CONVERGENCE TO PERFECTLY COMPETITIVE EQUILIBRIA

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We give a simple and fairly general demonstration that individual monopoly power vanishes as the number of traders increases, provided traders have bounded endowments, the number of traded goods is finite, and marginal rates of substitution do not, in the aggregate, exhibit abrupt changes.

The proof proceeds from "first principles" -- in particular, those underlying the Separation and Support Theorems in R^ℓ. Our definition of perfectly competitive equilibrium makes essential use of the arguments employed to demonstrate these Theorems and our conclusions amount to versions of them.

The result is intimately related to the Core Convergence Theorem (Edgeworth [1881], Shubik [1959], Debreu and Scarf [1963], Hildenbrand [1970], Bewley [1973], Anderson [1977], and especially the versions of Hansen [1969] and Nishino [1971]); but, our criterion for perfect competition is stronger. The Theorem we prove implies core convergence but core convergence can be demonstrated for sequences of economies that would, according to our criterion, fail to be perfectly competitive.

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(1) $$w^j \notin \text{int}_+ R^j(X),\ j = 1,\ldots,k.$$

It is useful to give the dual version of (1) in terms of the geometry of separating hyperplanes. Let $Q = \{q \in R^{\ell} : ||q|| = 1\}$ be the set of vectors of unit Euclidean norm. Suppose $w^j \notin \text{cl}\ R^j$, the closure of R^j. Let y^j be an element of cl R^j that is closest to w^j in terms of the norm $||\cdot||$ and let $z^j = y^j - w^j$. This construction may be used to show that,

(2) $$q = \frac{z^j}{||z^j||}$$

is the normal to a separating hyperplane between w^j and R^j ($qw^j \leq qR^j$) and $q \in Q$.

If $w^j \in \text{cl}\ R^j$ and $w^j \notin \text{int}_+ R^j$, a sequence $\{y_n\}$, $y_n \in R^{\ell}_+$, may be found such that $y_n \notin \text{cl}\ R^j$ and $y_n \to w^j$, such that $q_n y_n < q_n R^j$ and $q_n \in Q$. Because Q is compact, we may extract a convergent subsequence, $q_{n_h} \to q$, and show that q is a supporting hyperplane for R^j at w^j ($qw^j = \inf qR^j$).

Let $Q^j(X) = \{q \in Q : qw^j \leq qR^j(X)\}$ and let $P^j(X)$ be a subset of $Q^j(X)$ defined by

(3) $$P^j(X) = \begin{cases} \{q\}, \text{ where } q \text{ is defined by (2), when } w^j \notin \text{cl}\ R^j(X) \\ Q^j(X), \text{ when } w^j \in \text{cl}\ R^j\ (X). \end{cases}$$

We now have the following characterization of (1): $w^j \notin \text{int}_+ R^j(X)$ if and only if $Q^j(X)$ is not empty and $w^j \notin \text{cl}\ R^j$ if and only if $Q^j(X) \neq P^j(X)$. If $Q^j(X) \neq P^j(X)$, and therefore, $||z^j|| > 0$, we shall say that relative to the allocation X, agent j is contributing a positive surplus to other agents. If j were to demand a more favorable share of the total allocation than it is receiving in X, competitive forces would not necessarily be able to impugn the demand.

The remaining assumptions on E are,

(III) $$\Sigma w_i \in \text{int} R^{\ell}_+,$$

and, for all $j = 1,\ldots,k$,

demonstrated, using the hypothesis that there exist allocations for which all $Q^j(X)$ are non-empty, that $\bigcap_j P^j(X)$ "becomes non-empty" as the number of agents increases.

