ASYMMETRICAL PRODUCTION POSSIBILITIES, THE SOCIAL GAINS FROM INEQUALITY, AND THE OPTIMUM TOWN

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Discussion Paper #78
August 1976

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ABSTRACT

Mirrlees demonstrated that in a town in which land is a consumer good, identical individuals should not in general have equal utilities at the social welfare optimum. One aim of this paper is to provide a simple exposition and intuitive explanation of this result, and to investigate the determinants of the distribution of utilities at the social welfare optimum. The cause of this inequality is shown to be an individual-specific asymmetry in aggregate production possibilities. Another aim of the paper is to demonstrate that the essential results of Mirrlees' optimum town paper generalize to all situations with this form of production asymmetry.

ASYMMETRICAL PRODUCTION POSSIBILITIES, THE SOCIAL GAINS FROM INEQUALITY, AND THE OPTIMUM TOWN*

In a seminal article in residential location theory, Mirrlees discussed the welfare economics of "The Optimum Town." In the abstract of the paper, he states:[1]

> The paper discusses the welfare economics of a model in which land is a consumer good. The inhabitants of towns all work at the center of the town, but have to live some distance away, since they like living-space. The main conclusions are, first, that in general it is not optimal for identical people to have equal utility;--

That the optimum should yield different levels of utility to identical people has puzzled many, since one ordinarily expects identical individuals to be treated in the same way with a symmetric social welfare function.

Mirrlees, in attempting to explain this result, makes the following observation:[2]

> I sometimes think it is obvious that this result should be so: people cannot, when their locations have to be related to a single central point, be treated identically, and there is no reason, from a purely utilitarian standpoint, why different treatment should lead to the same utilities.

If locations have different characteristics and each individual must be assigned to a single location, it should be clear that there is no presumption that individuals will have the same utility at the social welfare optimum. An extreme example makes the point graphically. A ship is sinking. There are five survivors, but only room for four in the lifeboat. The two locations are "in the lifeboat" and "not in the lifeboat." With a Benthamite social welfare function, one would perish while the other four survive.

While this example demonstrates the plausibility of inequality at the social welfare optimum, it is not immediate that the distribution of utility in the optimum town should have any particular form. Mirrlees, while demonstrating that inequality was the rule rather than the exception, did not discuss the utility distribution in detail. More recently Riley [5] has shown that, when land is a normal good and when there is no preference for location *per se*, individuals further out will enjoy greater utility at the social welfare optimum. However, he was unable to provide an intuitive explanation of this result.

In the first section of this paper we focus on the central elements of the problem utilizing a simple example. When there are transportation costs and no locational preferences, the results derive from an asymmetry in production possibilities--fewer resources are needed to provide a person further from the center of the town with an extra unit of land than a person closer to the center. In the second section we demonstrate that under weak assumptions, whenever there is a production asymmetry in the sense that one or more commodities can be produced more cheaply for a particular individual, the latter receives higher utility at the social optimum. The urban problem and also the optimal income taxation problem with skill differentials, considered by Mirrlees [4] are then shown to exhibit such an asymmetry and the implications for the distribution of utility are derived.

I. The Optimum Town: An Example

In this section we first work through a specific numerical example. Then we examine more generally the determination and characteristics of the social welfare optimum in the optimum town.

We want to derive the utility-possibility frontier over the two locations, and show that a social welfare contour will not, in general, be tangent to the frontier when utilities are equal.

To derive the utility-possibility frontier we maximize individual 1's utility with individual 2's utility fixed, subject to a land constraint and a community income constraint.

