Optimal Water Pricing with Cyclical Supply and Demand

by

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1. Introduction

Since the seminal analysis by a group of French economists (see for example Boiteux (1949), Massé (1946), there has been a continuing stream of papers on the optimal pricing of a commodity when demand fluctuates in a periodic fashion. Topics analysed in this "peak load pricing" literature range from the Steiner-Hirshleifer exchange (1957-58) over whether marginal cost pricing is discriminatory, to the examination by Wellisz (1963) and Bailey and White (1974) of the impact of "fair return" regulation, and Panzer's discussion (1975) of optimal pricing with input substitution.[1]

There is also an extensive engineering-economic literature on the optimal release of storable commodities, with emphasis on water. An early paper by Koopmans (1957) examines optimal water storage with variable inflow, given the objective of satisfying some exogenously determined demand. Many of the major contributions to this literature are mentioned in Gablinger (1971), Roefs (1968), Becker and Yeh (1974), and Sobel (1975).

Unfortunately there has been little formal development of the links between these two areas of study. When optimal water storage is examined, a benefit or "value" criterion is usually assumed without any discussion of whether associated pricing policies for water on power have feedback effects on such a criterion. The relation between prices and welfare is, of course, central to the peak load literature, although until recently, the latter has focused almost exclusively on the pricing of a non-storable commodity.

An exception to this is an interesting paper by Jackson (1973) which combines the pricing and storage issues in a simple two period model in which

[1]A more complete list of references and discussion of recent developments are summarized in Bailey and Lindberg (1975).

unused electric power generating capacity in off-peak periods can be used to store energy in a pumped storage plant, which then generates during the peak period. He shows first that the simple rule that peak users should pay all marginal capacity costs is no longer optimal, and then develops a static model of evaluating plant expansion programs. However the simplicity of Jackson's model precludes an examination of the optimal timing of price changes and water release rates over a complete cycle. The interrelationship between pricing and storage has also been considered by Nguyen (1976) in a multi-period, linear cost, production-inventory model. Nguyen finds that introduction of storage reduces peak price and increases off-peak price, and reduces the optimal number of producing technologies used.

Although Nguyen mentions water storage and Jackson clearly uses it for periodic energy storage, neither author is concerned with optimal storage and pricing of water as a naturally occuring renewable resource. We shall consider this case of peak (seasonal) pricing in this paper. In particular, we consider the case where the periodic inflows are exogeneous inputs. This sets our work apart from Nguyen's study and the well developed literature in deterministic and stochastic inventory analysis [Scarf, Arrow and Karlin, 1950], for all of these assume the input to storage is "instantly" manufactured at some known cost. In these storage models, input to storage is a decision variable rather than an exogeneous "natural" input.

In addition, we depart from the standard assumption of "n" discrete demand periods and consider a continuous (price-sensitive) demand approach which yields naturally articulated demand periods. In this regard we follow Joskow's (1975) suggestion that some thought should be given to definition of relevant demand periods. The season-specific[2] prices produced by this analysis are consistant with the result obtained by Jackson (1973) and by Wenders (1976) in their treatment of a discrete demand curve, namely that some capacity charges are borne by off-peak users as well as peak users. The model studied is still relatively simple, but the insights gained are fundamental and relevant to the important policy decisions continually being made in water storage and allocation.

2. A Model of Water Supply Pricing with Storage

To focus on essentials we begin with a single community with rights to a natural surface water supply with periodically fluctuating flow rate. Presumably the community wishes to invest in storage facilities so as to allocate the flow over the period to meet demand in some optimal manner. However, as we suggest in the introduction, a fundamentally optimal allocation of social resources (concrete and steel, as well as water) must allow the opportunity cost of storage and treatment to feed back on demand via the price mechanism. Hence the community water planners are faced with the problem of developing an optimal investment-operating-pricing policy.

Most peak load pricing models assume two (or at least a finite number of) pricing periods. While this is primarily to keep things simple, it is also obvious that although flows and demands are continuous, a continuous adjustment of prices over the cycle would be operationally impractical.

[2] We are concerned with seasonal pricing and over-season storage in large expensive reservoirs. Water demand also varies with time of day, but these variations are accomodated by small water storage facilities (usually water towers) whose capacities are governed by fire fighting flow requirements that dominate regular peak season daily peak flow demands. See Hanke (1975).

