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AS A
CHARACTERIZATION OF PERFECTLY COMPETITIVE EQUILIBRIUM

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In the Walrasian definition of competitive equilibrium attention is confined to the consistency of individuals' plans on the presumption that each agent regards itself as a price-taker. This means that Walrasian equilibrium (WE) may exist where there is no supporting evidence for that presumption -- e.g., in a two-person "economy." The self-imposed limitations of the definition imply that WE describes necessary but not sufficient conditions for a perfectly competitive equilibrium (PCE) -- a WE for which the presumption of price-taking is justified. In this paper, we consider the question "When is a WE also a PCE?" The question may be divided in two: First, "What is a PCE?" and, second, "When will a PCE exist?" A third question with which this study began will also be examined: "How are the answers to the above two questions related to Equivalence of the core and WE?"

Informally, there can be no doubt as to what constitutes the traditional, textbook answer to the first question. An equilibrium is perfectly competitive when each agent is (actually) facing perfectly elastic demands and supplies for the goods he sells and buys. However, a formal characterization of this state of affairs has not been taken up by general equilibrium theorists. In the following section we shall discuss one source of this neglect and from it we shall attempt to justify what might appear to be the ill-considered choice of confining our attention throughout much of a paper devoted to PCE to (exchange) economies with only a finite number of traders.

I. Market Economies Versus Economy-Wide Gains from Trade

The most straightforward explanation as to why general equilibrium theorists have not bothered with a formal characterization of PCE is that it is so obvious. From the beginning, WE was interpreted as an equilibrium of market demands and supplies and these demands and supplies exist by definition only if there is a large number of small-scale buyers and sellers in each market. In such a setting, maximizing behavior by individual agents -- buying low and selling high -- should inevitably lead to a WE market-clearing (relative) price in each market.

The transparency of this conclusion leans heavily upon the partial equilibrium origins of general equilibrium theory -- analysis in terms of markets. According to this perspective, goods may be taken to exist independently of the agents trading them so that economy-wide equilibrium is simply the existence of simultaneous equilibrium in some fixed number of markets (≡ goods). Of course, PCE does not depend on the presence of markets per se, as it might, for example, if the term "market" were used to denote opportunity for trade. By relabelling commodities so that each seller is the supplier of its own goods, we may convert an economy with markets into one without; however, we would not wish to conclude that by this relabelling we could change a WE that is a PCE into one that is not. The point is that if an economy with a large number of traders can be represented as a market economy, we have a considerable amount of information about substitution possibilities among agents, enough for us casually to conclude that a WE is a PCE. In fact, the possibilities for substitution in a market economy are quite overwhelming.

Define an allocation of available goods to the members of an economy as displaying economy-wide gains from trade if it is impossible for groups consisting of very small fractions of the population to achieve, with only their own resources, all of the gains that are possible through direct and indirect participation in trade with the rest of the economy. This definition simply extends to small fractions of the population the familiar notion of gains from trade to the individual. Define the absence of any economy-wide gains from trade when all of the gains from participation in an economy-wide allocation may be obtained by partitioning the economy into isolated sub-economies such that each represents only a small fraction of the total population.

The notion of economy-wide gains from trade, which may be present in a world without production, can be compared to production possibilities in a single industry. The presence of economy-wide gains would be analagous to the presence of economies of scale for at least one firm while the absence of such gains would be similar to the absence of any scale economies so that maximum efficiency of the trading group or the production unit could be achieved at minimum scale. With production possibilities, scale economies would be one source of economy-wide gains from trade, a source that is particularly obvious because the gains to the economy as a whole may be traced directly to one of its members, a production unit. Less obvious, but not clearly less important, are the gains to the economy as a whole that are not internal to any of its members but depend on the many small-scale complementarities which may be appropriated by the members of the economy through (external) trade. It is the latter source of economy-wide gains on which we shall focus and this permits us to limit our attention to exchange economies in studying them.

The comparatively recent work by Aumann ['64], Vind ['64] and others which shows how formally to characterize a market economy makes it clear that a WE in a market economy implies the absence of any economy-wide gains from trade. This assertion is an immediate corollary of a mathematical result known as Liapunov's Theorem on the range of vector-valued measure.[1] It can be interpreted as saying that if each agent's endowment of any good is a very small fraction of the economy's so that the total supply of any commodity, c, can be represented as the integral $\int w_c(i)dv(i)$, where $w_c(i)dv(i)$ is the (infinitesimal) fraction of the total initially held by i, then there are numerous ways in which the economy can be partitioned into small-scale, isolated sub-economies each of which can achieve its allocation in a WE.

The point of departure for the present study is the belief that while the absence of economy-wide gains from trade might be characteristic of a single perfectly competitive market, where for example agents trade apples for oranges, it is a fallacy of composition to regard such overwhelming possibilities for substitution as characteristic of the economy as a whole. To make room for economy-wide gains from trade we must select a mathematical framework for general equilibrium theory that does not admit a representation as a market economy.

In a market economy, there are many buyers and sellers of each good. By comparison, a "non-market" economy will therefore require either (A) a reduction in the number of traders relative to goods or (B) an increase in the number of goods relative to traders. Case A is simply an economy with a finite number of agents while in case B there may be as many as a continuum of agents as long as there is also a continuum of goods. An example of B would be a world with large numbers of traders who are geographically dispersed so that in the presence of transport costs goods must be distinguished by the location of their suppliers.

It might appear that the answers to the questions posed in the Introduction will vary with the choice of a model of type A or B. For example, if we choose to study A, a finite economy, the reader might wish to conclude that no matter what is the precise answer to our first question (What is a PCE?), the answer to the second question (When will a PCE exist?) is "Never." While we shall show that this is not true, it is not far off. Nevertheless, a study of economies of type B, even though they may contain a non-atomic continuum of traders, has so far revealed the tentative conclusion that the choice between A and B will not materially affect the answers to our questions.

Anticipating the results below in which we follow A, as well as those of a forthcoming paper examining case B, we find that there is a characterization of

PCE that is independent of the number of traders (finite or infinite) and that it is the presence or absence of economy-wide gains from trade, not the presence or absence of large numbers *per se*, on which the existence of a PCE (and Equivalence of the core and WE) generally depends.

A brief, heuristic explanation connecting the presence or absence of economy-wide gains from trade and the absence or presence, respectively, of a PCE is that an agent's monopoly power originates in the extent to which it can supply goods that are at least partially complementary to those available from others. By definition, the absence of economy-wide gains from trade in market economies precludes such complementarities while the presence of such gains in non-market economies (finite or infinite) implies them.

II. Summary

The mathematical model depicting WE in a finite economy does not depend for its validity on any well-defined concept of (competitive) markets. The most recent general formulations of Gale-MasColell ['75] and Shafer-Sonnenschein ['75] make it clear that a WE may be regarded as a non-cooperative equilibrium among agents that individually maximize their gains from trading subject, of course, to the requirement that the prices of the goods traded are outside their control. To characterize a WE as a PCE it only remains to formulate conditions, without any appeal to markets, such that Walrasian quantities would continue to be a non-cooperative equilibrium when the prices of each supplier's goods are within its control.

This characterization is provided by the no-surplus (NS) condition: an economy-wide allocation of goods among individuals -- call it X -- satisfies the NS condition if each subset consisting of one fewer than all of the members can achieve with its own resources an allocation in which they are as well off as they are with X. Taken one at a time, the participation of each member of the economy is inessential to the welfare of others, relative to the allocation X.[2]

In Section III, the NS condition is more precisely defined and an analogy is drawn between the property of an NS allocation in an exchange economy and the distribution of factor rewards in a production economy according to their marginal products that satisfies the production exhaustion, or adding-up, property.

In Section IV, a characterization of PCE in terms of prices and perfectly elastic demands, called perfectly determinate (PD) prices, is given and it is shown that the price characterization (PD) and the quantity characterization (NS) are equivalent.

In finite exchange economies, a WE is rarely NS; but, we shall establish in Section VII for a special class of market economies, those produced by indefinite replication of a given finite exchange economy, that a WE is rarely other than NS; thus, confirming that WE in market economies are perfectly competitive.

In Section VI and part of VII we compare the NS condition and Equivalence as alternative characterizations of PCE. Each attempts to answer the question "When is a WE stable?" although the two conceptions of stability are quite different. For finite economies, whose WE allocations generally admit economy-wide gains, Equivalence neither implies nor is implied by the condition that every WE is NS (see Examples 2 and 3) although it is possible in some economies for these two distinct approaches to stability to coincide (Example 1). For market economies, whose WE allocations do not admit economy-wide gains from trade, we show that the two approaches almost always coincide.

In Section VIII, we use the results of this paper to give another perspective on the Equivalance Theorem. The proofs of those Propositions not given in the previous sections are collected in Section IX.

III. Notation and Definitions

The concepts and results below are formulated for a finite exchange economy $E = \{(P_i,w_i)\}_{i=1}^{i=n}$, where P_i describes the preferences and w_i the initial endowment of a typical agent i. It is assumed throughout that the relevant commodity space is R_+^ℓ. P_i is a binary relation on $R_+^\ell \times R_+^\ell$ and $P_i(x_i)$, a subset of R_+^ℓ, indicates the set of x_i' that i prefers to x_i while $R_i(x_i)$ will indicate the set of x_i' at least as desirable as x_i. Consistent with these interpretations, it is assumed that P_i is irreflexive $(x_i \notin P_i(x_i))$ and R_i is reflexive $(x_i \in R_i(x_i))$.

For each i=1, ..., n and any $x_i \in R_+^\ell$, it is assumed that:

(A.1) $P_i(x_i)$ is non-empty, convex and P_i has open graph in $R_+^\ell \times R_+^\ell$.

(A.2) $R_i(x_i)$ is the closure of $P_i(x_i)$.

These assumptions, which do not include transitivity of P_i or completeness of R_i, are similar to those minimally necessary to prove the existence of WE.[3] Assumption (A.2) amounts to local non-satiation -- i.e., within any neighborhood of x_i, there is an element of $P_i(x_i)$.

No assumptions are made about w_i, other than $w_i \in R_+^\ell$. In particular, we do not assume that w_i lies in the interior of R_+^ℓ.

An n-tuple of vectors (x_i), i=1, ..., n and $x_i \in R_+^\ell$, is denoted by X and its vector-sum Σx_i is denoted by x. An <u>allocation</u> in E is any feasible rearrangement of initial endowments $W = (w_i)$ -- any $X = (x_i)$ such that $x = w (= \Sigma w_i)$.

