

THE TREASURY BILL FUTURES MARKET

BY

Rodney L. Jacobs and Robert A. Jones

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It is common practice in analyzing the term structure of interest rates to characterize a long term loan as a one period loan plus a series of forward one period loan commitments extending out to the obligation's maturity date.^{1/} To each set of interest rates on forward loans there corresponds one yield curve for current loans of different maturities. Explaining the equilibrium yields on forward loans is, therefore, theoretically equivalent to explaining the equilibrium term structure. Since January 6, 1976, the International Monetary Market Division of the Chicago Mercantile Exchange has operated a futures market in 13 week Treasury bills. The prices established in this market provide data potentially distinct from that embodied in the term structure for exploring how forward interest rates are determined.

This paper develops a model of the Treasury bill futures market of the risk premium augmented expectations variety. Agents' expectations about future spot market interest rates, their anticipated positions as borrowers or lenders in those markets and their attitudes toward risk interact to determine equilibrium futures prices. Two hypotheses about expectations are examined. The first hypothesis is that individuals form expectations about the spot rate and its rate of change (drift) from current and past spot rates through an adaptive learning algorithm. The second hypothesis is that subsequent realized spot rates are unbiased indicators of previous expectations.

The following tentative conclusions indicate the range of empirical

issues addressed in the paper: (1) The interest rates on T-bills for future delivery deviate significantly from the corresponding forward rates implicit in the spot market yield curve; (2) The hypothesis that expectations about the level of and drift in interest rates are formed adaptively from past spot rates fits the futures market data significantly better than the hypothesis that expectations are on average correct; (3) The risk premium component of futures yields varies directly with time to delivery of the T-bills and negatively with the level of interest rates; (4) There was a significant shift in expected future interest rates in January of 1977 when the new administration's economic policies were announced.

Section I presents a theoretical model of futures market equilibrium, section II the expectations adjustment algorithm and section III the estimation method and empirical results.

I. FUTURES MARKET EQUILIBRIUM

The theoretical model of the futures market is developed in three stages. First, the anticipated future lending of a single agent is related to his expectations about future interest rates in a riskless world. Next, his hedge demand for forward loans when interest rates are uncertain is derived from forward interest rates, the degree of uncertainty about future interest rates and his attitude toward risk. Finally, the interest rates which clear the forward loan markets are obtained from the individuals' forward loan demand functions. Equilibrium forward rates turn out to be a risk premium augmented reflection of average market expectations.

1. Anticipated Lending

Suppose there is only one fixed interest financial asset, riskless one-period loans. Let r_τ be the known one period interest rate on loans issued at time τ . Consider a single agent with initial financial asset holdings q_0 and planning horizon T seeking to maximize the objective

$$(1) \quad U(c_1, c_2, \dots, c_T, q_T) \equiv u_1(c_1) + \dots + u_T(c_T) + B(q_T),$$

where q_T is terminal financial asset holdings and c_τ is disbursements from financial assets in period τ . For an individual c_τ may indicate consumption over and above period τ wage income and u_τ a one period utility function. For a firm c_τ may indicate borrowing to finance investment and u_τ the increase in the value of the firm which results from the investment. The terminal asset and one period utility functions, B and u , are assumed increasing, concave and twice differentiable. Asset holdings q_τ at the

end of period τ are constrained by

$$(2) \quad q_{\tau+1} = (1 + r_{\tau})q_{\tau} - c_{\tau+1}.$$

Maximization of equation (1) with respect to c_{τ} , subject to constraint

(2), requires^{2/}

$$(3) \quad \frac{u_{\tau}(c_{\tau}^*)}{B(q_{\tau}^*)} = \prod_{t=\tau}^T (1 + r_t) \text{ for } \tau = 1, \dots, T$$

and

$$(4) \quad q_T^* = q_0 \prod_{t=0}^T (1 + r_t) - \sum_{\tau=1}^T c_{\tau}^* \prod_{t=\tau}^T (1 + r_t).$$

Let $q_{\tau}^S(r)$ denote the resulting assets held at time τ -- the position the agent anticipates taking in the spot loan market at time τ as a function of the vector r of one-period yields. How does a change in interest rates, say a rise in r_k , affect q_{τ}^S ? Generally there will be both substitution and income effects which tend to offset each other if q_k^S is non-zero. However, the substitution effect alone can be easily determined. If the agent is compensated so as to achieve the same level of utility U after the rise in r_k , then not all of $c_1^*, \dots, c_T^*, q_T^*$ can move in the same direction. But since the right-hand side of (3) is unaffected by the change for $\tau > k$, $c_{k+1}^*, \dots, c_T^*, q_T^*$ must move in the same direction. Since the right-hand side of (3) rises by the same proportion for all $\tau \leq k$, it follows that c_{τ}^* falls for all $\tau \leq k$ and rises for $\tau > k$ and that q_T^* increases. The agent's asset holdings would, therefore, increase at all points in time. In what follows we assume this substitution effect dominates any income effects; i.e., $\partial q_{\tau}^S / \partial r_k \geq 0$ for all τ, k .