Let U^i be the utility of individual i, i = 1, 2

y_i be the length of the lot occupied by individual i, i = 1, 2

c_i be the amount of the composite good given to individual i, i = 1, 2

The land constraint specifies that land allocated to the two individuals together must be consumed by either one or the other individual.

$$y_1 + y_2 = 1$$

The community income constraint states that community income must be spent on either the composite good or on transportation costs. Individual 1's expenditure on transportation is $0.5y_1$ and individual 2's $y_1 + 0.5y_2$. Thus,

$$c_1 + c_2 + y_1 + 0.5(y_1 + y_2) = 1.5$$

Substituting the land constraint into the community income constraint yields

$$c_1 + c_2 + y_1 = 1$$

The maximization problem is therefore

$$\max_{c_1, y_1} \quad c_1^{1/2} y_1^{1/2} \quad \text{s.t. } \bar{U}^2 = (1 - y_1 - c_1)^{1/2}(1 - y_1)^{1/2}$$

From the first-order conditions we obtain

$$c_1 = 2y_1 - 2y_1^2 \quad \text{and} \quad c_2 = 1 - 3y_1 + 2y_1^2$$

Thus,

$$U^1 = \sqrt{2}\, y_1(1 - y_1)^{1/2} \quad \text{and} \quad U^2 = (1 - y_1)(1 - 2y_1)^{1/2}$$

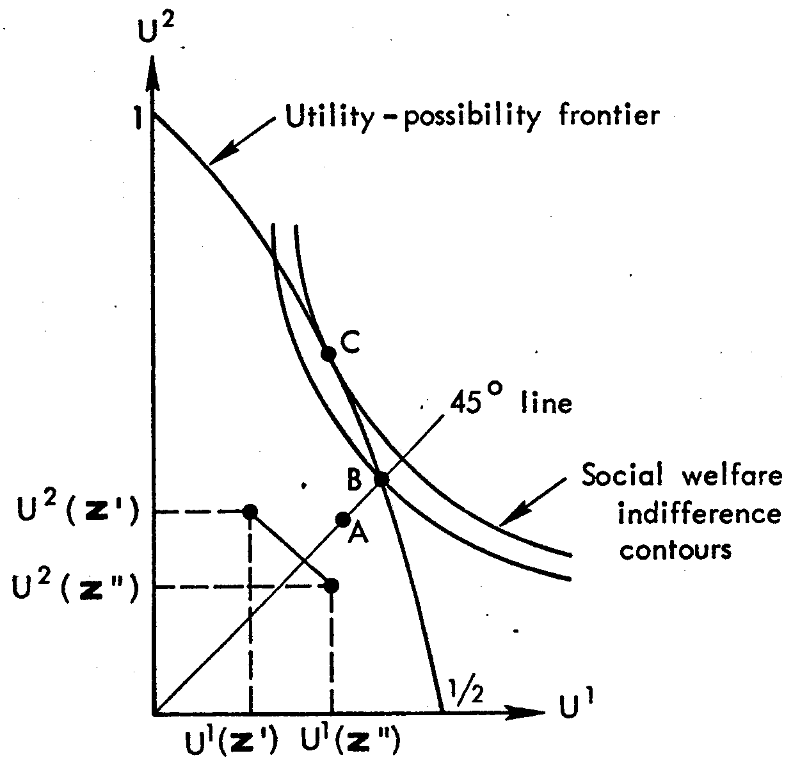
Differentiating U^1 and U^2 with respect to y_1 yields

$$\frac{dU^2}{dU^1} = -\left(\frac{2 - 2y_1}{1 - 2y_1}\right)^{1/2}$$

When $U^1 = U^2$, $y_1 = 1/3$, and $\frac{dU^2}{dU^1} = -2$. At the equal utilities Pareto optimum, therefore, $\frac{dU^2}{dU^1}$, the slope of the utility-possibility frontier is less than minus 1. For all y_1 such that $\frac{dU^2}{dU^1}$ is negative[4] it is easy to check that this slope is decreasing in y_1 and that U^1 is increasing. Then the set of feasible utility levels is convex.