For any X, let $P(X) = \Sigma P_i(x_i)$ be those quantities of total resources, x', that could be distributed among the members of E such that each could receive a vector $x_i' \in P_i(x_i)$ and $x' = \Sigma x_i'$. A similar interpretation applies to $R(X) = \Sigma R_i(x_i)$.

An allocation X is <u>Pareto-optimal</u> (PO) for E if there is no other allocation that all the members of E would prefer -- i.e., $x \notin P(X)$.[4] By (A.1), P(X) is non-empty, convex and open and therefore the basic separation for convex sets implies that the allocation X is PO for E if and only if there exists a $p \in R^\ell$, $p \neq o$, such that $px \leq pP(X)$. By (A.2), $px \leq pP(X)$ if and only if

(EM) $px \leq pR(X)$

and we shall call a pair (X,p) <u>expenditure-minimizing</u> (EM) because the above inequality implies that for each $i=1, \ldots, n$, if $x_i' \in R_i(x_i)$, then $px_i \leq px_i'$.

A <u>quasi-equilibrium</u>[5] (QE) for $\mathcal{E}$ is a pair (X,p) that is EM and also satisfies the budget-balance condition for all $i=1, \ldots, n$,

$$px_i = pw_i.$$

A WE is a QE (X,p) with the added property that for each $i=1, \ldots, n$,

$$x_i' \in P_i(x_i) \text{ implies } px_i' > px_i$$

A WE is a QE but the converse need not be true. However, if (X,p) is a QE and for all $i=1, \ldots, n$,

$$px_i \neq \min pR_+^{\ell}$$

it is well-known that the two definitions coincide.[6] Rather than postulating any additional conditions to guarantee $px_i \neq \min pR_+^{\ell}$, we shall simply formulate our results for the weaker definition of QE and take the position that as far as the issues with which this paper is concerned, nothing is lost by regarding the concepts of QE and WE as interchangeable.

In the following discussion of perfect competition, we shall be concerned with the influence of a single agent, j, on the rest of the economy and it will be convenient to have a special notation in which $(\cdot)^{(j)}$ indicates the corresponding term $(\cdot)$ in $\mathcal{E}$, without agent j. Thus, $\mathcal{E}^{(j)}$ will denote the economy $\{(P_i, w_i)\}_{i \neq j}$ and $w^{(j)} = \sum_{i \neq j} w_i$ the total resources available to $\mathcal{E}^{(j)}$. Similarly, $X^{(j)} = (x_i)_{i \neq j}$, $P^{(j)}(X^{(j)}) = \sum_{i \neq j} P_i(x_i)$, $R^{(j)}(X^{(j)}) = \sum_{i \neq j} R_i(x_i)$, and $x^{(j)} = \sum_{i \neq j} x_i$.

IV. The No-Surplus Condition and the Production Exhaustion Property of Perfectly Competitive Equilibrium

What will be an equilibrium configuration of purchases and sales in an exchange economy $\mathcal{E}$ when each agent looks only to its own gain? We shall pose the problem so as to emphasize its similarity to that of deciding upon the distribution of rewards to the different factors of production, for which it is well-established that the perfectly competitive solution is given by the marginal productivity theory.

Trade is productive for all the members of $\mathcal{E}$ whenever their exists an allocation $X = (x_i)$ such that,

(1) $x \in P(W)$.

According to (1), each agent is able to achieve a surplus through trade. We may say that the surplus produced through trade among all members of $\mathcal{E}$ is maximal when X is PO,

(2) $w \notin P(X)$.

Assuming that allocations satisfying (1) exist, can we find one among them satisfying (2) such that we can impute the total surplus produced through trade to the separate contributions of the participating agents?

We may go part of the way towards an answer by saying that in such an allocation no one should be paid more than they are worth. In a production economy, this would mean that no one should receive more than their marginal product. For an exchange economy, we shall say that no agent is receiving more than it is worth in the allocation X, if for each $j=1, \ldots, n$,

(NNS) $w^{(j)} \notin P^{(j)}(X^{(j)})$.

Such an allocation will be said to satisfy the non-negative surplus (NNS) condition.

NNS places an upper bound on the extent to which an agent can exploit its monopoly power by saying, in effect, that no seller j can enforce an outcome in which his customers would do better by refusing to deal with him and going elsewhere.[7] Although we may stipulate that among those allocations satisfying (1) and (2) we may only choose those satisfying NNS, these restrictions will not generally suffice to remove the problem of indeterminacy of equilibrium. While each agent may receive no more than it is worth, some may receive less. But, which ones should they be and how much less should they receive?

In the marginal productivity theory of distribution, each factor of production receives no more and no less than it is worth. For an allocation X in $\mathcal{E}$, we may say that each agent is receiving no less than its marginal product if for all j=1, ..., n,

(NS) $w^{(j)} \in R^{(j)}(X^{(j)})$

Otherwise, if $w^{(j)} \notin R^{(j)}(X^{(j)})$, j is contributing a positive surplus and can therefore claim to be receiving less than it is worth to the rest of the economy. Call an allocation satisfying NS <u>no-surplus.</u>

To continue the analogy with the marginal productivity theory, if each agent is paid its marginal product will the sum of the payments just exhaust the total product? For an exchange economy, the question is: Does there exist a PO allocation that satisfies NS? One means of confirming that for a finite economy $\mathcal{E}$ a perfectly competitive equilibrium does not generally exist is to recognize **that a PO-NS allocation does not generally exist. For example when n=2, PO and NS are** inconsistent with gains from trade, (1).

As in the marginal productivity theory, the source of the inconsistency has to do with a kind of increasing returns and as a further parallel it is has to do with increasing returns at the margin. As long as the marginal unit of any factor of production does not raise the average product so that the last unit yields constant returns, the intra-marginal units may yield increasing returns. A similar results holds for exchange economies. Gains from trade are a form of increasing returns but they are not incompatible with NS as long as the contribution of any j, regarded as the marginal agent, does not increase the average gains -- i.e., as long as $w^{(j)} \varepsilon R^{(j)}(X^{(j)})$.

However, when there is no PO allocation satisfying NS there is no way to impute the total gains to the separate contributions of the members of E and the attempts by the agents to obtain as much as they are worth must prove inconsistent, just as the attempt by each factor to obtain its marginal product, when the last unit raises the average, more than exhausts the total product. If X does not satisfy NS we cannot justify why the distribution of the gains from trade should be according to X rather than some other allocation, and this difficulty remains even though X might be a WE allocation.

However, if there does exist a PO-NS allocation, WE comes into its own. Under such conditions, the distribution of rewards (purchases and sales) according to the marginal productivity principle that exhausts the total product (gains from trade) is precisely a QE.

PROPOSITION 1: Let X be a PO-NS allocation for E; then there exists a $p \varepsilon R^{\ell}$ such that (X,p) is a QE.

To demonstrate, since X is PO there exists a p such that (X,p) is EM -- i.e., $px \leq pR(X)$. Since X is NS, $w^{(j)} \varepsilon R^{(j)}(X^{(j)})$ and, therefore, for all j=1, ..., n, $px^{(j)} \leq pw^{(j)}$, and since px = pw, this implies that for all i=1, ..., n, $px_i = pw_i$.

V. The No-Surplus Condition and Perfectly Determinate Prices

In the previous section we gave a characterization of PCE in terms of quantities, whereas the usual description is in terms of prices and perfectly elastic demands. In this Section we shall develop a price-elasticity characterization and show that it is equivalent to the NS condition.

Informally, demands are perfectly elastic at the price vector p when the attempt by any agent to set prices for the goods it supplies at levels higher than in p results in the loss of all sales and "markets" for the goods supplied by others clear without any adjustment in the prices charged by others. To make possible this experiment, each agent must have control over the prices of the goods it supplies; otherwise, if two agents are supplying the same good, an increase in the price by one agent would compel the other to go along. To avoid this problem, we take advantage of an indeterminacy in the dimension of the commodity space for E. If agents i and j have positive endowments of commodity c in the original specification of initial endowments, we may introduce a new commodity index with commodities c_i and c_j where $w_{ic_i} = w_{ic}$ and $w_{jc_j} = w_{jc}$. Continuing in this way, if there are k distinct goods it will require not more than $\ell \leq kn$ revised goods to bring the specification of initial endowments into the appropriate form of a personalized commodity space for E where $w_{ic} > 0$ implies $w_{jc} = 0$, all $j \neq i$. Of course, preferences must be modified to correspond to the relabelling of goods. Since we have neither assumed that endowments are strictly positive nor that preferences are strictly convex,[8] there is nothing to prevent us from assuming that this relabelling has been undertaken and the matrix of initial endowments in E has each agent as the unique supplier of its own goods.

For any agent j, we may divide the ℓ commodities into those which could be supplied by j and those that could not. For $p \in R^\ell$, let p_j be the vector of prices of the former and $p(j)$, the prices of the latter. For any p and j, p can be written as $(p(j), p_j)$ where $p(j)$ and p_j are projections of p onto the appropriate subspaces of R^ℓ. If will be convenient to denote by $q(p;j)$ any vector $q = (q(j), q_j) \in R^\ell$ such that $q(j) = p(j)$ and $q_j \gg p_j$. At $q(p;j)$ the prices of j's goods are higher than at p while the prices of the remaining goods are the same.

Now, more formally, we shall say that the price vector p is <u>perfectly determinate</u> (PD) if for any $j=1, \ldots, n$ and $q(p;j)$ if $x_i \in R^{\ell}_{+}$, $q(p;j)x_i = q(p;j)w_i$, $i \neq j$, there exists $y_i \in R^{\ell}_{+}$, such that:

(PD.1) $q(p;j)y_i = q(p;j)w_i$.

(PD.2) $y_i \in R_i(x_i)$ and $y_i \in P_i(x_i)$ if $(q(p;j) - p)x_i \neq 0$.

(PD.3) $\sum_{i \neq j} y_i = w^{(j)}$.

To interpret, the market opportunities available to $i(\neq j)$ are strictly smaller when they are defined by the price vector $q(p;j)$ than the price vector p since the price of all goods other than j's are the same but j's prices are higher. However, PD.1-3 indicates that if i is a purchaser from j -- i.e., $(q(p;j) - p)x_i \neq 0$ -- i has not made the best use of this smaller opportunity set since i could have refused to deal with j and, trading at the same prices, $q(p;j)$, have found buyers and sellers willing to make exchanges leading to a vector y_i preferred to x_i.

We shall regard the notion of perfectly elastic demands as synonomous with the existence of a p that is PD.

In the previous Section we showed that although an allocation X corresponding to a QE pair (X,p) does not generally satisfy NS, if there is a PO-NS allocation X it is part of some QE pair. Similarly, although a price vector corresponding to a QE pair (X,p) does not generally satisfy PD, if p is PD it is part of some QE pair.