Nevertheless, rather than subdivide the period of analysis into a finite number of subperiods it is not only analytically convenient, but practical as well to go to the limit and consider the problem in continuous time. It is practical because this continuous approach actually yields a pricing policy that may be naturally articulated in a very few discrete sub-periods.

To begin the analysis, we let X(t) be the cumulative reservoir inflow of water over the interval [o, t], where t is less than or equal to T, the length of the cycle (e.g., T=1 year).[3] We define the instantaneous inflow rate, I(t), at time t as the time rate of change of X(t). In integral form,

$$X(t) = \int_0^t I(\tau)d\tau.$$

Suppose a dam of capacity V_c is to be constructed and with it water treatment facilities (for purification) of capacity Q_c. Writing the instantaneous flow of water to the community at time t as Q(t) and the instantaneous stock of water behind the dam as V(t) the following feasibility conditions must be satisfied.

$$Q_c - Q(t) \geq 0 \quad (1)$$

$$V_c - V(t) \geq 0 \quad (2)$$

$$V(t) \geq 0 \quad (3)$$

$$I(t) - Q(t) - V'(t) \geq 0 \quad (4)$$

The first two conditions are capacity constraints and the last is the requirement that the net rate of increase in stored water cannot exceed the difference between inflow and consumption. If the inequality is strict, excess water is being spilled over the dam.

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[3]Presumably the appropriate cycle-period would be a year with spring run off and summer low flows. However, the results are general, for any period or cycle of flow.

For simplicity evaporation is ignored. However, as a first cut at taking the latter into account we could assume evaporation is equal to some seasonal rate of evaporation, $\beta(t)$ per acre of surface area. Since surface area, S, is a function of reservoir volume, we have $S(V(t))$, and evaporation is $\beta(t)S(V(t))$. We could then rewrite the flow constraint as:

$$I(t) - \beta(t)S(V(t)) - Q(t) - V'(t) \geq 0. \qquad (5)$$

The gross benefit $G(Q(t),t)$ of a flow $Q(t)$ at time t is measured as the area under the inverse demand curve $p = D^{-1}(Q(t),t)$. That is

$$G(Q(t),t) = \int_0^{Q(t)} D^{-1}(Q(t),t)dQ(t). \qquad (6)$$

Then total gross benefit over the cycle is given by:[4]

$$\int_0^T G(Q(t),t)dt, \qquad (7)$$

To complete the model, a description of costs is necessary. Greater storage capacity requires a bigger dam and hence larger initial capital investment and associated interest charges as well as greater maintenance costs. In addition, a larger storage capacity increases the opportunity costs associated with other uses of the flooded land.[5] These storage capacity costs are assumed to be incurred at a rate which is constant over time, $A(V_c)$. Finally it is assumed that the cost of operating the

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[4] Implicit in such a formulation is an assumption that the demand for water at one moment is unaffected by the price of water at all other times. While this is an approximation to reality it should be recalled that the time periods here are seasons, not hours or days. Hence, for the most part, the demand in one season will be independent of price in other seasons, because most consumers cannot or will not inventory water during one season inorder to use it in an adjoining season.

[5] We do not consider recreational use of the lake. However, since the value of such use is often highly dependent on lake level (high water -- low water differential) this consideration would make an interesting extension, especially if there were a lake level-dependent demand curve for recreation involved.

purification plant at the rate $Q(t)$ is $kQ(t)$ with interest and maintenance charges accumulating at the rate $B(Q_c)$.

3. Optimal Storage and Pricing

The net social gain from a release policy $Q(t)$ satisfying constraints (1) - (4) is the area under the demand curve less costs integrated over the cycle, that is:

$$S = \int_0^T G(Q(t),t)dt - \int_0^T [A(V_c) + B(Q_c) + kQ(t)]dt \qquad (8)$$

To solve for the optimal policy we form a Lagrangian expression which, for this continuous model, is an integral over the cycle.

$$L(Q_c,V_c) = \int H[(Q(t), V(t), \lambda(t), \mu(t), \nu(t), \theta(t); Q_c, V_c]dt, \qquad (9)$$

where

$$H = G(Q(t),t) - A(V_c) - B(Q_c) - kQ(t) + \lambda(t)(I(t)-Q(t)-V'(5)) +$$

$$\mu(t)(Q_c-Q(t)) + \nu(t)(V_c-V(t)) + \theta(t)V, \qquad (10)$$

where $\lambda(t)$, $\mu(t)$, $\nu(t)$ and $\theta(t)$ are time-dependent Lagrangian multipliers. The two time dependent variables $Q(T)$ and $V(t)$ must satisfy the first order Euler conditions and the end conditions $Q(0) = Q(T)$, $V(0) = V(T)$. Furthermore the two "capacity" variables must satisfy first order integral conditions.