A utility possibility frontier with these characteristics is drawn in Figure 1. Also shown in the figure are social welfare indifference contours. We assume that the social welfare function is symmetrical and quasi-concave, in which case a contour must have a slope which is no less than minus 1 for any pair (U^1, U^2) such that U^1 is greater than U^2. Then unless the indifference contour is L-shaped (implying infinite social aversion to inequality) or kinked, individual 2 is favored at the social welfare optimum.[5]

In order to understand the above result it is useful to consider as a starting point the allocation which leaves each individual with the same amount of land and composite good (point A in Figure 1). Why is this allocation Pareto inferior? The shadow price of land at some location, where there are no preferences for location _per se_ and where the opportunity cost of land at the boundary of the city is zero, can be shown to equal the additional transportation costs incurred by the economy if the individual at



U^2
1
Utility-possibility frontier
C
45° line
B
Social welfare indifference contours
$U^2(\mathbf{z}')$
A
$U^2(\mathbf{z}'')$
1/2
U^1
$U^1(\mathbf{z}')$ $U^1(\mathbf{z}'')$

Fig. 1

that location is given an additional unit of land, everyone else's lot size remaining unchanged. Thus, it is cheaper to give land to an individual further from the center of the city. This can be described as an asymmetry in production since it is cheaper to "produce" land for individual 2 than for individual 1. The shadow price of providing the composite good is the same for both individuals. A condition for Pareto optimality is that the marginal rate of substitution between two commodities for any individual equal the ratio of their shadow prices. At A, the MRS's between land and the composite good for the two individuals are the same, but the ratio of the shadow prices are not; therefore, the Pareto optimality condition cannot hold for both individuals simultaneously. The optimal allocation when both individuals have the same utility (point B in Figure 1) entails giving more land to 2 than to 1, and more of the composite good to 1 than to 2, as shown in Figure 2.[6] The absolute value of the slope of $U = \bar{U}$ gives the ratio of the shadow prices. Then the shadow price of land relative to the composite good is higher for 1 than for 2.

Having shown that the equal utilities point B involves different consumption bundles, it remains to demonstrate that under weak conditions the utility possibility frontier is not symmetrical but favors individual 2. In geometrical terms it must be shown that the absolute value of the slope of the frontier at B exceeds unity.

Suppose that the marginal utility of consumption of C is diminishing ($U_{cc} < 0$) and that the two goods are Edgeworth complements ($U_{cy} > 0$). It is easy to check that these conditions ensure that Y is a normal good.[7]

We now note that at any point on the frontier a small reallocation of C from individual 1 to individual 2 is feasible (at least in one direction).

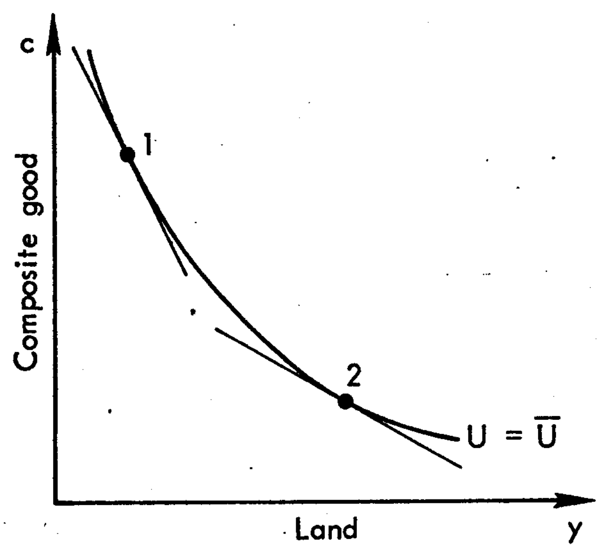


Fig. 2

Then the steepness of the utility possibility frontier at any point is simply the ratio of the marginal utilities of C; that is

$$- \left.\frac{dU^2}{dU^1}\right|_{UPF} = \frac{\partial U^2}{\partial c_2} \Big/ \frac{\partial U^1}{\partial c_1}$$

But we have already seen that at the point of equality $c_1^* > c_2^*$. Moving along the indifference curve from point 1 to point 2 in Figure 2 then involves increasing Y and decreasing C. Given our assumptions, both work in the direction of increasing the marginal utility of C. Then the right hand side of the above equation is strictly greater than unity.