PROPOSITION 2: <u>If p is PD for E, then provided R_i is transitive, there exists an allocation X such that (X,p) is a QE.</u>

As a representation of perfectly elastic demands, the fact that p is PD means that the area under each seller's demand curve, sometimes called the consumers' surplus, vanishes. The non-existence of this kind of surplus is clearly related to the existence of an NS allocation. In fact, the two are equivalent.

PROPOSITION 3: Let (X,p) be an EM pair for E. If p is PD, then X is NS: and if X is NS then, provided R_i is complete and transitive and $px_i \neq \min pR^{\ell}_{+}$, p is PD.

One example of a PO-NS allocation and a PD price vector occurs when there are no gains from trade. To show that there are more interesting possibilities we give the following:

EXAMPLE 1: Let the tastes of agent $i = 1, 2, 3$ be represented by the utility function $u_i(x_i) = u_i(x_{i1}, x_{i2}, x_{i3})$, initial endowments by the matrix W and final allocations by the matrices X_α, where

$$u_1(x_1) = x_{11}(x_{12} + x_{13})$$
$$u_2(x_2) = x_{22}(x_{21} + x_{23});$$
$$u_3(x_3) = x_{33}(x_{31} + x_{32})$$

$W =$

i \ c		
2	0	0
0	2	0
0	0	2

; $X_\alpha =$

i \ c		
1	α	$1-\alpha$
$1-\alpha$	1	α
α	$1-\alpha$	1

The allocations X_α, $0 \leq \alpha \leq 1$, represent the set of all PO-NS allocations. They are at the opposite extreme from the no-trade example of NS since their attainment requires that all agents participate in trade. Of course, because they are NS and there are only three traders any pair of individuals could also do as well by themselves. Each individual's participation in an NS allocation is essential only to itself.

It is easily verified that there is only one WE price vector for this example given by $p = (r,r,r)$, $r > 0$.

To illustrate the above three Propositions, we have

Proposition 1: Each PO-NS allocation X_α is a WE.

Proposition 2: The price vector p is PD. If agent j(= 1,2,3) were to raise the price of its good c(= j) above its value in p, j would sell nothing and excess demands among the other agents would be zero without disturbing the prices of their goods.

Proposition 3: (X_{α}, p) is EM and p is PD for each X_{α} and conversely.

We conclude this Section with two remarks about the NS-PD characterization of PCE. The first concerns the possibility that our criteria might be too strong and the second that they might be too weak.

1. Why insist on perfectly elastic demand as a requirement for price-taking equilibrium? If we are interested only in characterizing those price-quantity situations in which no seller could profitably depart from the WE prices by choosing an alternative offer within the class of uniform price offers, sufficient elasticity short of perfect will do, although how much will depend on the particular circumstances.[9] However, if we wish to derive uniform prices as an equilibrium condition when sellers can make all sorts of discriminatory offers, nothing short of perfect elasticity will do. Without it, profitable discriminatory pricing would exist as a consequence of the failure of NS.

To demonstrate, suppose everyone else is announcing Walrasian uniform price offers and there is some inelasticity in the demand schedule facing seller j. Then, by Proposition 3, the NS condition does not hold at the WE allocation. Therefore j's contribution to the WE allocation is essential to the welfare of at least one other individual i, and j should attempt to appropriate this surplus by for example, some all-or-nothing offer. However, if j faces perfectly elastic demands at WE prices, any offer to i that j would find more profitable than the Walrasian allocation X could and should be refused by i since we know that if p is PD, X is NS.

2. Another approach to price-taking behavior is provided by Roberts and Postlewaite (RP) [1976]. They ask whether it might not be in an agent's interest to misrepresent its excess demands to manipulate a more favorable market-clearing prices. Unless no agent finds it profitable to manipulate, a WE is not in their terms "incentive compatible."

Our notion of PCE as an NS-PD Walrasian equilibrium is incentive-compatible according to our underlying model of how trade takes place but it is not necessarily incentive-compatible when placed in the RP context. The explanation may be summarized by noting that RP look for conditions under which quantity-choosers can or cannot manipulate <u>the</u> market prices in a auctioneer-price-setting model while we are looking for conditions under which individual price-makers, who are not even restricted to charging uniform prices, will or will not choose to take WE prices.

VI. The No-Surplus Condition, Balanced Sets and the Core

Our discussion of the NS condition in Section IV began with the NNS condition as an upper bound on what each individual might expect to gain through trade. Edgeworth, and later on game theorists, were willing to go further in their efforts to find an equilibrium by extending the NNS condition from single individuals to arbitrary coalitions of individuals by saying that an equilibrium must be in the core. In this Section we compare Equivalence of the core and WE with equivalence of PD-NS and WE allocations for a particular class of economies in which WE allocations generally admit economy-wide gains from trade, namely finite exchange economies. The comparison is aided by the fact that the NS condition has a representation in terms of balanced sets, a concept integral to the core. In the following Section, the same comparison will be made for a particular class of economies in which WE allocations do not admit economy-wide gains from trade.

To emphasize the connections between the core and the NNS condition, let S be a non-empty subset of $I = \{1, 2, \ldots, n\}$ and let $w^S = \sum_{i\varepsilon S} w_i$, $x^S = \sum_{i\varepsilon S} x_i$, $P^S(X^S) = \sum_{i\varepsilon S} P_i(x_i)$, and $R^S(X^S) = \sum_{i\varepsilon S} R_i(x_i)$. An allocation X is in the <u>core</u> of $\mathcal{E}$ if

$$w^S \notin P^S(X^S) \text{ for all } S \subseteq I.$$[10]

Alternatively, an allocation is in the core if for each S, the complement (I-S) is not getting more from S than it is worth -- i.e., for each S, (I-S) is contributing a non-negative surplus.

If we denote $I - \{j\}$ by S_j, we may rewrite NNS as a part of the core conditions, where for each $j = 1, \ldots, n$, $w^{S_j} \notin P^{S_j}(X^{S_j})$.

Sufficient conditions for an n-person game or an exchange economy to have a non-empty core have been given by Shapley ['67] and Scarf ['67]. The basic tool for these results is the notion of balanced sets. Let $B = \{S\}$ be a collection of subsets of I and let $B_i = \{S: i\varepsilon S\}$. B is a <u>balanced collection</u> if there are non-negative numbers b_S, $S \varepsilon B$, such that for each $i=1, \ldots, n$,

$$\sum_{S\varepsilon B_i} b_S = 1.$$

The NNS and NS conditions are defined with respect to one particular balanced collection $B = \{S_j\}$ with weights $b_{S_j} = \frac{1}{n-1}$.

The exchange economy $\mathcal{E}$ represents a <u>balanced game</u> if for any $\tilde{X} = (x_i)$, where $\tilde{X}$ is not necessarily an allocation,

if $w^S \varepsilon R^S(\tilde{X}^S)$ for all S in any balanced collection B, then $w \varepsilon R(\tilde{X})$.

When preferences of players in an n-person game are representable by utility functions, Scarf has shown that a balanced game has a non-empty core and that if these players are agents in an exchange economy having convex preferences the economy is a balanced game so that the core of an exchange economy with convex numerically representable preferences is not empty.

Proposition 1, above, may be interpreted as a "substitute" for the considerably deeper results of Shapley and Scarf. It says that if there exists a PO allocation X for which the balance condition holds for the balanced collection $\{S_j\}$ -- i.e., X satisfies NS -- there is no need to verify any of the other conditions defining a balanced game to conclude that E has a non-empty core. Proposition 1 shows that this allocation is a QE and, provided that any one of a number of sufficient conditions obtains for converting a QE into a full WE, we can infer the existence of a WE which is, with locally non-satiated preferences, always in the core. Corollary to Proposition 1 is: If all core allocations satisfy the NS condition, the core coincides with the QE. Example 1 is an illustration of the Corollary.

Does equality of the core and QE (or WE) imply that every core allocation is NS? If so, then considering Proposition 3, there would be an equivalence between the perfect determinacy of prices at WE and Equivalence (of the core and WE). We shall establish that such is not the case by showing that Proposition 1 can also be viewed as a special case of a more general set of sufficient conditions for the existence of a QE and that if each core allocation satisfies some one of these more general conditions the core and QE will also coincide. At this point the reader may wonder why we have bothered to single out the NS condition for special attention. The reason is that of these more general conditions relating the core to WE, none but the NS condition appears to have anything to do with the characterization of a PCE. In Proposition 4, below, we shall generalize Proposition 1 but we shall not obtain a generalization of Proposition 3. The core may coincide with WE allocation when the economy is composed of imperfect competitors -- i.e., when WE prices do not satisfy PD.[11]

For any balanced collection $B = \{S\}$ with non-negative weights b_S, there is an associated $n \times n$ matrix $A(B) = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} \sum\limits_{\substack{S \in B_i \\ j \in S}} b_S, & \text{whenever } j \in \bigcup\limits_{S \in B_i} S \\ 0, & \text{otherwise.} \end{cases}$$

A is symmetric since $j \in \bigcup\limits_{S \in B_i} S$ if and only if $i \in \bigcup\limits_{S \in B_j} S$ and, since B is balanced, $0 \leq a_{ij} \leq 1$ and $a_{ij} = 1$. The number $a_{ij} (= a_{ji})$ may be loosely interpreted as a measure of the "inseparability" between i and j in the balanced collection B. For example, when $a_{ij} = 1$, i and j always appear in the same elements S of B and when $a_{ij} = 0$ they never appear in the same elements.

We shall say that a balanced collection B is <u>regularly balanced</u> if the matrix A(B) is regular (has an inverse). Not all balanced collections are regularly balanced -- e.g., partitions are not. Examples of regularly balanced collections are the sets of all subsets of I consisting of (n-k) elements, $1 \leq k \leq n-1$, for which $a_{ij} = \frac{n-k-1}{n-1}$, $i \neq j$. Thus, the sets $\{S_j\}$ describing the NS conditions are regularly balanced. In fact, for $n \leq 3$, the regularly balanced sets and those describing NS coincide. For n = 4, the reader may wish to verify that the collection $\{S\} = \{\overline{123}, \overline{14}, \overline{24}, \overline{34}\}$ is also regularly balanced.

To extend Proposition 1, we shall say that an allocation X for E satisfies the <u>generalized NS (GNS) condition</u> if there exists a regularly balanced set B such that for all $S \in B$

(GNS) $\quad w^S \in R^S(X^S)$

PROPOSITION 4: <u>Let</u> X <u>be</u> PO <u>for</u> E. <u>If</u> X <u>satisfies</u> GNS, <u>there exists a</u> p <u>such that</u> (X,p) <u>is a</u> QE.