2. Forward Loan Demand

Suppose the agent is uncertain about the time path that interest rates will follow and that a market for future loans exists. Assume that his beliefs about future interest rates can be represented as a joint probability distribution on r with mean r^e and covariance matrix Ω and that he may enter into forward loan commitments at yields r^f . Further assume, to avoid an awkward dynamic programming problem, that the agent behaves as if his future spot market loans are fixed at $q_\tau^s(r^e)$ for $\tau < T$ and not alterable on the basis of subsequent interest rate developments. In other words he views all interest rate risk as falling on terminal assets q_T .^{3/} The agent may then be supposed to make forward loans which maximize the expected value of $B(q_T)$.

Denote by $q^s(r^e)$ the column vector $(q_1^s, \dots, q_{T-1}^s)'$ of anticipated spot market positions and by q^f the vector $(q_1^f, \dots, q_{T-1}^f)'$ of forward loan commitments made at time zero. The difference $q^f - q^s$ is the agent's speculative (non-hedging) position in future loans. The (uncompounded) sum of the expected gains over periods 1, ..., T-1 attributable to this speculative position is $\pi = (q^f - q^s)'(r^f - r^e)$. The variance of this gain is $(q^f - q^s)' \Omega (q^f - q^s)$.^{4/} Adopting mean-variance criteria to obtain an approximation to the strategy which maximizes $E[B(q_T)]$, the agent chooses only those q^f which minimize the above variance for a given level of π .

From the Lagrangian

$$(5) \quad L(q^f, \lambda) = (q^f - q^s)' \Omega (q^f - q^s) + \lambda [\pi - (r^f - r^e)' (q^f - q^s)]$$

we obtain the first order condition for minimum variance

$$(6) \quad 2\Omega(q^f - q^s) - \lambda(r^f - r^e) = 0.$$

The multiplier λ indicates the increase in minimum variance per unit increase in expected gain associated with a given π . It can be shown to equal $2/a$ at the level of π maximizing $E[B(q_T)]$ for small risks, where $a = -B''/B'$ denotes the agent's index of absolute risk aversion with respect to terminal wealth. From (6) we obtain the demand functions for forward loans

$$(7) \quad q^f = q^s + \frac{\Omega^{-1}}{a} (r^f - r^e).$$

The agent will thus tend to hedge his anticipated loans more fully the more risk averse he is (larger is a), the more closely forward rates coincide with his expectations (smaller is $r^f - r^e$) and the more uncertain he is about his expectations ("larger" is Ω).

3. Futures Market Equilibrium

Finally, let us determine the forward loan rates r^f which clear the market in this partial equilibrium setting. Assuming that Ω is common to all agents (for notational simplicity) but designating the i^{th} agent's mean expectations, risk aversion index and anticipated asset holdings for the first $T-1$ periods by r_i^e , a_i and q_i^s (r_i^e) respectively, the forward loan position taken by agent i at time zero is

$$(8) \quad q_i^f = q_i^s + \frac{\Omega^{-1}}{a_i} (r^f - r_i^e).$$

Aggregating over all agents, setting $\sum_i q_i^f = 0$ for market clearing, and solving for the vector of equilibrium forward rates gives

$$(9) \quad r^f = \sum_i \left(\frac{a_i^{-1}}{\sum_j a_j^{-1}} \right) r_i^e - \Omega \left(\frac{\sum_i q_i^s}{\sum_j a_j^{-1}} \right) = R^e - \bar{a} \Omega \bar{q}^s.$$

The vector R^e is the inverse risk aversion weighted average of market

participants' expectations about future spot interest rates, \bar{q}^s is their unweighted average anticipated asset positions and \bar{a} is the harmonic mean of their absolute risk aversion indices. The first term in (9) will be called the expectation component of the forward loan rates and the second term (positive or negative) the risk premium component.^{5/} Notice that only one agent's risk aversion index need approach zero (in the absence of limitations on the size of positions taken) to drive \bar{a} toward zero, and eliminate the risk premium component of equilibrium forward loan rates. Moreover, to the extent that absolute risk aversion declines with increasing wealth, wealthier agents' expectations will be weighted more heavily in R^e .

In addition to universal risk averse behavior, the existence of a risk premium requires a net imbalance in the forward market participants' lending plans. Assume for the moment that Ω is a positive matrix whose elements increase with row number (this will be discussed further in section II). If, as Hicks (1946) conjectured, there is generally a shortage of lenders relative to borrowers in forward markets, then $\bar{q}^s < 0$ and the risk premium is positive. Forward rates are upward biased indicators of average expected spot rates. Conversely, a relative abundance of forward lenders, $\bar{q}^s > 0$, results in a negative risk premium. To the extent that the elements of Ω increase with row number, indicating greater subjective uncertainty about more distant future interest rates, the risk premium component increases in absolute value with the delivery date of the loan.^{6/}

The risk premium is influenced by expected future interest rates through their effect on \bar{q}^s . Recall from section I.1 that $\partial q_T^s / \partial r_k^e > 0$ for each agent. An increase in the interest rate expected to prevail in any future period increases all elements of \bar{q}^s and thus decreases the liquidity premium

for all delivery dates. The higher anticipated spot rate raises anticipated future lending relative to borrowing, and the resulting forward lending to avoid interest rate risk pushes down forward loan rates. This occurs whether the risk premium is positive or negative. But, since the excess demand for particular loans depends on the entire vector of expected rates, a change in the risk premium for any one date cannot be attributed just to changes in expectations about that date.^{7/}

This model, therefore, leads to two tentative hypotheses about the risk premium embodied in forward interest rates. First, the risk premium falls as some positively weighted average of expected future interest rates rises. Second, the risk premium should rise in absolute value, whatever its sign, with time to delivery of the forward loan.