For any Q_c, V_c we can in principle calculate the optimal profiles. Thus we can write:

$$S = S(Q_c,V_c) = \int_0^T H(Q_c,V_c)dt. \qquad (11)$$

At the optimum we must also have:

$$\frac{\partial S}{\partial Q_c} = \int_o^T \frac{\partial H}{\partial Q_c}\, dt = 0. \tag{12}$$

Then from (10) we have:

$$\frac{\partial S}{\partial Q_c} = \int_o^T \mu(t)dt - T \cdot B'(Q_c) = 0. \tag{13}$$

At the optimum we also have:

$$\frac{\partial S}{\partial V_c} = \int_o^T \frac{\partial H}{\partial V_c} dt = 0, \tag{14}$$

and from (10):

$$\frac{\partial S}{\partial V_c} = \int_o^T \nu(t)dt - T \cdot A'(V_c) = 0. \tag{15}$$

Finally the complementary slackness conditions require that each Lagrange multiplier (which must be non-negative) is zero if its corresponding constraint is slack. If the instantaneous demand curve $D^{-1}(Q(t),t)$ is finite and downward sloping for all non-negative Q, the benefit function $G(Q(T),t)$ is concave. Hence for any given (Q_c,V_c) the Euler conditions are also sufficient. In addition, given the linearity of the constraints, it can be shown that benefits are a concave function of (Q_c,V_c). Then if $A(V_c)$ and $B(Q_c)$ are convex the first order conditions define a unique global maximum.

Since H is not a function of $Q'(t)$ we have from the Euler conditions:

$$\frac{\partial H}{\partial Q} = \frac{d}{dt}\left(\frac{\partial H}{\partial Q'(t)}\right) = 0, \tag{16}$$

so from (10) we have:

$$\frac{\partial G}{\partial Q} - k - \lambda(t) - \mu(t) = 0. \tag{17}$$

Also

$$\frac{\partial H}{\partial V} = \frac{d}{dt}\left(\frac{\partial H}{\partial V'(t)}\right) , \tag{18}$$

so from (10) we have:

$$\lambda'(t) = \nu(t) - \theta(t) \tag{19}$$

The major results may now be derived from these equations. From eqn. (13), we find the continuous time equivalent of the capacity condition in the standard peak load problem. "$\mu(t)$" is the shadow value of increasing water treatment hydraulic capacity at time t, so the integral is the total value of additional capacity over the cycle. At the optimum this shadow value is equated with marginal capacity costs. Similarly (15) is the requirement that the shadow value of additional storage capacity over the cycle must equal the marginal cost of storage. We now return to the Euler conditions. First we note from (6) that $\frac{\partial G}{\partial Q}$ is the ordinate of the demand curve when the supply delivered (flow) of water is Q(t). In other words, $\frac{\partial G}{\partial Q}$ is the equilibrating price level, $p^*(t)$. Then from (17) we have the following pricing rule:

$$p^*(t) = k + \lambda(t) + \mu(t). \tag{20}$$

Since the Lagrange multipliers (shadow prices) are non-negative, eqn. (20) indicates that the optimal price of water never drops below treatment operating costs, k. Furthermore, from eqn. (13) there must be at least one interval over which $\mu(t)$ is positive when treatment capacity is fully utilized. Then over this interval the optimal price is strictly greater than k.

These results also tell something about $\lambda(t)$, the shadow cost of reservoir spilling. When surplus water spills over the dam, eqn. (4) is slack, and $\lambda(t) = 0$. Since the reservoir is full, it cannot be empty, so $\theta(t) = 0$. Furthermore, from (15) there must be at least one interval over which $\nu(t) > 0$ (that is, an interval over which the lake is full). Then from eqn. (19), $\lambda'(t) > 0$, so $\lambda(t)$ is increasing. When the reservoir is neither full nor empty, $\theta(t) = \nu(t) = 0$, and $\lambda'(t) = 0$, so $\lambda(t)$ is constant. Finally, given the cyclical nature of the problem, if $\lambda(t)$ increases, it must also decrease. For this to occur, $\lambda'(t)$ must somewhere be negative, requiring $\theta(t) > \nu(t)$. But this condition holds only when the reservoir is empty. Hence we have $\lambda(t)$ decreasing when the reservoir is empty.