Corollary to Proposition 4 is: If each core allocation satisfies GNS for some regularly balanced set, the core coincides with the QE.

The following Proposition summarizes the fact that in finite exchange economies Equivalence and NS-PD are distinct characterizations of the competitiveness of WE. It is demonstrated by two examples showing, in effect, that GNS does not imply NS (Example 2) so that Equivalence may obtain without NS and that for one core allocation to satisfy NS does not imply GNS for all core allocations (Example 3).

PROPOSITION 5: <u>Equivalence of the core and WE allocations neither implies nor is implied by the condition that every WE satisfies NS.</u>

EXAMPLE 2 (Equivalence $\not\Rightarrow$ NS): Let tastes be defined by the utility functions $u_i(x_i) = u_i(x_{i1}, x_{i2}, x_{i3})$, $i = 1, 2, 3$ and 4, and let initial and final allocations be given by W and X^*, respectively, where

$$u_1(x_1) = (x_{11}x_{12}x_{13})^{\frac{1}{3}}$$
$$u_2(x_2) = (x_{21}x_{22}x_{23})^{\frac{1}{3}}$$
$$u_3(x_3) = (x_{31}x_{32}x_{33})^{\frac{1}{3}}$$
$$u_4(x_4) = \frac{x_{41} + x_{42} + x_{43}}{3}$$

; W =

i \ c		
3	0	0
0	3	0
0	0	3
1	1	1

; X* =

i \ c		
1	1	1
1	1	1
1	1	1
1	1	1

With little difficulty, the reader may verify that X* is a PO allocation which yields the utility vector $u^* = (u_1^*, u_2^*, u_3^*, u_4^*) = (1, 1, 1, 1)$. For the regularly balanced set $B = \{S\} = \{\overline{123}, \overline{14}, \overline{24}, \overline{34}\}$, $w^S \in R^S(X^{*S})$, so that X* satisfies GNS and by Proposition 4, is a QE. Here it is also a WE at the price vector $p^* = (r, r, r)$, $r > 0$.

An outline of the argument that X* is the only allocation in the core is as follows: The coalition $\overline{123}$ can achieve any (u_1, u_2, u_3) such that $u_1 + u_2 + u_3 = 3$, and 4 can achieve $u_4 = 1$ on its own. It follows that for any $u = (u_1, u_2, u_3, u_4)$ in the core $u_1 + u_2 + u_3 = 3$ and $u_4 = 1$, and if $u_i < 1$, $i \neq 4$, then because $w^s \in R^s(X^{*s})$, $s = \overline{14}$, the coalition $\overline{14}$ will upset u. This means that $u = (1,1,1,1)$ and this is achievable only by the allocation X*.

X* does not satisfy NS -- $w^S \notin R^S(X^S)$ for $S \in \{\overline{123}, \overline{134}, \overline{234}\}$.

The core does not capture the monopoly power of individuals 1, 2 or 3. For example, without 1, the WE price vector for the economy $E^{(1)}$ is $p = (2r, r, r)$. In terms of prices, 1's contribution or marginal product may be measured by the decrease in the WE price of the commodity he supplies. From a non-cooperative point of view, there is no reason why 1 should surrender all of the surplus represented by this price decrease to the other traders.

EXAMPLE 3 (NS $\not\Rightarrow$ Equivalence): Tastes, endowments and WE allocations are given by (u_i), W and X_α, where

$$
\begin{aligned}
u_1(x_1) &= \min(x_{11}, x_{12} + x_{13}) \\
u_2(x_2) &= \min(x_{22}, x_{23} + x_{24}) \\
u_3(x_3) &= \min(x_{33}, x_{34} + x_{31}) \\
u_4(x_4) &= \min(x_{44}, x_{41} + x_{42})
\end{aligned}
\quad ; \; W = \begin{array}{c|cccc} i \backslash c & & & & \\ \hline & 0 & 1 & 1 & 0 \\ & 0 & 0 & 1 & 1 \\ & 1 & 0 & 0 & 1 \\ & 1 & 1 & 0 & 0 \end{array}
\quad ; \; X_\alpha = \begin{array}{c|cccc} i \backslash c & & & & \\ \hline & 1 & \alpha & 1-\alpha & 0 \\ & 0 & 1 & \alpha & 1-\alpha \\ & 1-\alpha & 0 & 1 & \alpha \\ & \alpha & 1-\alpha & 0 & 1 \end{array}
$$

The allocations X_α, $0 \leq \alpha \leq 1$, are PO and NS -- i.e., $w \notin P(X_\alpha)$ and $w^S \in R^S(X^S)$ for $S \in \{\overline{123}, \overline{124}, \overline{134}, \overline{234}\}$. By Proposition 1, there exists a p such that (X_α, p) is a QE. Here it is a WE and the $p = (r, r, r, r)$, $r > 0$. Further, p is the only WE price vector and, therefore X_α constitute the entire set of WE allocations.

The core contains allocations other than X_α -- e.g., the core contains allocations yielding all permutations of the vector of utilities $(u_1, u_2, u_3, u_4) = (0, \frac{4}{3}, \frac{4}{3}, \frac{4}{3})$.[12]

VII. The No-Surplus Property of Walrasian Equilibria in (Replica) Market Economies

So far, analysis of the NS condition has been confined to finite economies $\mathcal{E}$. In this Section we consider the economy $\mathcal{E}_k$, the k-fold replica of $\mathcal{E}$, and a definition of asymptotic-NS for the sequence of economies $\{\mathcal{E}_k\}$. Enlarging the economy in this way, we can achieve for almost all replica market economies, $\mathcal{E}_\infty$, what is almost never true of $\mathcal{E}$ — the equivalence of WE and NS allocations.

For each $i = 1, \ldots, n$ in $\mathcal{E}$ there are exactly k agents in $\mathcal{E}_k$ whose preferences and endowments are identical to i — i.e., $P_{i_h} = P_i$ and $w_{i_h} = w_i$, $h = 1, \ldots, k$. If X is an allocation in $\mathcal{E}$, let X_k be the replicated allocation in $\mathcal{E}_k$, where for each $i = 1, \ldots, n$, $x_{i_h} = x_i$, $h = 1, \ldots, k$.

Denote by $\mathcal{E}_k^{(j)}$ the "economy" consisting of all the members of $\mathcal{E}_k$ excluding one member of type j, $j = 1, \ldots, n$. Let $w_k^{(j)} = w^{(j)} + (\frac{k-1}{k})w_j$ and let $R_k^{(j)}(X) = R^{(j)}(X^{(j)}) + (\frac{k-1}{k})R_j(x_j)$. Then $kw_k^{(j)}$ represents the sum total of initial endowments in $\mathcal{E}_k^{(j)}$ and $kR_k^{(j)}(X)$ represents the set of total resources which must be available to the members of $\mathcal{E}_k^{(j)}$ in order that each of them be able to receive a vector of goods as least as desirable as they receive in X_k

If $z_k^{(j)} = z^{(j)} + (\frac{k-1}{k})z_j$, $z_k^{(j)} \in R^\ell$, satisfies

$$w_k^{(j)} + z_k^{(j)} \in R_k^{(j)}(X),$$

then $kz_k^{(j)}$ would represent sufficient resources which when added to the sum total $kw_k^{(j)}$ already available, would allow the members of $\mathcal{E}_k^{(j)}$ to do at least as well by trading amongst themselves as they could do by trading with j to obtain the allocation X_k. If we measure the size of $z_k^{(j)}$ by the Euclidean distance of the vector $z_k^{(j)}$ from the origin of R^ℓ, denoted by $||z_k^{(j)}||$, then the minimum value of $k||z_k^{(j)}||$ gives a rough indication of the monopoly power of agent j relative to the allocation X_k. It is a rough measure of how far X_k is from being an NS allocation, with respect to agent j, and it coincides with NS when $||z_k^{(j)}|| = 0$.

We shall say that the sequence of allocations $\{X_k\}$ for $\{\mathcal{E}_k\}$ is <u>asymptotically no-surplus</u> (ANS) if for each $j = 1, \ldots, n$, there exists a sequence $z_k^{(j)}$ such that

$$w_k^{(j)} + z_k^{(j)} \in R_k^{(j)}(X), \text{ and}$$

(ANS)

$$k||z_k^{(j)}|| \to 0.$$

The requirement $k||z_k^{(j)}|| \to 0$ means that to satisfy ANS not only must the surplus contributed by any agent to any other agent go to zero as k increases but it must diminish <u>sufficiently rapidly</u> so that the surplus contributed by any agent to <u>all</u> other agents goes to zero.

It is perhaps not so obvious that a WE for a sequence of replica economies will satisfy ANS because an increase in k, although it increases the number of directly competing suppliers, also increases the number of agents from whom one might be able to extract a small surplus. Nevertheless, with one mild qualification, we may obtain the desired conclusion as a consequence of the overwhelming substitution possibilities in market economies -- absence of economy-wide gains from trade.[13, 14]

The qualification we shall need amounts to ruling out sudden changes in marginal rates of substitution or kinks in indifference (hyper-) surfaces. Such kinks, if they occur at a WE allocation, are an obstacle to the required speed of convergence for $||z_k^{(j)}||$.

For an allocation X for $\mathcal{E}$, define $x^{(j)}[k] = (\frac{k-1}{k})x^{(j)} + \frac{1}{k} w^{(j)}$, a convex combination of the sum of the resources received by $\mathcal{E}^{(j)}$ in $X^{(j)}$ and the resources with which they are initially endowed. The allocation X satisfies <u>preference-price continuity</u> (PPC) if for each $j = 1, \ldots, n$ and sequence $\{p_k^{(j)}\}$, $p_k^{(j)} \in R^{\ell}$, $||p_k^{(j)}|| = 1$ such that,

(PPC) $\quad p_k^{(j)} x^{(j)}[k] \leq p_k^{(j)} R^{(j)}(X^{(j)})$, then $p_k^{(j)} \to p$.

Note that since $x^{(j)}[k] \to x^{(j)}$, PPC implies that (X,p) is an EM pair.

In Section III it was observed that for finite economies the indeterminacy of equilibrium and the related failure of a WE allocation to satisfy NS is due to the fact that the NNS condition, which places a hypothetical upper bound on what each agent could expect to gain from trade, does not generally yield a determinate lower bound. To exhibit the remarkable differences between finite and market economies,[15] we show that for the latter NNS is all that is needed to establish determinacy. From the following two Propositions we may conclude that as the condition of being a market economy is approached, the upper bound on monopoly power converges to the lower bound of NS, or no monopoly power at all.