Consider a sequence of economies $\{E^k\}$ and a corresponding sequence of allocations $\{X^k\}$ where $X^k = (x_1,\ldots,x_k)$ is an allocation for E^k -- i.e., $x^k \equiv \Sigma x_i = \Sigma w_i \equiv w^k$. Define the following averages: $\bar{w}^k = (k^{-1})w^k$, $\bar{x}^k = (k^{-1})x^k$ and $\bar{R}^k(X^k) = (k^{-1})\Sigma R_i(X^k)$. Let $j(k)$ be a selection of exactly one element from $\{1,\ldots,k\}$ for each $k = 1,2,\ldots$. For each k, let $j = j(k)$ and define $\bar{w}^{j(k)} = (k^{-1})w^j$, $\bar{x}^{j(k)} = (k^{-1})x^j$, $\bar{R}^{j(k)}(X^k) = (k^{-1})R^j(X^k)$, $\bar{w}_{j(k)} = (k^{-1})w_j$, and $\bar{R}_{j(k)}(X^k) = (k^{-1})R_j(X^k)$. These definitions yield the following identity:

$$k(\bar{x}^{j(k)} - \bar{w}^{j(k)}) = w_{j(k)} - x_{j(k)}. \tag{6}$$

For the sequence of initial endowments in $\{E^k\}$, it is assumed that there exists numbers α and β, $o < \alpha \leq \beta$, such that for any selection $j(k)$, $k = 1,2,\ldots,$

$$\alpha \leq ||w_{j(k)}|| \leq \beta, \tag{V}$$

and, that there exists $y^o \in \text{int}R_+^\ell$ such that,

$$(\bar{w}^k - y^o) \in R_+^\ell \tag{VI}$$

Assumption (VI) simply extends (III) to the sequence of economies. Its primary role is to ensure that for any $q \in Q_+$, $\lim\inf q\bar{w}^k > 0$. (V) says that individual endowments are bounded above and below. This will be required to preclude the persistence of individual monopoly power as k increases. Note that either $\alpha \leq ||w_{j(k)}||$ or (VI) ensures that the number of agents with non-trivial endowments is increasing with k. It follows from $||w_{j(k)}|| \leq \beta$ that,

$$||\bar{w}^{j(k)} - \bar{w}^k|| \to 0. \tag{7}$$

(ii) $$\lim k[p^k(\bar{x}^{j(k)} - \bar{w}^{j(k)})] = \lim [p^k(w_{j(k)} - x_{j(k)})] = 0.$$

For any $p \in Q$, $k[p\bar{w}^k - \inf p\bar{R}^k(X^k)] \leq 0$. If $\limsup k[p^k\bar{w}_{j(k)} - \inf p^k\bar{R}_{j(k)}(X^k)]$ $= \limsup [p^k(w_{j(k)} - \inf p^k R_{j(k)}(X^k)] > 0$, for $p^k \in Q_+$, then $\liminf k[p^k(\bar{w}^{j(k)} - \inf p^k\bar{R}^{j(k)}(X^k)] < 0$. But this contradicts (ii) and, therefore, using (ii),

(iii) $$\lim [p^k w_{j(k)} - \inf p^k R_{j(k)}(X^k)] = 0.$$

Adding (i) and (iii),

$$\lim \{p^k(\bar{w}^{j(k)} + w_{j(k)}) - \inf p^k[\bar{R}^{j(k)}(X^k) + R_{j(k)}(X^k)]\} = 0.$$

Letting $y^k \in S_{j(k)}(X^k) \subset \text{int}_+ R_{j(k)}(X_k)$, $(\bar{x}^{j(k)} + y^k) \in \text{int}_+[\bar{R}^{j(k)}(X^k) + R_{j(k)}(X^k)]$. By (V), $\bar{w}^{j(k)} \to \bar{w}^k$ and by IV and (VI), $\liminf p^k\bar{w}^{j(k)} > 0$. Using (ii) and the hypothesis of (b) implies,

$$0 < \liminf [p^k(\bar{x}^{j(k)} + y^k) - p^k(\bar{w}^{j(k)} + w_{j(k)})] = \liminf [p^k(y^k - w_{j(k)})].$$

Note that the selection in Lemma 1 is arbitrary and it may be chosen so that $|p^k(w_{j(k)} - x_{j(k)})| = \max_{1 \leq j \leq k} |p^k(w_j - x_j)|$. Thus, our definition of convergence to Walrasian equilibria implies uniform convergence in terms of the price-weighted values of net trades. (This is an obvious property of the Core Convergence Theorem when the sequence of economies consists of replicas of a given finite economy.) However, just as the definition of Walrasian equilibrium in (4) with a fixed, finite number of agents does not imply that the equilibrium is perfectly competitive, definition (8) does not, by itself, imply that individual monopoly power is necessarily vanishing as k increases.