To be able to describe the actual welfare maximizing profiles of stored water and prices requires complete quantitative specification of the benefit, cost and water supply functions. However the qualititative behavior of $V^*(t)$ and $p^*(t)$ can be deduced from rather general considerations. For example, suppose that the rate of inflow $I(t)$ and water demand profile with price equal to marginal operating cost k, are approximately 180° out of phase. Two such profiles are depicted as dotted curves in figure 1a. At the beginning of the cycle ("winter") the rate of inflow is greatest and demand is smallest. At the midpoint ("summer") the inflow rate is at a minimum and unadjusted demand is at a peak.

Initially we assume purification capacity costs are negligible. Then the optimal price is, from (20):

$$p^*(t) = k + \lambda(t).$$

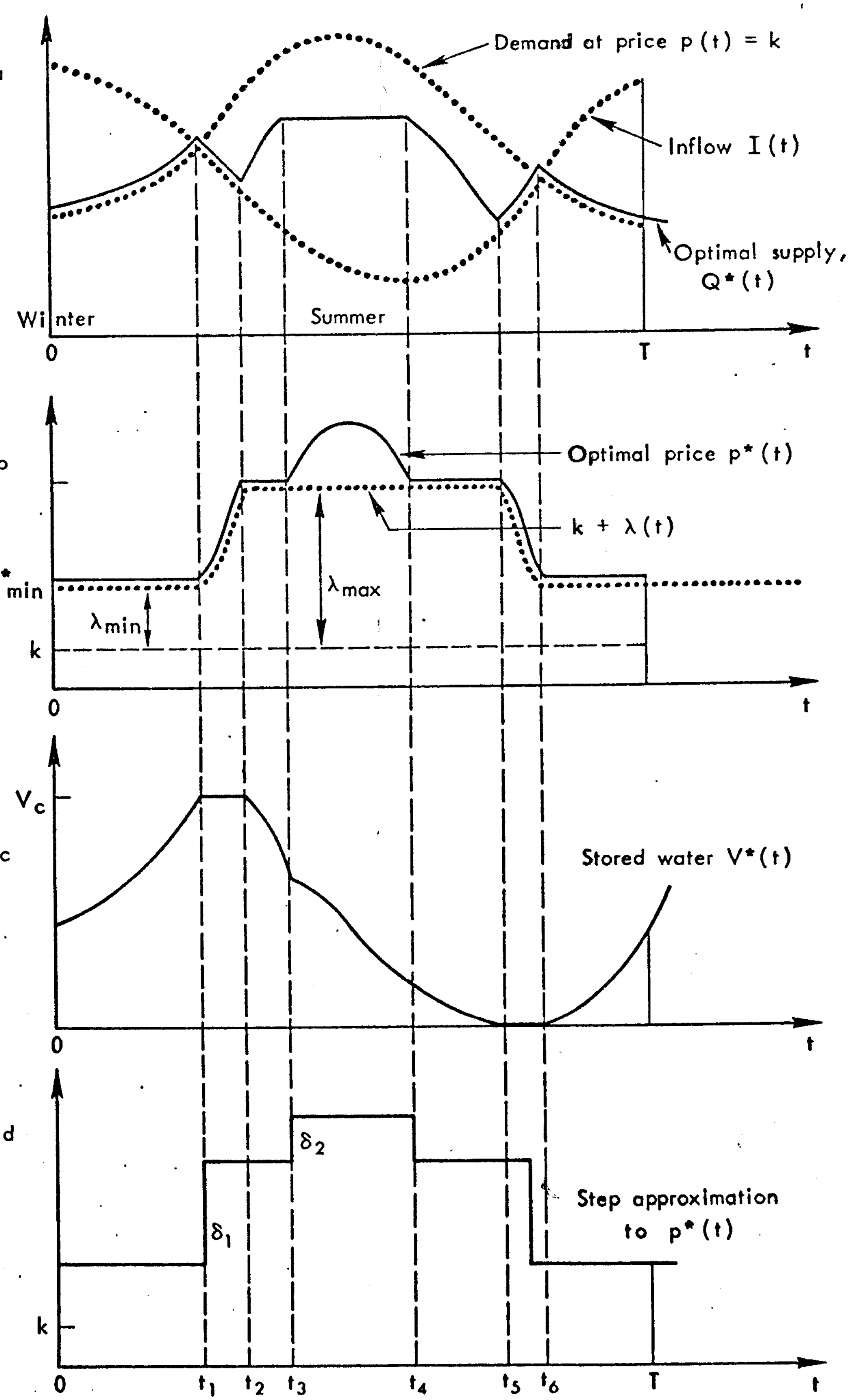
During periods when the lake is neither full nor empty the shadow prices $\nu(t)$ and $\theta(t)$ are both zero so from (19), $\lambda(t)$ and hence $p^*(t)$ are constant.