Mirrlees' model differs somewhat from the one considered so far since he includes location in the utility function, rather than treating transportation costs explicitly. Inequality results here for a rather different reason--the indivisibility of commodities. One individual must be given the more preferred location, another the less.[8]

II. General Results

In the previous section we examined in detail one example in which asymmetrical production possibilities result in inequality at the social welfare optimum. In this section we show that whenever production possibilities are asymmetrical in the sense that a commodity can be produced more cheaply for some consumers than for others, the individual for whom production is cheaper is favored at the social welfare optimum if and only if the good is normal. We then apply the analysis to two examples. The first case is the optimal income tax problem treated by Mirrlees [4]; the second, the optimum town problem considered in the previous section.

In order to focus on essentials we consider the simplest of economies: two individuals and two commodities. Generalizations to a many-person, many-commodity world are straightforward.

Let $z_i = (c_i, y_i)$ be the consumption vector of individual i (i = 1, 2). All members of society are assumed to have the same function of utility $U(z_i)$. The notion of a production asymmetry is incorporated in the following description of aggregate production possibilities.

$$S = \{(c, y_1, y_2) \mid c = \sum_1^2 c_i,\ F(c, y_1, y_2) \leq 0,\ \frac{\partial F}{\partial y_1}(c, y_1 y_2) \neq \frac{\partial F}{\partial y_2}(c, y_1, y_2) \text{ where } y_1 = y_2\}$$

It is convenient to impose the further restriction that the production possibility frontier F is a concave function implying that the production set S is convex.

Consider two feasible allocations (z_1', z_2'), (z_1'', z_2''). Given our assumption that S is convex, it follows that any convex combination

$$(\lambda z_1' + (1 - \lambda) z_1'',\ \lambda z_2' + (1 - \lambda) z_2'') \qquad 0 < \lambda < 1$$

is also feasible. Moreover, given the concavity of U we must have:

$$\begin{bmatrix} U(\lambda z_1' + (1 - \lambda) z_1'' \\ U(\lambda z_2' + (1 - \lambda) z_2'' \end{bmatrix} \geq \lambda \begin{bmatrix} U(z_1') \\ U(z_2') \end{bmatrix} + (1 - \lambda) \begin{bmatrix} U(z_1'') \\ U(z_2'') \end{bmatrix} \qquad (1)$$

We can represent all the vectors on the right hand side of (1) as vectors in utility space on the line joining $(U(z_1'), U(z_2'))$ and $(U(z_1''), U(z_2''))$, as in Figure 1. From (1) all points on this line can definitely be achieved by the feasible allocation $\{(\lambda z_i' + (1 - \lambda) z_i''); i = 1,2\}$. To summarize, we have demonstrated:

Lemma 1: If U(z) is concave and the set of feasible allocations is convex, the utility possibility set is convex.

We now assume that at all output levels the cost of production of Y for consumption by individual 1 is higher than for individual 2. Expressing this as the opportunity cost in terms of commodity C, we require:

$$\frac{\partial F}{\partial y_1} / \frac{\partial F}{\partial c} > \frac{\partial F}{\partial y_2} / \frac{\partial F}{\partial c} \tag{2}$$

We can now derive:

Lemma 2: If U(c,y) is quasi-concave and the cost of production of Y for consumption by individual 1 is higher than for individual 2, the allocation at the point of equality on the utility possibility frontier satisfies $c_1^* > c_2^*$, $y_1^* < y_2^*$.

Proof: The necessary conditions for Pareto efficiency are that each individual's marginal rate of substitution should equal the marginal rate of transformation, that is:

$$- \frac{dc_1}{dy_1}\Bigg|_{U = U^1} = \frac{\partial U^1}{\partial y_1} / \frac{\partial U^1}{\partial c} = \frac{\partial F}{\partial y_1} / \frac{\partial F}{\partial c} \tag{3}$$

$$- \frac{dc_2}{dy_2}\Bigg|_{U = U^2} = \frac{\partial U^2}{\partial y_2} / \frac{\partial U^2}{\partial c} = \frac{\partial F}{\partial y_2} / \frac{\partial F}{\partial c} . \tag{4}$$

Utilizing (2), the right hand side of (3) is greater than the right hand side of (4).