Assume $\{X^k\}$ is a sequence of non-negative surplus allocations and assume $\bar{w}^{j(k)} \notin \text{cl}\, \bar{R}^{j(k)}(X^k)$. Construct $\bar{y}^{j(k)} \in \text{cl}\, \bar{R}^{j(k)}(X^k)$ just as $y^j \in \text{cl}\, R^j(X)$ was

To obtain a result on vanishing monopoly power (equivalently, convergence to no-surplus, or perfectly competitive Walrasian equilibrium), we must show that there exists $\{(X^k, p^k)\}$ such that $\inf ||p^k - P^{j(k)}(X^k)|| \to 0$.

LEMMA 2: Let $\{E^k\}$ be a sequence of economies satisfying (V) and let $\{X^k\}$ be a sequence of non-negative surplus allocations such that for any selection $j(k)$, $\{P^{j(k)}(X^k)\}$ is a sequence of non-empty subsets of Q_+. If $\{p^k\}$ is a sequence such that $\inf ||p^k - P^{j(k)}(X^k)|| \to 0$, then $k||\bar{z}^{j(k)}|| \to 0$.

PROOF: Choose $q^k \in P^{j(k)}(X^k)$ such that $||p^k - q^k|| \to 0$. Without loss of generality assume $k||\bar{z}^{j(k)}|| > 0$, all k. By construction

$$0 \leq k||\bar{z}^{j(k)}|| = kq^k\bar{z}^{j(k)} = kq^k(\bar{y}^{j(k)} - \bar{w}^{j(k)}) = k[\inf q^k\bar{R}^{j(k)}(X^k) - q^k\bar{w}^{j(k)}]$$
$$\leq kq^k(\bar{x}^{j(k)} - \bar{w}^{j(k)}) = q^k(w_{j(k)} - x_{j(k)})$$

By the argument used to establish (ii) in the proof of Lemma 1,
$q^k(w_{j(k)} - x_{j(k)}) \to p^k(w_{j(k)} - x_{j(k)}) \to 0.$

One more assumption will be required from which the desired conclusion will readily follow. Let $\{X^k\}$ be such that for any selection $j(k)$, $\{Q^{j(k)}(X^k)\}$ is a sequence of non-empty subsets of Q_+. For a particular selection $\hat{j}(k)$, let $p^k \in Q^{\hat{j}(k)}(X^k)$. Therefore, $p^k\bar{w}^{\hat{j}} \leq p^k\bar{R}^{\hat{j}(k)}(X^k)$. Since $p^k \in Q_+$ and $\bar{R}^k(X^k) = \bar{R}^{\hat{j}(k)}(X^k) + \bar{R}_{\hat{j}(k)}(X^k)$, we have $p^k\bar{w}^{\hat{j}(k)} \leq \inf p^k\bar{R}^k(X^k) \leq p^k\bar{x}^k$; and, since $\bar{w}^{\hat{j}(k)} \to \bar{w}^k (= \bar{x}^k)$,

$$\text{(11)} \qquad \lim [p^k\bar{w}^{\hat{j}(k)} - \inf p^k\bar{R}^k(X^k)] = 0.$$

Now, let $j(k)$ be an arbitrary selection, and let $\{q^k\}$ be a sequence such that $\limsup ||q^k - p^k|| > 0$. The final assumption is:

$$\text{(VII)} \qquad \limsup [q^k\bar{w}^{j(k)} - \inf q^k\bar{R}^k(X^k)] > 0.$$