Recall that the allocation X satisfies NNS for $\mathcal{E}$ if for each $j = 1, \ldots, n$, $w^{(j)} \notin P^{(j)}(X^{(j)})$. Therefore, X_k satisfies NNS for $\mathcal{E}$ if for each j,

$$w_k^{(j)} \notin P_k^{(j)}(X) = P^{(j)}(X^{(j)}) + \left(\frac{k-1}{k}\right)P_j(x_j)$$

PROPOSITION 6: <u>If the allocation</u> X <u>satisfies</u> PPC <u>for</u> $\mathcal{E}$ <u>and</u> $\{X_k\}$ <u>satisfies</u> NNS <u>for</u> $\{\mathcal{E}_k\}$, <u>then</u> X <u>is a</u> QE.[16]

To establish that NNS implies ANS and to show the equivalence of QE and ANS in market economies, we have:

PROPOSITION 7: <u>Let</u> X <u>be</u> PO <u>for</u> $\mathcal{E}$. <u>If</u> $\{X_k\}$ <u>is</u> ANS, <u>there is a</u> p <u>such that</u> (X,p) <u>is a</u> QE; <u>and, if</u> (X,p) <u>is a</u> QE <u>for</u> $\mathcal{E}$ <u>then, provided</u> X <u>satisfies</u> PPC, $\{X_k\}$ <u>is</u> ANS.

As in Section IV, we could go on to give a definition of almost perfectly elastic demands or prices that we asymptotically PD. Unlike the construction in IV, it would not be necessary to enlarge the commodity space indefinitely so as to permit each agent to be the unique supplier of its own goods. We need only distinguish between the goods initially held by one j and all other goods, held by members of $\mathcal{E}_k^{(j)}$. With this construction an analogue of Proposition 3 may be demonstrated that for an EM pair (X,p) in $\mathcal{E}$, $\{X_k\}$ is ANS if and only if p is asymptotically PD. Rather than proceed with the formal details we shall give a geometrical argument on which the formal proofs of Propositions 6 and 7 also rely.

Suppose there are only two types of agents, type 1 initially owning commodity 1 and type 2, commodity 2. We plot the situation for a typical member of type 2 in Figure 1, where p is WE price vector and w_2 and x_2 are, respectively, the initial and WE allocations for a member of type 2.

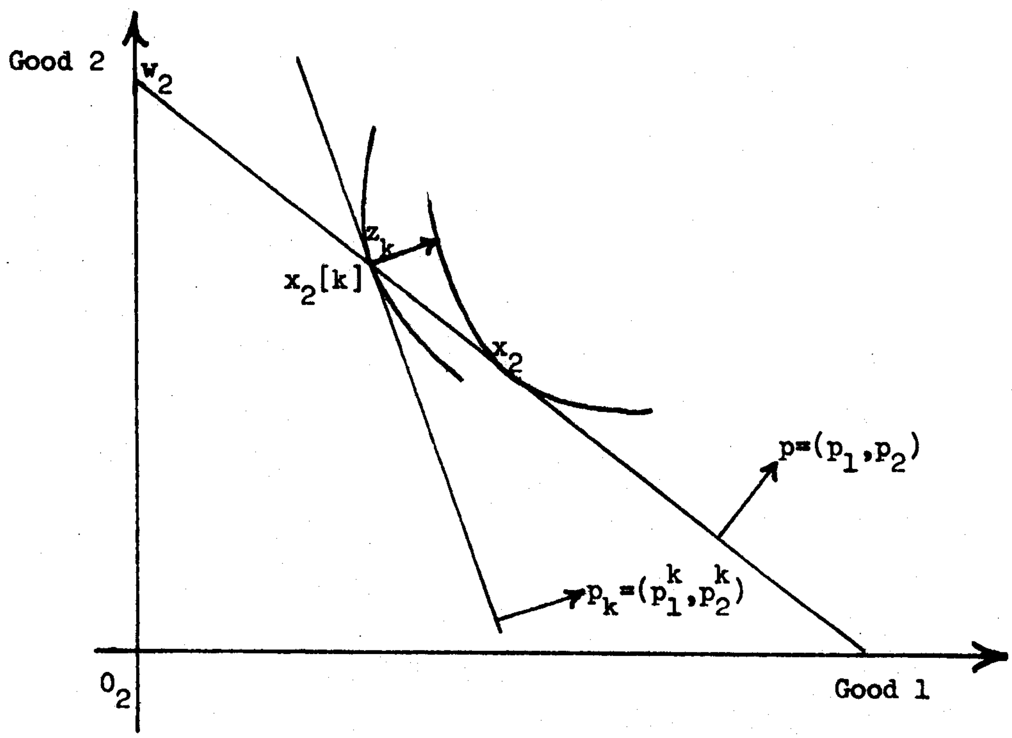

Figure 1

To establish that p is PD and that x_2 is part of an NS allocation, at least asymptotically, consider a single member of type 1 attempting to impose a more favorable exchange rate for itself than p_1/p_2. If all other traders are transacting at the rate p_1/p_2, all traders of type 2 will first go to the other and lower-priced type 1 traders. Assuming that they have similar experience in making exchanges, the type 1 traders will be able to sell all they wish at the WE exchange rates while the type 2 traders will be able to obtain most of what they would like. If there are k of each type, each type 2 trader will be able to obtain

$$x_2[k] = (\frac{k-1}{k})x_2 + (\frac{1}{k})w_2 = x_2 - \frac{1}{k}(x_2 - w_2)$$

After having gone to the lower-priced type 1 traders, they may look to the remaining trader of type 1. If the latter sets an exchange rate $\rho > p_1^k/p_2^k$, where $p_k = (p_1^k, p_2^k)$ is orthogonal to the tangent vector at $x_2[k]$, his customers will not find additional purchases advantageous and he will sell nothing. At $\rho > p_1^k/p_2^k$ type 2 traders would wish to sell, not buy, commodity 1.

We may take $p_k z_k$ as an upper bound on the total revenue that the non-conforming member of type 1 could possibly extract from each member of type 2, where z_k satisfies

$$\min_{||z_k||} (x_2[k] + z_k) \in R_2(x_2).$$

If the WE is ANS, $k||z_k|| \to 0$ and therefore $kp_k z_k \to 0$ so that the elasticity of demand increases without bound. Conversely, if the elasticity of demand is increasing without bound, $kp_h z_k \to 0$.

To show that PPC is necessary for our results we show how ANS fails without it.

EXAMPLE 4: Consider the economy of equal numbers of type 1 and type 2 traders illustrated by the Edgeworth box of Figure 2.

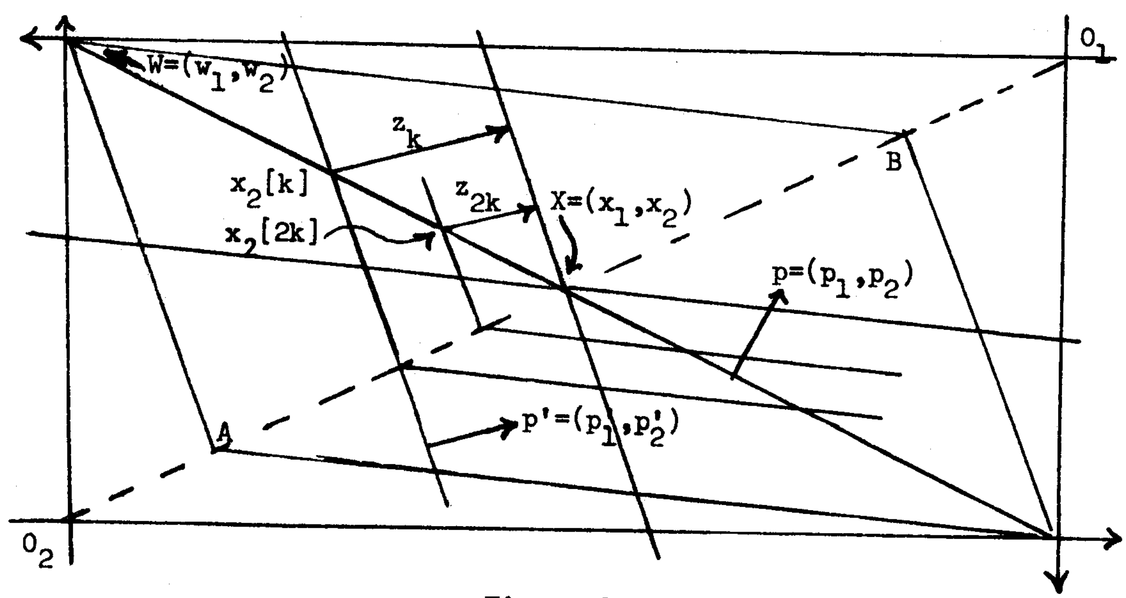

Figure 2

No matter what the value of k, the core and WE allocations of $\mathcal{E}_k$ coincide. They consist of all points all the line AB. If we consider the economy $\mathcal{E}_k^{(1)}$ or the economy $\mathcal{E}_k^{(2)}$, then no matter what the value of k the core and WE allocations consist only of the point A or B, respectively.

The example is of some historical interest because of its similarity to Edgeworth's master-servant example in which each master has need of only one servant and one servant cannot serve two masters. If the minimum wage at which servants would offer themselves is a and the maximum wage at which masters would accept in b, Edgeworth noted that as long as there are an equal number of masters and servants, then no matter how many there are the equilibrium wage is indeterminate, lying somewhere between a and b ($a < b$). If there is one more master than servant (or one more servant than master) indeterminacy disappears and the wage rate becomes b (or a). Edgeworth attributed the indeterminacy to the indivisibility of the good "domestic service" and on the basis of his example modified his proposition that large numbers of traders in a market lead to a determinate outcome only when goods are divisible. The self-evident similarity of the master-servant example to Example 4, in which goods are divisible, suggests that some other explanation is called for. What is common to Edgeworth's example and Example 4 is the failure of (A)NS.

To see this, consider the WE allocation $X = (x_1, x_2)$ in Figure 2 replicated k times for $\mathcal{E}_k$ and consider how close the members of $\mathcal{E}_k^{(1)}$ can come to doing as well. From the preceeding comments, all of the type 1 traders can be given x_1 and each of the type 2's may receive $x_2[k] = x_2 - \frac{1}{k}(x_2 - w_2)$. It would then require z_k additional resources to minimize, with respect to Euclidean distance, the amounts of additional resources necessary for type 2 agents to be as well off as they are with x_2. If we double k, we may decrease the additional resources to make each type 2 agent as well to z_{2k}, but we cannot decrease the amounts by more than one-half -- i.e., $2||z_{2k}|| = ||z_k||$. Thus, the influence of any type 1 agent, measured in this way, does not decrease as k increases, and $\{X_k\}$ is not ANS. Consequently, elasticity of demand will not increase with k.