Then in any Pareto-efficient allocation, that is an allocation corresponding to a point on the utility possibility frontier, individual 1's marginal rate of substitution of y for c is greater. Moreover at the point of equality both individuals' consumption bundle must lie on the same indifference curve, as depicted in Figure 2. Since we have shown that individual 1 has a higher marginal rate of substitution of y for c, it follows immediately that his consumption bundle, labelled 1, must lie to the north-west of the consumption bundle of individual 2. Q.E.D.

Then differentiating with respect to the utility level U* we have:

$$\frac{\partial y^*}{\partial U^*} = \frac{\partial^2 M^*}{\partial U^* \partial p_y} = \frac{\partial}{\partial p_y}\left(\frac{\partial M^*}{\partial U^*}\right)$$

The expression $\frac{\partial M^*}{\partial U^*}$ is the additional income needed to maintain a unit increase in the level of utility, that is, the inverse of the marginal utility of income. Then the marginal utility of income falls with a compensated increase in the price of Y if and only if $\frac{\partial y^*}{\partial U^*}$ is positive. But this last expression is simply the marginal effect on consumption of Y associated with increasing the level of utility at constant prices. It is therefore a move out along the income expansion path and is positive if and only if Y is a normal good.

Q.E.D.

If the utility possibility set is convex, it follows immediately that at all points on the frontier such that $U_1 > U_2$, the slope is also less than minus 1. From Lemma 1 this is assured if the feasible product set is convex and U(c,y) is concave (and hence quasi-concave).

Then for any symmetric quasi-concave social welfare function, welfare is maximized with individual 2 receiving higher utility than individual 1. We therefore have finally:

Theorem 2: Under the assumptions of Lemmas 1 and 2 the optimal allocation yields lower utility to individual 1 for any symmetric quasi-concave social welfare function.

It remains to demonstrate that the urban and skill distribution models indeed exhibit an individual-specific asymmetry in production.

In the simplest version of the skill difference model, individuals consume a good (C) and leisure (Y). Each individual has a fixed number of total hours $\bar{y}_i$ and his marginal productivity per hour worked is θ_i. Then the aggregate production set can be written as:

$$c \leq \sum_i \theta_i(\bar{y}_i - y_i).$$

Rearranging we have:

$$c + \sum_i \theta_i y_i - \sum_i \theta_i \bar{y}_i \leq 0.$$

The individual with higher productivity (larger θ) is the individual for which there is a higher marginal cost of leisure. Then if preferences are identical and if leisure is a normal good it follows from Theorem 2 that the more able individual is worse off in the social optimum.[9]

In the rudimentary urban model n individuals are located along a line. The first individual lives adjacent to the "business district" on a property of length y_1. Immediately behind him lives individual 2 on a property of length y_2 and so on. Each lives at the center of his property. For each unit of travel t units of commodity C are used up. Then if in a day the individuals produce $\bar{c}$, aggregate production possibilities are given by:

$$\sum_i c_i + t \sum_i (-0.5y_i + \sum_{j=1}^{i} y_j) \leq \bar{c}$$

Rearranging we have

$$c + \sum_i t(n - i + 0.5)y_i - \bar{c} \leq 0$$

Then if each indvidual has the same concave utility function of the produced commodity C and property size Y it follows from Theorem 2 that the individual further from the business district is favored at the social welfare optimum if land is a normal good.

Concluding Remarks

In this paper we have examined the economic logic behind the Mirrlees result that in the optimum town identical individuals should receive different utilities at the social welfare optimum.

With transportation costs, it is less expensive to give land to individuals further away from the center of the city because this causes a smaller increase in aggregate transportation costs. Unless land is an inferior good, and without locational preferences, individuals closer to the center of the city should receive lower utility at the social welfare optimum. With a preference for location and no transportation costs, the individual living at the preferred location is better off at the social welfare optimum unless location is an inferior good.

We demonstrated generally that with asymmetrical production possibilities and individuals with identical utility functions there is inequality at the social welfare optimum. Also, in these circumstances the individual for whom the cost of producing a commodity is cheaper has higher utility at the social welfare optimum if and only if that commodity is normal in demand.