II. INTEREST RATE EXPECTATIONS

Of the five variables appearing in equation (9), only the forward loan rates are directly observable. R^e and $\bar{a}\bar{\Omega}q^s$ must be related to things observable before the empirical usefulness of equation (9) can be explored. Section 1 below develops an adaptive learning model relating interest rate expectations to past spot rates. Section 2 obtains the form of the risk premium implied by this model. Section 3 specifies an alternative expectations hypothesis and the various forms of the model to be estimated.

1. An Adaptive Learning Model

The maintained assumptions of our adaptive learning model are as follows: (1) The value to individuals of basing forecasts of future interest rates on information beyond just the past history of spot rates is outweighed by the cost of collecting and processing the additional information.^{8/} (2) Individuals behave as if the reduced form stochastic process generating spot rates belongs to the class of ARIMA processes described below. (3) Individuals incorporate new observations into their beliefs about future spot rates in an optimal Bayesian fashion.

Letting r_t denote the observed spot rate for one period loans prevailing at time t , suppose that the stochastic process generating spot rates is

$$\begin{aligned} r_t &= \bar{r}_t + u_t \\ \bar{r}_t &= \bar{r}_{t-1} + \bar{d}_t + v_t \\ \bar{d}_t &= \alpha \bar{d}_{t-1} + w_t \end{aligned} \tag{10}$$

in which u_t , v_t , w_t are zero mean random variables, distributed independently of each other and over time with constant variances, and α is a parameter

between 0 and 1. \bar{r}_t may be thought of as the true underlying interest rate at time t , \bar{d}_t as the trend or drift in interest rates between times $t-1$ and t , u_t as a transitory disturbance to interest rates (or observation error), v_t as a permanent disturbance to interest rates or a transitory disturbance to drift and w_t as a permanent disturbance to the drift. The parameter α indicates the rate of decay of the existing drift in the spot rate. The model of (10) may be thought of as rationalizing the tendency of individuals to view spot rate changes as either transitory, permanent or signalling the beginning of a trend.^{9/}

If u_t , v_t and w_t are normally distributed and if prior beliefs about (\bar{r}_t, \bar{d}_t) are represented by a bivariate normal distribution with mean (r_t^e, d_t^e) , it can be shown that observing r_t results in a normal Bayesian posterior distribution on (\bar{r}_t, \bar{d}_t) with mean^{10/}

$$r_t^{e'} = r_t^e + \lambda_1(r_t - r_t^e)$$

(11)

$$d_t^{e'} = d_t^e + \lambda_2(r_t - r_t^e).$$

After observing r_t , an agent who believes in the stochastic process (10) has a mean forecast of spot rates and trend to prevail at time $t + \tau$ of

$$r_{t+\tau}^e = r_t^{e'} + (\alpha + \alpha^2 + \dots + \alpha^\tau) d_t^{e'}$$

(12)

$$d_{t+\tau}^e = \alpha^\tau d_t^{e'}.$$

It can further be shown that the coefficients λ_1 , λ_2 and the posterior covariance matrix - which depends on the prior covariance matrix and the variances of u , v , w - approach constants over time with $1 \geq \lambda_1 \geq \lambda_2 \geq 0$.

This convergence is presumed to take place before the sample period.

Combining equations (11) and (12) gives the following adaptive algorithm by which agents revise their mean interest rate forecasts

$$r_{t+\tau}^e = r_t^e + \lambda_1(r_t - r_t^e) + \frac{1 - \alpha^\tau}{1 - \alpha} d_{t+1}^e$$

$$(13) \quad d_{t+1}^e = \alpha[d_t^e + \lambda_2(r_t - r_t^e)].$$

Remember that r_t^e refers to the expectation of r_t as of time $t-1$ and that $r_{t+\tau}^e$ denotes the expectation at time t (after observing r_t) of $r_{t+\tau}$ for all $\tau \geq 1$. We shall assume that individual deviations from these forecasts are uncorrelated with differences in attitudes towards risk across the population and, hence, use $r_{t+\tau}^e$ as an unbiased estimate of element τ of market average expectations R^e .