The hypothesis for (VII) describes $\{p^k\}$ as a sequence of unit normals to hyperplanes separating $\bar{w}^{\hat{j}(k)}$ and $\bar{R}^k(X^k)$. Since $\bar{w}^{\hat{j}(k)} \to \bar{x}^k \in \bar{R}^k(X^k)$, the

It is also clear that in the setting in which it is placed, (VII) is not very restrictive. It will obtain for "most" economies satisfying the other assumptions. As long as $\text{int}_+\bar{R}^k(X^k)$ is not empty, the set of points on the boundary where cl $\bar{R}^k(X^k)$ admits a unique supporting hyperplane is dense and its complement is a set of measure zero. (See Rockafeller [1970], pp. 241-250.) Further, since $\bar{R}^k$ is actually the average of many R_i's, even if individual preferences are not differentiable, aggregate preferences "frequently" will be. (An early example of this phenomenon of smoothing by aggregation is given in Houthakker [1955]).

At the R_+^ℓ-boundary of $\bar{R}^k(X^k)$ there is a natural kink creating a source of non-uniqueness that is not made less likely by aggregation, but this has been ruled out by (VI). The absence of differentiability at the boundary may be symptomatic of the fact that there is a strong demand for goods not in abundant supply so that even though the suppliers of these goods operate on a small scale, their monopoly power may not vanish. (Nevertheless, Hart [1977] has provided an example of a sequence of economies with $\bar{w}^k \varepsilon \text{ int}R_+^2$ where $\bar{w}^k$ is converging to the boundary of R_+^2 that is, in our terms, perfectly competitive.)

REMARK: We are presuming that the likelihood of (VII) is identical to the likelihood that a boundary point of a convex set will have a unique support. This is the case when the commodity space is finite-dimensional or, more generally, when the relevant dual space of (normalized) linear functionals is compact. (See Mas-Colell [1975] for a model with an infinite number of commodities for which the analogous Q_+ is the set of non-negative, continuous functions defined on a compact set that are equicontinuous.) When the relevant Q_+ is not compact, it requires more than uniqueness of supporting hyperplanes to obtain the analogue of (VII). We pursue this in a forthcoming paper.

II. Concluding Remarks

In the absence of external effects, the above result may be regarded as a specialized and stronger version of the Core Convergence Theorem. (If core bargaining means uninhibited cooperation among agents, the definition of Walrasian equilibria will bear little connection to the core in the presence of external effects since Walrasian equilibria are not generally even Pareto-optimal. Despite this Core inequivalence, there is nothing about the concept of perfectly competitive equilibrium that would be logically incompatible with the presence of external effects.) The non-negative surplus condition is just one of the set of conditions for an allocation to be in the core. It says that any "one-fewer" group of agents cannot improve upon their share of an allocation using only their own resources. With these groups, and only these groups, imposing restrictions on the sequence of allocations, we have given conditions implying that the allocations must be approaching Walrasian equilibria. Since the latter is always in the core, *a fortiori* the core with its larger set of restrictions is converging.

When the one-fewer groups are decisive in determining the core, we may give a non-cooperative interpretation of core convergence. If any one agent were to attempt to extract a more desirable bundle of goods by, say, charging a higher price to its customers, we know that none of them need submit since $k||\bar{z}^{j(k)}||\to 0$ implies that they can find other sellers willing to supply them. Thus, a perfectly competitive Walrasian equilibrium is not vulnerable to threats by individual agents. By relaxing the boundedness assumption (V) or the aggregate smoothness assumption (VII) it is possible to achieve core convergence, without the one-fewer groups being decisive. (In general, this convergence would not be uniform.) In this case, we know that if an agent the relevant coalitions are pre-specified to be the one-fewer sets and the mode of convergence is uniform. It appears, therefore, that appeal to the Shapley-Folkman Theorem for reasons other than the demonstration of convexity of aggregate preferences is essential only to bring within the coverage of the Core Convergence Theorem those sequences of economies that are not, according to our criterion, perfectly competitive. (In the Aumann [1964]-Vind [1964] continuum of traders model, an analogous version of our results can be obtained without appeal to the Lyapunov Convexity Theorem.)