Many people, when confronted with the Mirrlees result, because they feel it wrong that identical individuals should be treated differently, argue that a maximin social welfare function is appropriate. Of course consistency then requires that they accept the application of the Rawlsian Maxi-min principle to the whole range of distributional issues.

It should be noted however, that if people can be moved costlessly the intertemporal social welfare optimum will involve inequality at any point in time, but equality (in discounted present value terms) over any finite period of time. Maximum social welfare is then achieved by maximizing average utility at any point in time which is equivalent to employing a Benthamite social welfare function.

Even if individuals' locations (or skills) are not interchangeable the argument for equality is not clear-cut. Behind the veil of ignorance individuals may be assumed to have identical ethical beliefs:

$$V^{ji} = V^j(U^1, \text{—}, U^j, \text{—}, U^N) \qquad i, j = 1, \text{—}, N$$

where V^{ji} relates the social welfare that individual i would attach to distributions of utilities if he were to find himself in situation j after the veil of ignorance were lifted.[10] Behind the veil of ignorance individuals vote to maximize the expected value of social welfare which is equivalent to maximizing $\sum_{j=1}^{N} V^j(U^1, \text{--}, U^j, \text{--}, U^N)$. If individuals care only about their own utility, $V^j = U^j$ and once again we have the Benthamite social welfare function. Concern for equality can of course be incorporated in the functions V^j [11]. However unless one accepts the Rawlsian position that behind the veil of ignorance the desire for horizontal equity is absolute, there will remain some degree of inequality at the social welfare optimum.

Footnotes

*This paper draws in part cn Arnott's doctoral dissertation [1], which benefited from the helpful suggestions of William Brainard, Katsuhito Iwai and Charles Wilson, and was supported by the Canada Council. Comments by John Hartwick, James MacKinnon and the editor, Lars Werin, are also gratefully acknowledged.

1. Mirrlees [3], p. 115.

2. Mirrlees [3], p. 23.

3. Transportation costs can be interpreted as the costs of commuting to the city center, to work or to shop, and/or the costs of transporting the composite good from the city center to the individual's residence.

4. Every allocation for which $y_1 > 1/2$ is Pareto inferior.

5. Choosing a log linear social welfare function, that is, $W = \log U^1 + \log U^2$, the optimum for the example is $y_1 = 1/4$, $U^1 = (3/32)^{1/2}$ $U^2 = (9/32)^{1/2}$.

6. A formal derivation of this result is provided in section II.

7. A weaker condition which utilizes the symmetry of the Slutsky matrix is derived in Section II.

8. Only a very simple model is required to demonstrate the Mirrlees result where there is a preference for location but no transportation costs. Consider two islands of equal area. Two identical individuals derive utility from location and the composite good. Island 2 is preferred to island 1. There is a fixed amount of the composite good to divide between the two individuals. Individual 1 is to live on island 1, individual 2 on island 2. Under what circumstances is individual 1 better off at the social welfare optimum, and under what circumstances is individual 2?

   For ease of exposition, let us consider a Benthamite social welfare function. If the utility function is separable in the composite good and location, then individual 2 is better off at the social welfare optimum. They receive the same amount of the composite good, and 2 has the preferred location. If the composite good complements location in the Edgeworthian sense, individual 2 is relatively better off than in the separability case.

9. This is exactly the result Mirrlees [4] obtains under the assumption that skill levels are continuously distributed. It is not to be confused with those for the widely studied case in which wealth transfers can only be achieved by the redistribution of earned income. See, for example, Mirrlees [2].

10. By utility here is meant the enjoyment an individual gets from his bundle of goods. The social welfare that an individual attaches to an allocation of goods depends not only on his utility but on others'. For an individual's utility to be independent of his measurement of social welfare it must be the case that the social welfare function he employs be separable between his utility and the utility of others. By situation we mean that the individual has acquired a specific personality (skills, tastes, etc.), not that the allocation of goods is fixed.

11. For example $V^j = U^j - \sum_i (\sqrt{U^j} - \sqrt{U^i})^2$. For the special case N = 2 the expected value of V becomes $\sqrt{U^1 U^2}$.

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