2. The Risk Premium

Our remaining task is to relate $\bar{a}\bar{q}^s$ to observable quantities. This will be done somewhat crudely to keep the number of parameters manageable. From equation (10) the actual spot rate observed at time $t+\tau$ can be written as

$$(14) \quad r_{t+\tau} = \bar{r}_t + \sum_{j=1}^{\tau} \bar{d}_{t+j} + \sum_{j=1}^{\tau} v_{t+j} + u_{t+\tau}$$

$$= \bar{r}_t + \frac{\alpha(1 - \alpha^\tau)}{1 - \alpha} \bar{d}_t + \sum_{j=1}^{\tau} v_{t+j} + u_{t+\tau} + (\text{terms linear in } w\text{'s})$$

Subtracting the agent's forecast given by equation (12) yields the anticipated error in forecasting τ periods ahead.

$$(15) \quad r_{t+\tau} - r_{t+\tau}^e = (\bar{r}_t - r_t^e) + \frac{\alpha(1 - \alpha^\tau)}{1 - \alpha} (\bar{d}_t - d_t^e)$$

$$+ \sum_{j=1}^{\tau} v_{t+j} + u_{t+\tau} + (\text{terms linear in } w\text{'s}).$$

The covariances of these forecast errors are the elements of Ω . The essential structure of Ω can be determined if we are willing to make assumptions about the relative magnitudes of the various sources of forecast error. Let us assume that transitory disturbances to spot rates u_t and permanent disturbances to their trend w_t have relatively low variance so that most of the forecast error in equation (15) results from the error in estimating current trend $(\bar{d}_t - d_t^e)$ and the cumulative permanent shocks to interest rates v_{t+j} .^{11/} Since the current error in estimating trend and the future permanent shocks to interest rates are independent, we obtain the following approximation for the $\tau\delta$ element of Ω by ignoring all but the second and third terms in equation (15):

$$(16) \quad \Omega_{\tau\delta} = E[(r_{t+\tau} - r_{t+\tau}^e)(r_{t+\delta} - r_{t+\delta}^e)]$$

$$\approx \frac{\alpha(1 - \alpha^\tau)}{1 - \alpha} \frac{\alpha(1 - \alpha^\delta)}{1 - \alpha} \text{Var}(\bar{d}_t - d_t^e) + \text{Min}\{\tau, \delta\} \text{Var}(v_t).$$

Equation (16) implies that $\Omega_{\tau\delta}$ is non-negative and increases with both τ and δ as asserted in I.3. Moreover, the first term of $\Omega_{\tau\delta}$ increases in direct proportion to $\alpha(1 - \alpha^\tau)/(1 - \alpha)$ as one moves down any column of Ω , while the second term increases until $\tau > \delta$ and then remains constant. This last fact suggests that the element τ of vector $\bar{a}\bar{\Omega}^{-1}\bar{s}$ is approximately proportional to a factor $\theta(1 - \theta^\tau)/(1 - \theta)$. The risk premium exponentially approaches some maximum level as the delivery date of the loan increases if $\theta < 1$ and it rises in direct proportion to τ if $\theta = 1$.^{12/} We will not constrain θ to equal α in order to partially accommodate the second term of equation (16). The hypothesis that $\theta = \alpha$ can then be tested when the model is estimated.

Our final step is to incorporate the effect of expected interest rates on the risk premium. A rise in any period's expected spot rate will increase every element of \bar{q}^s . Since an increase in the current spot rate increases expectations for every future period, we approximate the inner product of a representative row of Ω with \bar{q}^s by a linear function of r_t . Changes in r_t will, therefore, serve as a proxy in the determination of \bar{q}^s for changes in the entire vector of forward predictions. Given this approximation, the risk premium component at time t of interest rates for loans τ periods forward becomes

$$(17) \quad - (\bar{a}\Omega\bar{q}^s)_\tau = (\beta_0 + \beta_1 r_t)\theta(1 - \theta^\tau)/(1 - \theta).$$

3. Empirical Specification

Maintaining the hypothesis that expectations are formed according to the adaptive rule of equation (13), the theoretical model's predicted interest rate at time t for loans to be delivered at time $t+\tau$ becomes

$$(18) \quad r_{t+\tau}^f = r_t^e + \lambda_1(r_t - r_t^e) + \frac{1 - \alpha^\tau}{1 - \alpha} d_{t+1}^e + (\beta_0 + \beta_1 r_t)\theta\left(\frac{1 - \theta^\tau}{1 - \theta}\right),$$

where $d_{t+1}^e = \alpha[d_t^e + \lambda_2(r_t - r_t^e)]$. The observable variables are the current spot rates r_t and the current forward rates $r_{t+\tau}^f$. The unknown parameters to be estimated are β_0 , β_1 and θ for the risk premium and λ_1 , λ_2 and α (plus initial values of r_0^e and d_0^e) for the expectations hypothesis. The parameter ranges consistent with all aspects of the theoretical model are $1 \geq \lambda_1 \geq \lambda_2 \geq 0$, $\beta_1 \leq 0$, $1 \geq \alpha \geq 0$, and $\theta = \alpha$. The expectations hypothesis simplifies to adaptive expectations on interest

rates if λ_2 and d_0^e equal zero and to simple adaptive expectations on interest rate changes if λ_1 and α equal unity. The parameters of specification (18) are estimated and the various parameter restrictions are tested in Section III.