Assuming that other traders are transacting at the WE prices p corresponding to X, the maximum exchange rate a seller of type 1 can set, without losing all his business, does not go to p as k increases. After the type 2 traders have purchased all that the (k - 1) type 1 traders are willing to supply at p, the remaining type 1 trader can offer any exchange rate ρ such that $\frac{p_1}{p_2} < \rho < \frac{p_1'}{p_2'}$ and sell, when k is large, as much of commodity 1 as it likes.

It might be conjectured that the failure of ANS in Example 4 is solely a result of the failure of the WE price correspondence to be continuous. We conclude this Section with examples showing that such a conjecture is false in both directions.

EXAMPLE 5 (Price Continuity $\not\Rightarrow$ ANS): The economy described by Figure 3 represents a slight modification of the previous example such that the rays along which the indifference curves of the two traders are kinked intersect at only one point X which is the unique WE allocation at the price vector p.

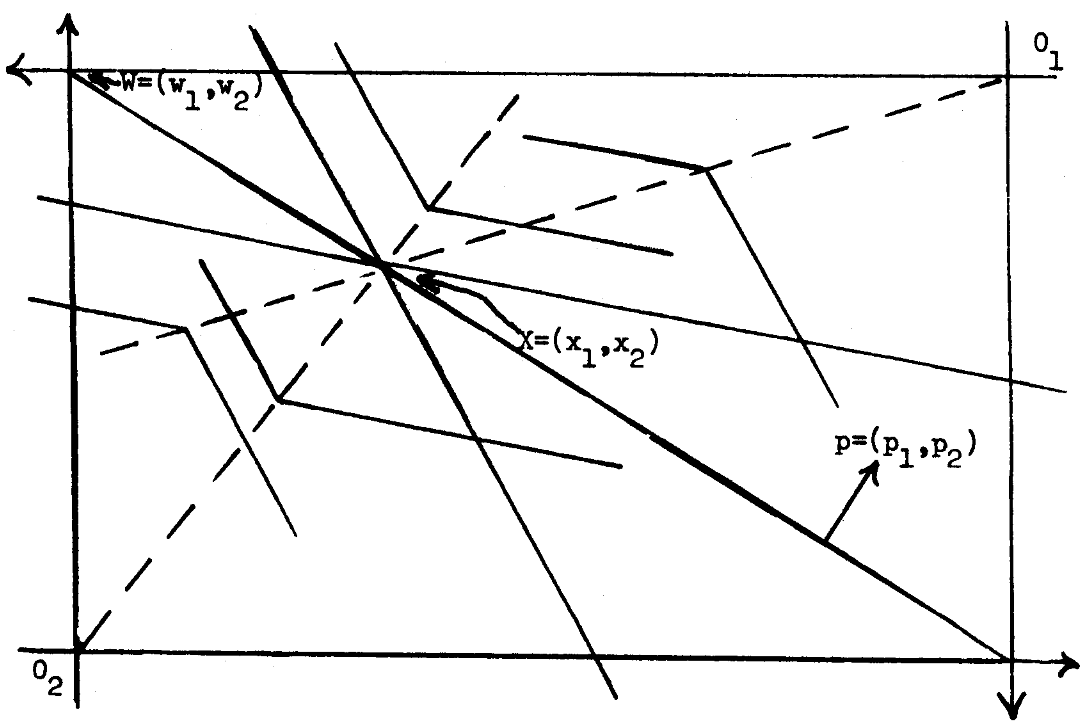

Figure 3

As we replicate the economy, the core continuously shrinks to X_k and if we consider the WE allocations and prices in $\mathcal{E}_k^{(1)}$ or $\mathcal{E}_k^{(2)}$, they will be unique and approach X_k and p continuously. Nevertheless, ANS does not hold and for precisely the same reasons as outlined for the economy of Example 4.

EXAMPLE 6 (ANS $\not\Leftrightarrow$ Price Continuity):[17] The example is illustrated by Figure 4. Indifference curve of both traders are drawn to be differentiable in the interior of R_+^2. The set of WE allocations lie along AB and the set of WE prices consist of all $q \in R_+^2$ such that $\frac{p_1}{p_2} \leq \frac{q_1}{q_2} \leq \frac{p_1'}{p_2'}$. Since PPC is satisfied at every WE allocation, by Proposition 7 as we replicate the economy any WE is ANS.

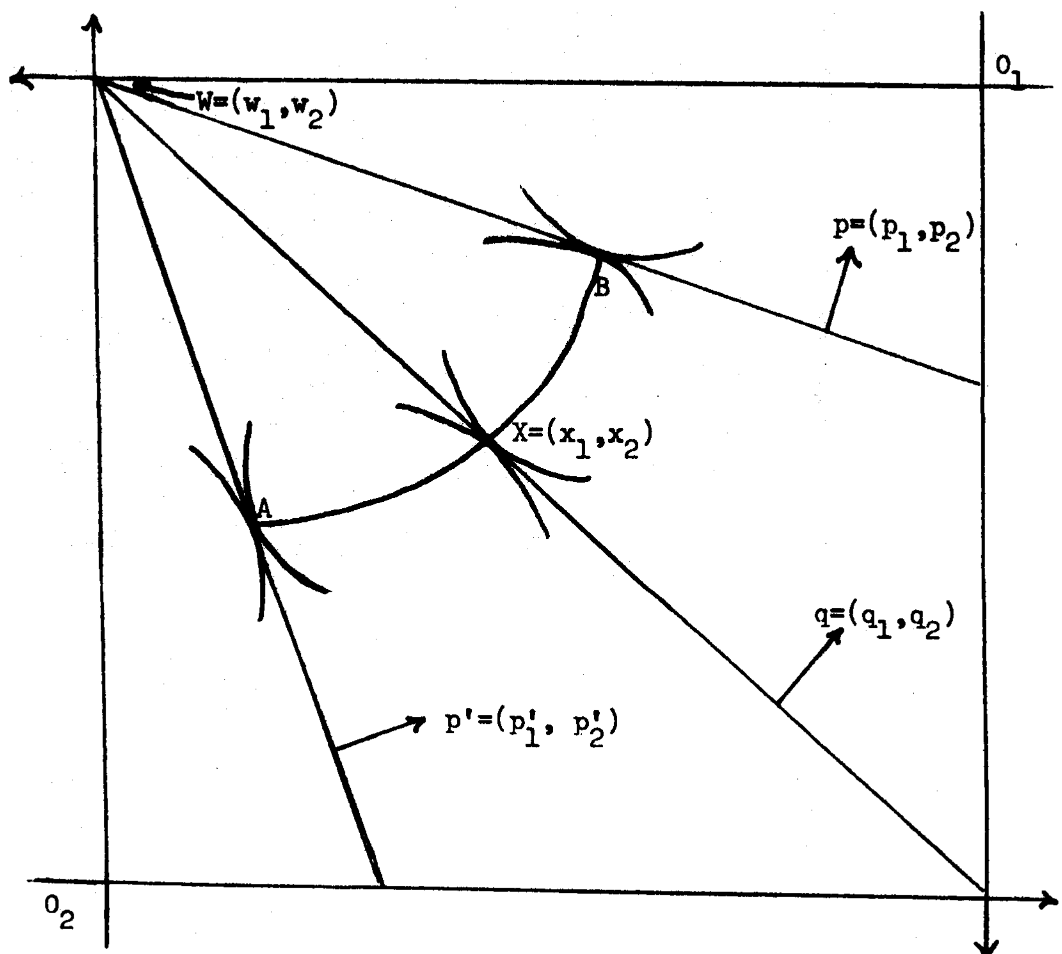

Figure 4

Let $e_1(q)$ be the excess demand by a type 1 trader for commodity 1 and let $e_2(q)$ be similarly defined for type 2, where $q \in R^2_+$. Since $e_1(q) + e_2(q) = 0$ for all q such that $\frac{p_1}{p_2} \leq \frac{q_1}{q_2} \leq \frac{p'_1}{p'_2}$, it follows that $(k-1)e_1(q) + ke_2(q) \neq 0$ for all such q. Therefore, all WE price vectors in $\mathcal{E}_k^{(1)}$ are far from say $q = \frac{1}{2}(p + p')$, a WE price vector in $\mathcal{E}_k$, no matter how large is k.

The discontinuity of WE prices in this example may be contrasted with the discontinuity in Example 4. In both cases, to obtain an exact WE in $\mathcal{E}_k^{(1)}$ might require a substantial change in prices. Nevertheless, the failure of ANS in Example 4 means that the economy $\mathcal{E}_k^{(1)}$ cannot come within a certain "distance" of obtaining all the gains from trade possible in a WE allocation for $\mathcal{E}_k$ involving trade with the additional member of type 1 and the latter may extract the surplus apparently attributable to him by setting a higher price for his supply of commodity 1. However, the presence of ANS in Example 6 means that $\mathcal{E}_k^{(1)}$ can come arbitrarily close, as k increases, to obtaining all the gains from trade in a WE of $\mathcal{E}_k$, even while trading at WE prices for $\mathcal{E}_k$, so that although exact market clearance in $\mathcal{E}_k^{(1)}$ might require a substantial increase in the price of commodity 1 and a loss in welfare to each of the type 2 traders, a single member of type 1 is powerless to exploit this.