Fig. 1a

Fig. 1b

Fig. 1c

Fig. 1d



Moreover, we know that there is a period while the lake is full in which $\lambda(t)$ rises and similarly a period when the lake is empty in which $\lambda(t)$ is falls. Then the profile of $k+\lambda(t)$ must be as depicted by the dotted curve in figure 1b. As drawn $\lambda(t)$ is everywhere strictly positive. Note from figure 1a that for part of the cycle, demand with price equal to the marginal cost of operating the purification plant is less than the inflow. Hence in the absence of storage capacity the minimum price would be k. But for the case depicted, the minimum "off-peak" price is always above the marginal operating cost of purification because the addition of storage makes water "scarce" throughout the cycle.

Whether or not this is optimal depends upon the size of the fluctuations in supply and demand and on the marginal cost of storage capacity. When the latter is close to zero, as much storage capacity as necessary can be built. Then unless demand aggregated over the cycle with water priced at k, is less than total supply, it is necessary to charge a higher constant price $p^*(t) = \lambda^* + k$. At the other extreme, with high marginal costs of storage, it is optimal to introduce only a small storage capacity. As a result there is spillage over the dam during the interval in which inflow exceeds consumption. Certainly during spillage, $\lambda(t)$ is zero. Hence for such cases there is an interval in which price falls to the marginal cost of purification.

We now describe the cycle in more detail, noting that there are two cases that must be considered. In the first ("scarce") case, the addition of storage makes water "scarce" throughout the year, as noted above. In the second case, water is sufficiently abundant (relative to demand) that reservoir spillage occurs in the spring prior to the high demand period. Reservoir capacity is required to carry water over from one season to another, but does not cause an overall scarcity. We discuss the scarce case first. We begin with the two intervals $[t_1,t_2]$, $[t_5,t_6]$ over which $\lambda(t)$ and hence $p^*(t)$ are changing and the lake is respectively full and empty. Since $V'(t) = 0$ and assuming $\lambda(t)$ is positive in these intervals, we have from (4) that inflow and consumption rates are equal, as depicted by the solid curve in figure 1a. The lake fills during the winter $[0,t_1]$ during which time the price p^* is set at the minimum level. It then rises to a new plateau while the lake remains full. Since $\lambda'(t) = 0$ when the lake is neither full nor empty, price $p^*(t) = k + \lambda_{min}$ is held at this plateau as the lake empties $[t_2,t_5]$ and then falls again when the low water mark is reached. The picture is completed with the introduction of purification capacity costs. We raise $p^*(t)$ just enough so that the integral of this price difference over time is equal to marginal purification capacity costs. Formally from (13) we require:

$$TB'(Q_c) = \int_0^T \mu(t)dt = \int (p^*(t) - k - \lambda(t))dt \ .$$

This effectively cuts the top off the peak. The optimal price profile with purification capacity costs is then the solid curve in Figure 1b.

In the second case, we have $\lambda(t) = 0$ during part of the period $[t_1,t_2]$, because the reservoir is spilling. Then neglecting treatment capacity again, $p^*(t) = k$ over this interval. Since $\lambda'(t) = 0$ before the reservoir is full, $\lambda(t) = 0$ during $[0,t_1]$ and hence during $[t_6,T]$. However, since some storage capacity is built, $A'(V_c) > 0$, so $\nu(t)$ must be somewhere positive by eqn. (15). Since $\theta(t)$ cannot be positive when $\nu(t)$ is positive, we have $\lambda'(t) = \nu(t) > 0$. Hence $\lambda(t)$ increases during $[t_1,t_2]$ after the reservoir ceases to spill. Then $p^*(t)$ increases until the reservoir begins to empty, at which point $\nu(t) = \mu(t) = 0$, and $\lambda'(t) = 0$, so $p^*(t)$ is constant at $k + \lambda_{max}$ during $[t_2,t_5]$. The rest of the cycle is the same as the scarce case, except that $p^*(t) = k$ during $[0,t_1]$ and $[t_6, T]$. In this second case off-peak users clearly do not pay capacity costs.

