,\dots,T-1\} \equiv J$.

Proof: The periodicity of $R(nD/S)$ follows from the fact that $TD(\frac{SxD}{G(S,D)})$ is divisible by S. Therefore $R((n+T)D/S) = R((nD+TD)/S) = R(nD/S)$. That $R(nD/S) = mG(S,D)$ for some $m \in J$ follows from the division algorithm of basic algebra. The rest of the proposition will be proved if we can show that: $R(nD/S) \neq R(n'D/S)$ when $n, n' \in \{1, \dots, T\}$ and $n \neq n'$. To show this, suppose the contrary. Then for some $m \in J$ and $k, k' \in I$; $mG(S,D) = nD - kS = n'D - k'S$. Suppose, with no loss in generality, that $n > n'$. Then $0 < (n-n')D = (k-k')S < TD$. But this contradicts the fact that, by construction, TD is the least common multiple of S and D. Q.E.D.

Proposition 2: Let $t_n = m + nD/y$ ($n \in I$) denote the date of the $(n+1)^{st}$ purchase. Then $M(t_n) = S - D - R(nD/S)$.

Proof: From (N.4) and (N.7) we have
 $M(t_n) = S - D - nD + [(nD + ym)/S]S$
 Because $ym \cong \frac{1}{2} G(S,D)$, it follows that $[(nD + ym)/S] = [nD/S]$.
 Therefore:
 $M(t_n) = S - D - (nD - [nD/S]S) = S - D - R(nD/S)$.
 Q.E.D.

Proposition 3: $G(S,D) = \inf_{n \in I} (S - R(nD/S))$.

Proof: If $S/D \in R^*$ the proof follows immediately from proposition 1. Suppose that $S/D \notin R^*$. Then there exist two sequences of positive integers $\{n^q\}, \{k^q\}$ such that $0 > n^q D - k^q S \rightarrow 0$ (cf. Niven and Zuckerman, pp. 134-139). Therefore $R(n^q D/S) \rightarrow S$, and $R(nD/S) < S$ for all $n \in I$.
 Q.E.D.

Proposition 4: Suppose S/D is rational. Then for any sequence $\{D^q, S^q\}$ such that (a) $D^q/S^q \neq D/S$ for all q ; (b) $D^q > 0$ and $S^q > 0$ for all q ; and (c) $\lim_{q \rightarrow \infty} (D^q, S^q) = (D, S)$, then $\lim_{q \rightarrow \infty} G(D^q, S^q) = 0$.

Proof: Because D/S is rational, there exist positive integers m and n such that $mD = nS$. Therefore, from (c),

$$\lim_{q \rightarrow \infty} mD^q = mD = nS = \lim_{q \rightarrow \infty} nS^q. \quad (d)$$

Take any q . If D^q/S^q is rational then, from (a) and (b), $mD^q/G(D^q, S^q)$ and $nS^q/G(D^q, S^q)$ are distinct integers, and $G(D^q, S^q) > 0$. Therefore $|(mD^q/G(D^q, S^q)) - (nS^q/G(D^q, S^q))| \cong 1$; i.e.,

$$|mD^q - nS^q| \cong G(D, S) \cong 0. \quad (e)$$

Obviously the inequality (e) will also be satisfied if D^q/S^q is irrational. Taking limits in (e) and noting that, from (d), $\lim_{q \rightarrow \infty} |mD^q - nS^q| = 0$, we get $\lim_{q \rightarrow \infty} G(D^q, S^q) = 0$. Q.E.D.

Proposition 5: If $S \cong D$, $d(n) = (1/y)\{\bar{D} - |\bar{D} - \mathcal{D}(t_n)|\}$. If $S \leq D$, $d(n) = (1/y)\{S - |S - \mathcal{D}(t_n)|\}$.

Proof: For definiteness suppose $D \leq S$ so that t_n represents a sale date and $\mathcal{D}(t'_n) = D \cong \mathcal{D}(t_n)$. Then if $(t_n - t'_n) \cong 0$, we have $\mathcal{D}(t_n) \cong \bar{D}$, and:

$$d(n) = y(t_n - t'_n) = \mathcal{D}(t'_n) - \mathcal{D}(t_n) = \bar{D} - (\mathcal{D}(t_n) - \bar{D}).$$

I.e.,
$$d(n) = (1/y)\{\bar{D} - |\bar{D} - \mathcal{D}(t_n)|\}.$$

Alternatively, if $(t_n - t'_n) < 0$, we have $\mathcal{D}(t_n) < \bar{D}$; hence

$$\begin{aligned} d(n) &= (t'_n - t_n) = (1/y)\mathcal{D}(t_n) \\ &= (1/y)\{\bar{D} - (\bar{D} - \mathcal{D}(t_n))\} \\ &= (1/y)\{\bar{D} - |\bar{D} - \mathcal{D}(t_n)|\}. \end{aligned}$$

A comparable proof establishes the same formula for the case $S < D$. Q.E.D.

Proposition 6: $I^2 = \frac{1}{3} (1-g^2) + g^2(1-\theta)^2$

Proof: Suppose that $S \leq D$ (the proof is analogous for $S > D$).