An alternative to using the adaptive rule to generate estimates of market expectations is to use observed spot rates as proxies for previously held expectations.^{13/} Such a procedure can be justified by assuming that rational agents efficiently use all current information and hence have zero mean forecast errors. The adaptive rule also had zero mean forecast error but was based on the efficient use of only the observed spot rate information. The desirability of using $r_{t+\tau}$ as an estimate of true market expectations $R_{t+\tau}^e$ hinges on whether the error due to omitting information not embodied in spot rates $(r_{t+\tau}^e - R_{t+\tau}^e)^2$ exceeds on average the forecast error when using all available information $(r_{t+\tau} - R_{t+\tau}^e)^2$. The additional information used must reduce the forecast error variance by one-half over the adaptive prediction for this to be the case.^{14/} The use of realized spot rates as proxies for prior expectations in equation (9) gives the alternative specification of the forward rate

$$(19) \quad r_{t+\tau}^f = r_{t+\tau} + (\beta_0 + \beta_1 r_t) \theta \left(\frac{1 - \theta^\tau}{1 - \theta} \right).$$

The power of equation (19) as a predictor of the actual futures rates is compared with that of equation (18) in the following section.

III. DATA AND ESTIMATION RESULTS

In this section we first examine the main characteristics of the Treasury bill futures data which need to be explained by the model. In particular, we note that the futures rate has always been at a premium with respect to the spot rate but that this premium has varied over time. We next consider the method employed to fit the data and finally analyze the empirical results. These results indicate that the model does an excellent job of reproducing the futures data if we allow for a shift in expectations at the time the new administration announced its economic policy. The time varying futures premium is accurately reproduced by variations in the interest rate drift and by that portion of the risk premium which depends on interest rates.

1. The Futures Data

Starting January 6, 1976, the International Money Market Division of the Chicago Mercantile Exchange has operated a market in 13 week Treasury bill futures contracts. Each contract is for a T-bill which has a maturity value of one million dollars and which is to be delivered in either March, June, September, or December. For any specific contract, trading terminates on the second business day following the T-bill auction for the week commencing on the third Monday of the contract month. Since the T-bill auction is usually on Monday, trading usually terminates on Wednesday and the contract is delivered on Thursday of the third week. For example, the December 1976 futures contract was delivered on Thursday, December 23, 1976. Each December contract required delivery of a one million dollar T-bill which matured 91 days later on February 24, 1977.

Since 13 week Treasury bills are issued once a week, we will take a week as the unit of time in equation (11). To maintain compatibility with the futures contracts, we assume that r_t is the market rate on Wednesday for T-bills which mature 91 days from the Thursday settlement date for that weeks T-bill auction. The yield on futures contracts is also taken from the Wednesday market data.^{15/} The futures yields at time t will be denoted as $\hat{r}_{t+\tau_j}$ where τ_j is the number of weeks from t to the delivery date of the j th contract and j is an index for the number of futures contracts traded at time t . The number of weeks to the first futures contract, τ_1 , can vary from zero to 13 weeks while the number of weeks to subsequent contracts is always τ_1 plus a multiple of thirteen.

The weekly Treasury bill spot rate and the futures yields are illustrated in Figure 1. The daily volume of trading in T-bill futures was relatively light during the first two months of trading. Beginning in March of 1976, however, there was a substantial increase in trading volume for all contracts so we have taken March as the start of the data interval. The data illustrate that the futures contracts have always sold at a premium compared to the current spot rate. Figure 2 depicts the futures premium for several selected days (ignore the solid lines for the moment). This data covers the range of the futures premium for the first two years data and illustrates the substantial variation that has occurred in the futures premium. If our model is to accurately fit the data, the time varying initial slope of the futures premium must be explained by variations either in d_t^e or in that portion of the risk premium which depends on interest rates. In addition, the curvature observed in the futures premium needs to be reproduced by α and θ .

The term structure for Treasury bills also implies a series of forward interest rates $r_{t+\tau_j}^*$. We are treating the futures rates as a different data set because of significant differences between the two rates. Figure 3 illustrates this difference by depicting $\hat{r}_{t+\tau_j} - r_{t+\tau_j}^*$ as a function of τ_j for several selected contracts.^{16/} Significant arbitrage profits appear to have gone unexploited during the trading period of most contracts.^{17/} The contracts introduced near the end of the data interval do not exhibit this phenomena to the same degree. Use of the futures data also offers two advantages over the use of the forward rates. First, the forward rates are very sensitive to errors in the spot rate for large τ . Second, the futures market provides rates further into the future because the maturity of existing T-bill is limited to one year.

2. Estimation Technique

Estimating the model requires solving the system of difference equations

$$(20) \quad \begin{aligned} r_{t+1}^e &= r_t^e + \lambda_1(r_t - r_t^e) + d_{t+1}^e \\ d_{t+1}^e &= \alpha[d_t^e + \lambda_2(r_t - r_t^e)] \end{aligned}$$

for r^e and d^e as a function of the time history of spot rates. To facilitate the solution we define the column vectors

$$X_t = \begin{pmatrix} r^e \\ d^e \end{pmatrix}_{t+1} \quad \text{and} \quad Z_t = \begin{pmatrix} \lambda_1 + \alpha\lambda_2 \\ \alpha\lambda_2 \end{pmatrix} r_t$$

and write the model in matrix form

$$(21) \quad X_t = [A]X_{t-1} + Z_t,$$

where [A] is the matrix

$$[A] = \begin{bmatrix} 1-\lambda_1 - \alpha\lambda_2 & \alpha \\ -\alpha\lambda_2 & \alpha \end{bmatrix}.$$

The solution of equation (21) is given by

$$(22) \quad X_t = [A]^t X_0 + \sum_{j=0}^t [A]^{(t-j)} Z_j,$$

where X_0 is the vector of initial conditions (r^e and d^e before observing the first interest rate). The model is stable if the characteristic roots of A lie within the unit circle,^{18/} in which case the initial condition effect damps to zero with time, and the time history of r_t governs the behavior of X_t .