VIII. An Alternative Interpretation of the Equivalence Theorem

The formulation of competitive bargaining as costless multilateral contracting and recontracting that we now call the core allowed Edgeworth to demonstrate two points he regarded as fundamental. First, he showed that with small numbers, the theory of value could say little because the outcome is indeterminate. He criticized Cournot's theory of duopoly:

> "... not merely because the solution given by Cournot in the particular case put by him -- namely, where two competitors deal in an article which, like the supplies from a mineral spring, can be multiplied without expense (Recherches, Art. 43) -- is [technically] erroneous, but rather because he has missed the general theorem: that the solution is indeterminate where the number of competitors is small."
>
> (Edgeworth, Collected Papers, Vol. 3, p. 110)

Second, he was able to demonstrate using the same theory as in the small numbers case that in a market with large numbers of buyers and sellers the outcome is determinate and is a WE. He criticized Walras' approach to stability:

> He describes a way rather than the way by which economic equilibrium is reached. For we have no general dynamical theory determining the path of the economic system from any point assigned at random to a position of equilibrium. We know only the statical properties of the position; as Jevons's analogy of the lever implies. Walras's laboured description of prices set up or "cried" in the market is calculated to divert attention from a sort of higgling which may be regarded as more fundamental than his conception, the process of recontract as described in these pages and in an earlier essay. It is believed to be a more elementary manifestation of the propensity to truck than even the effort to buy in the cheapest and sell in the dearest market. The proposition that there is only one price in a perfect market may be regarded as deducible from the more axiomatic principle of recontract (Mathematical Psychics, p. 40 and context).
>
> (Edgeworth, Collected Papers, Vol. 2, pp. 311-312)

Retracing the path Edgeworth mapped out, and later writers rediscovered and brilliantly extended, we find it unnecessary to invoke the core to make these points. We need only appeal to the absence of a PO-NS allocation with small numbers to establish indeterminacy and the presence of such an allocation in market economies to deduce determinacy. Going beyond this to obtain Equivalence, we find certain anomalies. Equivalence is neither necessary nor sufficient to rationalize the determinacy of WE, at least not if our definition of determinacy depends on perfectly elastic demands. Nevertheless, Examples 2-5, above, illustrating these anomalies are in one way or another exceptional and, broadly-speaking, Inequivalence does coincide with less than perfectly elastic demands for WE in finite economies simply because such economies rarely contain PD price vectors and Equivalence does coincide with perfectly elastic demands in market economies because such economies almost always satisfy ANS. One might therefore wish to summarize the results of this paper by saying that our approach represents a mathematical refinement of the core approach in which we look only to the equivalence or inequivalence of the NNS condition and WE -- i.e., NS. Conceptually, however, this change with its interpretation in terms of elasticities of demand rather than core bargaining is basic.

The Equivalence Theorem is interpreted as an explanation of the stability of WE where stability is defined with respect to all the core conditions not simply NNS. Put this way, the Equivalence Theorem exhibits the dependence of PCE on the notion of the core. Appealing to Propositions 6 and 7 and their connections with the traditional characterization of PCE in terms of elasticities of demand (Proposition 3), we may arrive at a quite different interpretation. Equivalence occurs in market economies because WE are perfectly competitive. Put this way, the Equivalence depends on PCE and the concept of PCE stands unsupported by the core. There is much to be said for this stance.

If we consider the core as a possible foundation for competitive equilibrium, we will have admitted at a fairly fundamental level an economic logic -- costless multi-lateral contract -- that is foreign to the entire superstructure of economic theory. To take just one illustration, recall the well-known quotation from A. Smith:

> People of the same trade seldom meet together, even for merriment and diversion, but the conversation ends in a conspiracy against the public, or in some contrivance to raise prices.
>
> (Wealth of Nations, ed. Canaan, p. 128)

According to the core there need be no cause for concern since a portion of the public can also meet with some members of that trade and conspire to disrupt the conspiracy! Such costless symmetry in the abilities of groups to cooperate was implicitly rejected by Smith and certainly the majority of later writers in their treatments of both perfectly and imperfectly competitive equilibrium. To go further, such a rejection is the starting point for any decentralized theory of resource allocation.

IX. Proofs

Proof of Proposition 2 (PD ⟹ QE): Let p be PD. By (A.1) there exists $\tilde{X} = (\tilde{x}_i)$ such that $(\tilde{X}, p)$ is EM, where $\tilde{X}$ is not necessarily a feasible allocation.

From the hypothesis that p is PD, for each j and any $q(p;j)$ and x_i, $i \neq j$, such that (a) $q(p;j)(x_i - w_i) = 0$ and (b) $(q(p;j) - p)x_i \neq 0$, $w^{(j)} \in \sum_{i \neq j} P_i(x_i)$. Since x_i can be chosen to satisfy (a) and (b) while being arbitrarily close to $\tilde{x}_i$, we have by (A.1) and (A.2) that $w^{(j)} \in R^{(j)}(\tilde{X}^{(j)})$ -- i.e., there exists $y_i(j)$, $i \neq j$, such that

$$y_i(j) \in R_i(\tilde{x}_i) \quad \text{and} \quad \sum_{i \neq j} y_i(j) = w^{(j)}.$$

Because $(\tilde{X},p)$ is EM, $py_i(j) \geq p\tilde{x}_i$; however, since $q(p;j)$ can be chosen arbitrarily close to p, we must have $py_i(j) = p\tilde{x}_i$, $i \neq j$.

Let $x_i = \frac{1}{n-1} \sum_j y_i(j)$. Then $X = (x_i)$ is an allocation for $\mathcal{E}$ because

$$x = \Sigma x_i = \frac{1}{n-1} \sum_i \sum_j y_i(j) = \frac{1}{n-1} \sum_j \sum_i y_i(j) = \frac{1}{n-1} \sum_j w^{(j)} = w.$$

Also, since x_i is a convex combination of elements of $R_i(\tilde{x}_i)$, $x_i \in R_i(\tilde{x}_i)$; and, $px_i = \frac{1}{n-1} p \sum_j y_i(j) = p\tilde{x}_i$.

To show that (X,p) is a QE it only remains to show that (X,p) is EM. Suppose the contrary that there exists an x_i' such that $px_i' < px_i$ and $x_i' \in R_i(x_i)$. Then $px_i' < p\tilde{x}_i$ and by transitivity of R_i, $x_i' \in R(\tilde{x}_i)$, contradicting the hypothesis that $(\tilde{X},p)$ is EM.

<u>Proof of Proposition 3 (NS $\Leftrightarrow$ PD)</u>: To show PD $\Rightarrow$ NS when (X,p) is EM and p is PD simply repeat the first two paragraphs of the proof of Proposition 2.

To show NS $\Rightarrow$ PD when (X,p) is EM and X is NS, we have from Proposition 1 that if (X,p) is EM and X is NS, then (X,p) is a QE, so that $px_i = pw_i$. Because X is NS, for each j there is a y_i, $i \neq j$, such that $y_i \in R_i(x_i)$ and $\sum_{i \neq j} y_i = w^{(j)}$ Therefore, $py_i \geq px_i = pw_i$ and the fact that $\sum_{i \neq j} y_i = w^{(j)}$ implies that for all $i \neq j$,

$$py_i = px_i = pw_i.$$

Let $\tilde{x}_i$, $i \neq j$, satisfy $q(p;j)(\tilde{x}_i - w_i) = 0$. By construction of $q(p;j)$, $p\tilde{x}_i \leq q(p;j)\tilde{x}_i = q(p;j)w_i = pw_i$. Further, $p\tilde{x}_i = pw_i$ if and only if $(q(p;j) - p)\tilde{x}_i = 0$.

To show that p is PD we must establish that $p\tilde{x}_i = pw_i$ implies $y_i \in R_i(\tilde{x}_i)$. If not, the assumption of completeness of R_i means $\tilde{x}_i \in P_i(y_i)$. By (A.1) and $pw_i = py_i \neq \min pR_+^\ell$, there exists x_i' such that $x_i' \in P_i(y_i)$ and $px_i' < py_i = px_i$. But by transitivity of R_i and the hypothesis that X is NS, so $y_i \in R_i(x_i)$, we would have $x_i' \in R_i(x_i)$, contradicting the fact that (X,p) is EM.

If $p\tilde{x}_i < pw_i$, which means $(q(p;j) - p)\tilde{x}_i \neq 0$, a similar argument shows that $y_i \in P_i(\tilde{x}_i)$, thus establishing that p is PD.

<u>Proof of Proposition 4 (GNS $\Rightarrow$ QE)</u>: From the hypothesis that X is PO, there exists a p such that (X,p) is EM. From the hypothesis that X satisfies GNS for the balanced collection $B = \{S\}$, $w^S \in R^S(X^S)$, and therefore $pw^S \geq px^S$. If $pw^S > px^S$ for at least one $S \in B$, then because $\sum_{S \in B_i} b_S = 1$, we would have

$$p \sum_{S\varepsilon B} b_S w^S = p \sum_i \sum_{S\varepsilon B_i} b_S w^S = p \sum_i w_i \sum_{S\varepsilon B_i} b_S = pw$$

$$> p \sum_{S\varepsilon B} b_S x^S = p \sum_i \sum_{S\varepsilon B_i} b_S w^S = p \sum_i x_i \Sigma b_S = px.$$

This contradicts w = x and therefore for all $S\varepsilon B$,

(1) $pw^S = px^S$.

Define $p(x_i - w_i) = r_i$, $i=1, \ldots, n$. From (1), for each $S\varepsilon B_i$,

(2) $r_i + \sum_{\substack{j\varepsilon S \\ j\neq i}} r_j = 0.$

Multiplying equation (2) by b_S for $S\varepsilon B_i$ and summing, we obtain

$$\sum_{S\varepsilon B_i} b_S(r_i + \sum_{\substack{j\varepsilon S \\ j\neq i}} r_j) = \sum_{\substack{S\varepsilon B_i \\ i\varepsilon S}} (b_S r_i) + \sum_{j\neq i} (\sum_{\substack{S\varepsilon B_i \\ j\varepsilon S}} b_S) r_j$$

(3) $$= a_{ij} r_i + \sum_{j\neq i} a_{ij} r_j = 0.$$

For each $i=1, \ldots, n$ there is an equation (3) and these n equations in the n unknowns $r = (r_1, \ldots, r_n)$ can be written in matrix form as

$$Ar = 0,$$

where A is the matrix A(B) associated with the regularly balanced collection B. Since A has an inverse, Ar = 0 implies r = 0. Therefore (X,p) is a QE.

<u>Proof of Proposition 6 (NNS ⟹ QE)</u>: Let

(1) $y_k^{(j)} = x^{(j)}[k] + (\frac{k-1}{k}) x_j = x^{(j)} - \frac{1}{k}(x^{(j)} - w^{(j)}) + (\frac{k-1}{k}) x_j.$

It is easily verified that $y_k^{(j)} = w_k^{(j)}$ and since X_k satisfies NNS, $y_k^{(j)} \notin P_k^{(j)}(X)$. Therefore, there exists $p_k^{(j)} \varepsilon R^\ell$ such that

(2) $p_k^{(j)} y_k^{(j)} \leq p_k^{(j)} P_k^{(j)}(X).$

By construction $y_k^{(j)} \to x$ and by PPC, $p_k^{(j)} \to p$. Therefore $px \leq pP(X)$ and by (A.2) we have $px \leq pR(X)$ so that (X,p) is EM. It only remains to show that $px_i = pw_i$ to show that (X,p) is a QE.