The conclusion that the optimal price of water should fall when the water level has reached its lowest point seems at first surprising. However it should be noted that the lake reaches the minimum level during an interval in which the water supply is rapidly increasing. In fact, it is the higher price level prior to this moment which slows the rate of consumption and hence the rate at which the lake empties until well after the mid-summer trough.

Essentially what has been achieved is a reduction in the fluctuations of the supply by storing water between the supply peak and the following trough. Since storage is costly it is not optimal to eliminate fluctuations in water supply entirely. Instead the dam is just large enough so that the benefits that would result from an increased storage capability are exactly offset by the cost of a higher dam. Then to avoid shortages, the price of water must be varied over the cycle in order to balance supply and demand.

To see just how large the price differential should be we note that the only interval in which $\nu(t)$ exceeds zero (and the lake is full) preceeds the interval in which $\theta(t)$ exceeds zero. Then integrating (19) and using (15) we have:

$$\lambda_{max} - \lambda_{min} = \int_0^T \nu(t)dt = T \cdot A'(V_c) \quad .$$

The right hand side is of course the marginal cost of storage capacity. Then in the absence of purification capacity costs price differential is just equal to the marginal cost of storing winter water for summer consumption. A further price adjustment takes place during the period of peak demand, reflecting non-zero marginal costs of purification capacity.

It is instructive to compare the optimal pricing policy with prices in the absence of storage. For simplicity the purification capacity constraint is ignored. Over the intervals $[0,t_1]$, $[t_6,T]$ inflow exceeds demand so without storage the price is set at k, that is, to just cover operating costs. In the interval $[t_1,t_6]$ the price is adjusted upwards to restrict the rate of consumption to the inflow rate. From figure 1a this implies that without storage, consumption is lower and price is higher over the interval $[t_2,t_5]$. Therefore storage achieves a reduction in price fluctuations over the cycle. Not only is the price at the peak lowered but in general the minimum price is raised.

From the above results it is clear that the simple rule that off peak users pay marginal operating costs while peak users incur in addition all marginal capacity costs, is not appropriate for this very important case of water supply (where storage is involved). While it is true that the summer price increase reflects the marginal cost of storage capacity, the winter users pay more than operating costs whenever the introduction of storage has the effect of making the constraint on the aggregate supply of water binding. It should not be inferred however that the marginal cost pricing foundation upon which the simple "peakers pay" rule was derived, is in any way weakened. In the model described above, all users in periods for which some constraint on capacity is binding (purification, storage or aggregate water supply) pay a share of the marginal cost of this capacity constraint.

Finally we note that there is a natural three step approximation to the optimal price profile $p^*(t)$. This is depicted in figure 1d. Consumers pay a minimum price $k + \lambda_{min}$ during the winter months, then the price is raised by δ_1 during the summer months in which the lake is emptying. The price differential is set just high enough to cover the marginal cost of storage capacity. In addition there is a further price hike of δ_2 around mid-summer which reduces peak demand. This is set just high enough to cover the marginal cost of purification capacity.

This natural approach to pricing is important, because it offers a solution to the problem of defining the number and duration of peak and off-peak seasons. In this regard, note that unlike the case of electric power, the cost of time-specific meters need not be included in the peak load pricing analysis, since present meters are generally read on a much shorter interval than the seasons considered herein.

4. Summary

This paper attempts to begin to bring together the parallel literatures on peak load (water) pricing and reservoir planning and operation. In particular, seasonal demands are assumed to be affected by the cost (water price) of meeting them. We consider the investment-operating-pricing problem in a static context using continuously varying price-sensitive demand. This approach to demand automatically yields a small set of periods during which price increases and then decreases: A peak load pricing policy. We have shown that introduction of storage lowers the prices that would be charged in the peak season and increases off peak season prices. In particular, we show that storage facilitates reallocation of natural supply in such a way that price in all seasons may exceed marginal operating costs, a deviation from the standard "peakers pay capacity costs" rule.

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