For a given time series of T-bill rates r_t , set of initial conditions r_0^e and d_0^e , and model parameters λ_1 , λ_2 and α , equation (22) is solved to obtain the theoretical forecast variables as a function of time. At each point in time theoretical values for the yields on all futures contracts traded at that time are computed from equation (18). The performance of the model is then evaluated from the data residuals $\hat{\Delta}r_{tj} = \hat{r}_{t+\tau_j} - r_{t+\tau_j}^f$. Specifically we seek the initial conditions and model parameters which minimize the sum square of residuals $SSR = \sum_{tj} \hat{\Delta}r_{tj}^2$.^{19/} The model is non-linear in the initial conditions and parameters so the minimization of SSR requires an iterative algorithm. We have used the technique of Marquardt (1963) which involves an interpolation between the Taylor series and gradient methods.^{20/}

3. Estimation Results

The first row of Table 1 presents estimation results for the entire data set. The model fits the data reasonably well as indicated by the value of R^2 . However, an examination of the residuals reveals that the model is not capable of reproducing either the large jump in the futures premium which occurs for the week of January 15, 1977, or the high premium which remains after that date. This sudden change in the futures premium was possibly caused by the announcement of the new administration's economic policy. In the context of our model this shift in the futures premium could be treated as a structural shift either in expectations formation or in the risk premium.

The data suggest that the futures premium shift can be attributed to a change in expectations formation. This is clearly indicated by the model estimates for each half of the data set presented in rows two and three of Table 1. For the first year's data the value of d_0^e represents an initial expected drift in the rate of interest of 2.4 percentage points/year. This drift and the value of α imply that individuals expected interest rates to rise by 1.95 percentage points over the next year. The expected drift declines with r_t and by January of 1977 the model suggests that individuals expected interest rates to fall by 0.6 percentage points over the next year. The positive risk premium, however, insured that the futures rate was always at a premium with respect to the spot rate. The estimation results in row four of Table 1 test whether the risk premium differed significantly from zero over the first year. The lower value of λ_2 and the higher initial drift partially compensate for the zero risk premium by maintaining a positive expected drift for the

entire data interval. The appropriate F-test at the 99% level of significance cannot reject the hypothesis of a zero risk premium.^{21/} Alternatively, if d_t^e is constrained to equal zero (d_0^e and λ_2 equal zero), SSR increases by a factor of five. The risk premium alone, therefore, is not capable of reproducing the futures premium for the first year's data.

The estimation results in row three of Table 1 yield approximately the same risk premium for the last years data as was obtained for the initial data set. There does not appear to have been any structural shift in the risk premium. There does, however, appear to have been a shift in expectations. The value of d_0^e represents a significant shift in the expected drift when compared with the negative drift obtained at the end of the initial data set. In addition there has been a significant drop in the size of λ_2 as if individuals believed that the variance of the permanent shocks to the drift had declined relative to the transitory shocks.^{22/} The estimates in row five indicate that the drift is not significantly different from zero. The effect of a positive drift which is revised very slowly can be approximated by a shift in the risk premium. Setting the risk premium to zero increases SSR by a factor of two so that the expectations algorithm by itself is unable to explain the futures premium.

Estimates presented in the last two rows of Table 1 compare the adaptive expectations algorithm with the hypothesis that realized spot rates serve as a proxy for prior expectations. The realized spot rates are only available for the first four contracts traded in 1976 so we have restricted our attention to this data set. The adaptive expectations model

specified by equation (18) provides an excellent fit of the data. The model parameters are not significantly different from those obtained from estimates of the data for the first six contracts. The estimates presented in the last row indicate that the drift model is vastly superior to the specification of equation (19) in which $r_{t+\tau}$ serves as a proxy for $r_{t+\tau}^e$. Restricting expectations to actual realized values increases SSR by a factor of fifteen and gives the wrong sign for the risk premium coefficients. Alternatively, the expectations model with zero drift (the expected future spot rate is equal to the current spot rate) yields an SSR of 26.49 and also gives the wrong sign for the risk premium.

The ability of the model to explain the entire data set with either a shift in expectations or a shift in the risk premium is examined in Table 2. The parameter Δd^e is a shift in the expected drift for the week of January 15, 1977. The parameter λ_2' is the drift adaptation coefficient and β' is a scale factor which multiplies the risk premium after that date. The remaining parameters are constrained to be equal across both data intervals.