From (1) and (2), $p_k^{(j)}[x^{(j)} - \frac{1}{k}(x^{(j)} - w^{(j)}) + (\frac{k-1}{k})x_j] \leq p_k(x^{(j)} + (\frac{k-1}{k})x_j)$. This reduces to $0 \leq p_k^{(j)}(x^{(j)} - w^{(j)})$ and by PPC to

(3) $\quad 0 \leq p(x^{(j)} - w^{(j)})$.

From (3) and the fact that $x = w$ it follows that $px_i = pw_i$.

<u>Proof of Proposition 7 (ANS ⟺ QE)</u>: By hypothesis X is PO. Therefore, there exists p such that (X,p) is EM.

To show ANS ⟹ QE, let $(w_k^{(j)}+z_k^{(j)}) \in R_k^{(j)}(X)$. Then

$$px_k^{(j)} = p(x^{(j)} + (\frac{k-1}{k})x_j) \geq p(w^{(j)} + (\frac{k-1}{k})w_j + z_k^{(j)}) = p(w_k^{(j)} + z_k^{(j)}).$$

Multiplying the above inequality by k and rearranging terms we obtain

$$kp(x^{(j)} - w^{(j)}) + (k-1)p(x_j - w_j) \leq kpz_k^{(j)}.$$

By ANS, $kpz_k^{(j)} \to 0$, from which it follows that since $kp(x-w) = 0$, $p(x_j-w_j) = 0$, for all $j = 1, \ldots, n$, and therefore (X,p) is a QE.

To show that if (X,p) is a QE, $\{X_k\}$ is ANS, let $y_k^{(j)} = x^{(j)}[k] + (\frac{k-1}{k})x_j$. Since $y_k^{(j)} = w_k^{(j)}$, if $x^{(j)}[k]$ $(= (\frac{k-1}{k})x^{(j)} + (\frac{1}{k})w^{(j)}) \in R^{(j)}(X^{(j)})$ for some k, then by convexity of $R^{(j)}(X^{(j)})$, $x^{(j)}[\bar{k}] \in R^{(j)}(X^{(j)})$ for all $\bar{k} > k$ and we may set $||z_k^{(j)}|| = 0$ such that $w_k^{(j)} + z_k^{(j)} \in R_k^{(j)}(X)$. Therefore assume $x^{(j)}[k] \notin R^{(j)}(X^{(j)})$.

Let $z_k^{(j)}$ satisfy $(x^{(j)}[k] + z_k^{(j)}) \in R^{(j)}(X^{(j)})$ and

(1) $\quad 0 < ||z_k^{(j)}|| \leq ||R^{(j)}(X^{(j)}) - x^{(j)}[k]||$.

Note that since $y_k^{(j)} = x^{(j)}[k] + (\frac{k-1}{k})x_j = w_k^{(j)}$, if $(x^{(j)}[k] + z_k^{(j)}) \in R^{(j)}(X^{(j)})$, then $(y_k^{(j)} + z_k^{(j)}) \in R_k^{(j)}(X)$ and to establish ANS we need only show that for the above construction of $z_k^{(j)}$, $k||z_k^{(j)}|| \to 0$.

The standard method of proof for the Separation Theorem shows that when $x^{(j)}[k] \notin R^{(j)}(X^{(j)})$, where $R^{(j)}$ is closed and convex, and $z_k^{(j)}$ is defined by (1), if we set

$$(2) \quad p_k^{(j)} = \frac{z_k^{(j)}}{||z_k^{(j)}||},$$

we shall obtain

$$p_k^{(j)} x^{(j)}[k] < p_k^{(j)}(x^{(j)}[k] + z_k^{(j)}) \leq p_k^{(j)} R^{(j)}(X^{(j)}).$$

Since $x^{(j)} \in R^{(j)}(X^{(j)})$, we have $0 < p_k^{(j)} z_k^{(j)} \leq p_k^{(j)}(x^{(j)} - x^{(j)}[k])$ which can be rewritten as

$$(3) \quad 0 \leq \frac{k p_k^{(j)} z_k^{(j)}}{p_k^{(j)}(x^{(j)} - w^{(j)})} \leq 1$$

after substituting the definition of $x^{(j)}[k]$.

By PPC, $p_k^{(j)} \to p$ and since $p(x^{(j)} - w^{(j)}) = 0$, because (X,p) is a QE, the denominator of (3), $p_k^{(j)}(x^{(j)} - w^{(j)}) \to 0$. Therefore, the numerator of (3) approaches zero which means after substituting (1) for $p_k^{(j)}$ that

$$k p_k^{(j)} z_k^{(j)} = \frac{k z_k^{(j)}}{||z_k^{(j)}||} z_k^{(j)} = k||z_k^{(j)}|| \to 0,$$

as was to be shown.

Footnotes

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**Much of the material presented here was worked out in discussions with Louis Makowski. Since a first draft of this paper was written, Makowski [1976] has shown that the no-surplus condition may be applied to economies with production.

Conversations with Juan Urrutia helped me to clarify the presentation of Sections VII and VIII and to streamline the proofs of Propositions 6 and 7.

I have also benefited from comments by Ted Bergstrom, Bryan Ellickson, Oliver Hart and Robert Jones.

1. Statements of Lyapunov's Theorem and its role in proving Equivalence and existence of WE in market economies are found in Vind [1964] and Aumann [1966].

The phenomenon of absence of economy-wide gains from trade in market economies may be seen in the papers of Schmeidler [1972] and Grodal [1972], although this is not their emphasis.

2. The importance of "one-fewer" sets of agents for competitive theory was suggested by an example that Robert Aumann showed me from the Aumann-Shapley [1974] theory of non-atomic games to refute my conjecture that absence of economy-wide gains from trade was a necessary condition for Equivalence.

3. R_i is complete if for any x_i, x_i', either $x_i \in R_i(x_i')$ or $x_i' \in R_i(x_i)$, and P_i is transitive if for any x_i, x_i', and x_i'', $x_i \in P_i(x_i')$, $x_i' \in P_i(x_i'')$ implies $x_i \in P_i(x_i'')$.

Although these assumptions are superfluous for the existence of a WE (see Gale and Mas-Colell [1975]), it does not appear possible to dispense with them entirely here because transitivity and completeness provide added possibilities for substitution in each agent's preferences which we find necessary to guarantee that a WE is a PCE. See Propositions 2 and 3, below.

4. The usual definition of PO is more demanding. It says that X is PO only if there is no other allocation that all the members of $\mathcal{E}$ would find as desirable and at least one member would find more desirable -- i.e., X satisfies for each $j=1,\ldots,n$, $x \notin (P_j(x_j) + \sum_{i \neq j} R_i(x_i))$.

5. See Debreu [1962].

6. Also, if $px_i \neq \min pR_+^\ell$, our weaker definition of PO coincides with the usual one.

7. It may be objected that as an upper bound on how much an agent could conceivably expect to gain from trade NNS is too conservative if it depends on the capacity of each member of $\mathcal{E}^{(j)}$ separately to perceive the disadvantages of dealing with j. This observation simply reinforces the argument for indeterminacy that we draw from NNS.

8. Preferences are strictly convex if for any two distinct elements of $R_i(x_i)$, the line segment joining them lies in $P_i(x_i)$.

9. Non-uniform pricing does not necessarily indicate non-price-taking behavior. Makowski [1975] has shown that if the costs of transacting vary with the number of traders with which one deals, perfectly competitive pricing will take the form of a price schedule that varies with quantity. Here, however, because we have assumed that preferences and consumption sets are convex and there are no costs of exchange, perfectly competitive pricing must be uniform.

10. The above definition of the core departs from the standard one in the same way that our definition of PO departs from the standard. (See footnote 4.) The standard definition requires not only $w^S \notin P^S(X^S)$, but for each $j \in S$, $w^S \notin (P_j(x_j) + \sum_{\substack{i \in S \\ i \neq j}} R_i(x_i))$. For our purposes, the distinctions may be ignored.

11. Such a finding has already been reported for economies composed of a (non-atomic) continuum of agents and large or atomic agents. See Gabszewicz and Mertens [1971] and Shitovitz [1973]. They constructed classes of economies exhibiting Equivalence despite the fact that the large traders would not be considered perfect competitors -- i.e., WE prices would not be PD. Example 2 shows that such <u>curiosa</u> also occur in purely atomic economies and Proposition 4 gives a set of sufficient conditions for constructing them.

12. To demonstrate that this vector of utilities is in the core, note that is it PO, so if a coalition can do better for itself it must contain individual 1. Clearly, 1 cannot do better by itself and the coalitions $\overline{1i}$, $i = 2, 3, 4$ cannot give i as much as $u_i = \frac{4}{3}$ because each agent i has at most one unit of commodity $c(= i)$ to share. Finally, any three-person coalition $\overline{1ij}$, $i \neq j = 2, 3, 4$ cannot give both i and j as much as $u_i = u_j = \frac{4}{3}$ because they do not have more than one unit of each of commodities $c(= i)$ and $d(= j)$ to share.

It would appear that the "kinks" in this Example, although they simplify the demonstration, are not essential because the two- and three-person coalitions lie outside of some ε-neighborhood of being able to improve upon the stated vector of utilities. If the utility functions were smoothed in the region of the kinks while keeping the radius of curvature small, similar results should also obtain.

13. The ANS property of WE is undoubtedly related to results on the speed of convergence of the core to WE at the rate $\frac{1}{k}$. See Debreu [1975]. Nevertheless, the relation is not as close as one might suspect. Shapley [1976] has constructed a class of examples of core convergence such that for any rate, however slow, there is an economy converging as slowly. But for each of Shapley's examples we have ANS or $k||z_k^{(j)}|| \to 0$. Further, for an example that converges infinitely rapidly -- see Example 4, below and also Shapley -- ANS fails.

14. In an economy where traders do not supply goods that simply duplicate what is already available from many others, then even though they may be numerous, the analogue of $k||z_k^{(j)}|| \to 0$, which would measure the surplus contributed by a small-scale trader relative to its size, would be problematic.

15. More generally, the differences are between economies with and economies without economy-wide gains from trade.

16. Proposition 6 is similar to a result obtained by Hansen [1969]. He showed that if X is PO and $\{X_k\}$ satisfies the non-negative surplus condition, not for each j=1,...,n, but for each group consisting of all the members of type j and all but one of the members of each of the other types -- i.e.,

$$(kw_j + (k-1) \sum_{i \neq j} w_i) \notin (kP_j(x_j) + (k-1) \sum_{i \neq j} P_i(x_i)),$$

then X is a QE. Hansen used this result to show that under conditions similar to PPC convergence of the core could be obtained while appealing only to a small subset of the core conditions, provided that preferences are complete, transitive and strictly convex.

17. This example is due to Guy Laroque.

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