(1) Suppose that $(S/D) \in R^*$. At any bunch point $t_n = nD/y + m$ ($n \in I$) we have $f(t_n) = yt_n - [yt_n/S]S = R(yt_n/S) = R((nD + ym)/S) = R(nD/S) + ym$. (The last equality follows from the fact that $ym < G(S,D)$.) Therefore, from Proposition 5 and (19) in the text,

$$I(t_n, t'_n) = (1-d(n)/d') = \left| 1 - \frac{R(nD/S)}{\bar{S}} - \frac{y\theta m^*}{\bar{S}} \right|.$$

Therefore, from Proposition 1, the definition of m^* , and (20) in the text,

$$I^2 = \frac{1}{T} \sum_0^{T-1} (1-2ng-\theta g)^2$$

where $T = \frac{1}{g} = \frac{S}{G(S,D)}$. Using standard formulae for sums of integers and sums of squares of integers we get:

$$\begin{aligned} I^2 &= (1-\theta g)^2 + 4(g^2/T) \sum_0^{T-1} n^2 - 4g^2(1-\theta g) \sum_0^{T-1} n \\ &= (1-\theta g)^2 + 4(g^2/T)(T-1)T(2T-1)/6 - 4g^2(1-\theta g)T(T-1)/2. \end{aligned}$$

By simple algebraic reduction, using the fact that $T = 1/g$, this expression can be shown to equal $\frac{1}{3}(1-g^2) + g^2(1-\theta^2)$.

(2) Suppose that $(S/D) \notin R^*$. Then we have $g=\theta=0$ and we must show that $I^2 = 1/3$. As in the rational case, we have:

$$I^2 = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_0^N \left(1 - \frac{R(nD/S)}{\bar{S}} \right)^2$$

We note (without proof) that the limiting distribution, as $N \rightarrow \infty$, of $R(nD/S)/\bar{S}$, $n \in \{0, \dots, N\}$, is a uniform distribution over the interval $[0, 2)$, so that the above expression is just the variance of this uniform distribution, which equals $1/3$. Q.E.D.

Proposition 7: $m = \min_{\{n \in I\}} d(n) \leq m^* = G(S,D)/2y.$

Proof: From (N.5), (N.6), Proposition 1 and Proposition 5, it follows that there is some bunch point (the k^{th} , say), at which a purchase precedes a sale by an interval of $G(D,S)/y$. As m increases the length of this interval decreases at the same rate. As long as $m \leq G(D,S)/2y$ the minimal $d(n)$ is equal to m and occurs at the first bunch point. But when m is increased beyond $G(D,S)/2y$, the value of $d(k-1)$ becomes less than $G(D,S)/2y$. Thus the minimal $d(n)$ can never exceed $G(D,S)/2y$.

FOOTNOTES

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1. See especially Hirshleifer (1973), Ostroy and Starr (1974), Fischer (1974) and Clower (1975). Explicit concern with the timing problem goes back at least to Fleming (1964), but is first emphasized in the context of inventory-theoretic models by Perlman (1971).

2. See Baumol (1952), Tobin (1956), Clower (1970), Johnson (1970), Perlman (1971), Feige and Parkin (1971), Fried (1973), Barro and Santomero (1976), Grossman and Policano (1975), Policano (1976). Some of these papers deal nominally with trading frequencies for more than one commodity, but in each such case trading frequencies of all goods are assumed to satisfy integer constraints so that each frequency is an exact multiple or divisor of every other.

3. A non-stationary version of the problem analyzed here was studied by D.W. Bushaw and five other mathematicians during a summer institute in applied mathematics at Washington State University in 1972. The difficulty of the problem is reflected in the paucity of unambiguous results obtained by this

group (which included some leading specialists in dynamical polysystems). One of us has made some progress on a simpler problem involving only two goods (see Howitt, 1977).

4. The divisor function is defined more precisely in the mathematical appendix. For a discussion of some elementary properties of the function, see Niven and Zuckerman (1972), pp. 4-6.

5. See Baumol (1952), Perlman (1971), Feige and Parkin (1971). Barro and Santomero (1976) deal with the determination of the payments period.

6. Cf. Grossman and Policano (1975).

7. See any of the references cited in footnote 2.

8. This formula was first established by Georges Monette of the University of Western Ontario for the special case where D and S are integers. Monette's proof is set out in full in an earlier version of the present paper (Howitt and Clower, 1974). Though it bears little resemblance to the proof given here, it played a crucial role in all of our early work.

9. Graphs corresponding to positive values of θ are simply vertical displacements of that illustrated in Figure 2.

10. These points correspond to the line with $g=1$ in Figure 3.

11. See Barro (1976).

12. For historical accounts of this question, see Gilbert (1953) and Patinkin (1965).

13. See Cagan (1956).

14. The failure of previous models to take into account bunching costs may also explain their failure to account for the absolute magnitude of the typical household's money holdings, which these models all tend to underestimate. Cf. Barro and Fischer (1975).

15. See Samuelson (1968), Friedman (1969), Feige and Parkin (1971).

16. See Gramm (1974).

17. The only exception would be the highly special case of zero bunching costs and payment of competitive interest on holdings of money balances (cf. above, p. 26).

18. Thus the result obtained by Grossman and Policano (1975, p. 1106 and footnote 9) follows not from any asymmetry between purchases and sales in this respect, but because only the longest of their transaction intervals, which happens to be a purchase interval, is long enough to support positive temporary bond holdings.

19. Cf. Grossman and Policano (1975, p. 1110).

20. For further elaboration, see Heller (1972).

21. Cf. Hahn (1973), pp. 229-34, and p. 241; also Perlman (1973) and Russell (1974).

22. This type of market arrangement is described in more detail by Howitt (1974), Clower (1975), and Clower and Leijonhufvud (1975).

23. This logical difficulty in specifying the standard budget constraint is discussed by Clower (1965); the existence of specialist traders permits primary traders to make "notional" plans "effective" except in abnormal situations where stocks of inventories held by specialists are temporarily exhausted or grossly in surplus.

24. For a more extensive account of these matters, see Clower (1975), pp. 14-18.

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