The first row of Table 2 considers a scale shift in the risk premium. For the first years data the risk premium is quite small and the model behaves very much like the model described by estimates in row four of Table 1. After January of 1977 the risk premium is $0.11-0.017 r_t$ and is similar to that obtained in Table 1 from the unconstrained estimates for each data set. These unconstrained estimates of Table 1, give a combined SSR of 17.85 compared to 21.53 when all parameters except for a risk scale shift are constrained to be equal across both data sets. We

obtain a test statistic of 5.42 distributed as $F(7,184)$ and reject the null hypothesis at the 99% level.^{23/}

The second row of Table 2 considers a shift in expectations formation. The parameters are very similar to those of the unconstrained fits and the appropriate F-test has a value of 1.61 distributed as $F(6,184)$. Since the 95% significance level is approximately 2.15, these results confirm the evidence of Table 1 that the observed shift in the futures premium can be reproduced by a structural change in expectations formation.^{24/}

Figure 2 illustrates the theoretical futures premium for several selected days and gives an indication of the model's ability to reproduce the time varying futures premium observed in the data. The size of the risk premium and its variation with interest rates is depicted in Figure 4.

The estimation results in the last row provide a standard of comparison against which our model can be judged. The two novel features of our approach are the expected drift in the rate of interest and the interest rate dependence of the risk premium. We see that these innovations greatly enhance our ability to explain the actual futures premium. Setting both d_t^e and β_1 equal to zero increases SSR by a factor of five.

Table 3 contains the results of additional tests of the model specification. Various constraints discussed in the theoretical sections above were imposed on the final model given in row two of Table 2. The first test examines the constraint $\theta = \alpha$ implied by the theoretical model of the risk premium. The constraint produces a negligible increase in SSR and there is no evidence that these two parameters are different. The second test examines the constraint $\lambda_1 = 1.0$ so that the expectations model

is equivalent to adaptive expectations on the rate of change of interest rates. We reject this hypothesis at the 95% level but not at the 99% level of significance. The remaining constraints are clearly rejected at the 99% level. These constraints involve zero interest rate dependence for the risk premium, zero decay in the forecast drift in interest rates, and zero drift in the expected rate of interest.

IV. CONCLUSIONS

Our apparently strong empirical results should be interpreted with some caution for several reasons. First, the model necessarily involves joint hypotheses about how expectations are formed and about how expectations and other factors determine interest rate futures prices. Testing one aspect of the model is conditioned on the validity of the other. Second, although we have an abundance of observations the variety of economic experience over the sample period was limited. A test over a complete business cycle would be desirable. Finally, as this relatively new market matured the type of agent participating in it may have changed. One might interpret the decline in the trend adaptation coefficient and in the discrepancy between futures yields and term structure forward rates as evidence of such a phenomenon.

Granted these qualifications, however, the data strongly support two hypotheses about interest rate futures prices. First, the prices behave as if agents form their expectations by adapting to trends in current spot rates. This means that unanticipated changes in short-term interest rates tend to be extrapolated rather than viewed as transitory aberrations. Second, the futures yields embody a positive risk premium which falls as interest rates rise. That this latter effect has not been noted by investigators of the term structure must be attributed either to differences in expectations hypotheses used (lack of expected drift) or to structural changes over time.

We were not concerned directly with the accuracy of market predictions of interest rates. The good predictive performance of the model using

adaptively generated expectations compared to using *ex post* correct expectations suggests that any additional information actually used in forming beliefs did not substantially reduce the average agent's forecast error. This suggestion has implications for empirical work in other areas in that rendering rational expectations operational by assuming perfect foresight may introduce far greater measurement errors than would occur if the information available were assumed limited.

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FOOTNOTES

¹See, for example, Roll (1970), Nelson (1972) and McCulloch (1975).

²Augment the given interest rates r_0, \dots, r_{t-1} by $r_T = 0$ for the purpose of (3).

³The difficulties which appear without these restrictive assumptions are developed by Stiglitz (1970). Our planning horizon will be pushed forward with time, of course, and future decisions will be made on the basis of then current information.

⁴Since the planning horizon T for our purposes will be the time to delivery of the most distant futures contract (less than two years) the interest earned on the speculative gains would be small.

⁵The expectations component is analogous to an expression for forward rates derived by Bierwag and Grove (1967).

⁶For this to occur when some elements of \bar{q}^s are positive and some are negative may require that the rows of Ω be proportional to each other. The expectations hypothesis developed in Section II suggests that this requirement may be approximately satisfied.

⁷Kessel (1965) argued that the risk premium should rise with interest rates rather than fall. Nelson (1972) presents an argument that they should fall based on the notion that interest rate risk is skewed (interest rates cannot fall below zero) and that the third derivative of utility functions are generally positive. Our current argument based on intertemporal substitution induced by interest rate changes provides, we feel, a stronger case for the negative effect.

⁸That is, we suppose that such behavior is "economically rational" in the sense of Feige and Pearce (1976).

⁹The process (10) is an ARIMA(0,2,2) process for $\alpha = 1$, ARIMA(1,1,2) for $0 < \alpha < 1$, and ARIMA(0,1,1) for $\alpha = 0$. With $\sigma_w^2 = \bar{d} = 0$ it is the process used by Muth (1960) to discuss the optimal properties of adaptive learning. With $\sigma_v^2 = 0$ and $\alpha = 1$ it is the process explored by Nerlove and Wage (1964).

¹⁰See Jacobs and Jones (1977).

¹¹If u_t is small then the error in estimating current underlying interest rates ($\bar{r}_t - r_t^{e'}$) will also be small. In this instance, \bar{r}_t will be close to observed r_t , the optimal value of λ_1 will be close to unity and the resulting $r_t^{e'}$ will also be close to r_t .

¹²This form of the risk premium was also adopted by McCulloch (1975).

¹³This procedure, or a variation on it, is used by Kessel (1965), Roll (1970) and McCulloch (1975) to test for the presence of risk premia in the term structure.

¹⁴Let $\epsilon = r_{t+\tau} - r_{t+\tau}^e$ be the agent's forecast error using only past spot rate information (see eqn. (15)) and $\epsilon^* = r_{t+\tau} - R_{t+\tau}^e$ be the forecast error when using an unspecified larger information set. $R_{t+\tau}^e - r_{t+\tau}^e = \epsilon - \epsilon^*$ indicates the effect on expectations of knowing the additional information and must be uncorrelated with ϵ^* if $R_{t+\tau}^e$ is based on efficient use of the information available (i.e.: $\text{Cov}(\epsilon - \epsilon^*, \epsilon^*) = \text{Cov}(\epsilon, \epsilon^*) - \text{Var}(\epsilon^*) = 0$). The variance of the measurement error resulting from use of $r_{t+\tau}$ for $R_{t+\tau}^e$ is $\text{Var}(\epsilon^*)$; the variance resulting from use of $r_{t+\tau}^e$ for $R_{t+\tau}^e$ is $\text{Var}(\epsilon - \epsilon^*) = \text{Var}(\epsilon) - 2\text{Cov}(\epsilon, \epsilon^*) + \text{Var}(\epsilon^*) = \text{Var}(\epsilon) - \text{Var}(\epsilon^*)$. Hence using $r_{t+\tau}$

results in a larger error in measuring $R_{t+\tau}^e$ than using $r_{t+\tau}^e$ if $\text{Var}(\epsilon^*) > \frac{1}{2}\text{Var}(\epsilon)$.

¹⁵All Wednesday market data are taken from the Thursday Wall Street Journal. For holidays we have interpolated to obtain data for a fictional Wednesday market day. The yields are reported as a banker's discount so we have maintained the bankers discount on a percent/annum basis as our definition of the rate of interest. The T-bill rate is the average of the bid and ask rates while the futures rates were determined from the market opening prices. If there was no opening price reported for any contract because of low trading volume, that contract was excluded from the weeks data. The number of futures contracts traded at any time has recently been increased to eight. We have restricted our attention to the first six contracts in order to maintain a uniform data set.

¹⁶The forward rate on a 91 day T-bill which matures at time $t+\tau$ was computed in the following manner. Let r_n^b denote the banker's discount (average of bid-ask) on a T-bill which matures n days from time t . Then r_τ^b and $r_{\tau+91}^b$ are the banker's discounts on T-bills which span the futures contract. Convert each banker's discount to a bill price $P_n = 100 - nr_n^b/360$ and then to a continuously compounded rate of return

$$r_n = \frac{36500}{n} \ln\left(\frac{100}{P_n}\right).$$

The continuously compounded implied forward rate is then

$$r_{t+\tau}' = \frac{(\tau + 91)r_{\tau+91} - \tau r_\tau}{91}.$$

Converting back to a T-bill price gives

$$P'_{t+\tau} = 100e^{-\frac{91r'_{t+\tau}}{36500}}$$

which yields the following implied forward rate in banker's discount form

$$r^*_{t+\tau} = \frac{360}{91}(100 - P'_{t+\tau})$$

¹⁷Capozza and Cornell also note this spread between the forward and future rates. They find that the difference is usually within the bounds of transaction costs for individuals who would need to borrow T-bills to arbitrage between the two markets. The differences, however, appear to offer significant profit opportunities to individuals who already hold a portfolio of T-bills.

¹⁸Stability of the model is insured by the theoretical restriction that $1 \geq \lambda_1 \geq \lambda_2 \geq 0$. Given that α is close to unity, the characteristic roots will have imaginary components if $\lambda_2 > 2 - \lambda_1 - 2\sqrt{1 - \lambda_1}$. In this case the initial condition effect will be a damped oscillation.

¹⁹We could substitute the difference equation solution represented by equation (22) into equation (18) for the theoretical futures rate and symbolically represent the resulting expression as a function of the model parameters and past interest rates

$$\hat{r}_{t+\tau} = G(r_t, r_{t-1}, \dots, r_0^e, d_0^e, \lambda_1, \lambda_2, \alpha, \beta, \beta_1, \theta, \tau) + U_{t+\tau},$$

where G denotes the theoretical value $r_{t+\tau}^f$ and $U_{t+\tau}$ represents an error term. Although we have suppressed the stochastic structure of the model in the theoretical section, the error term could arise because the

expectations algorithm had a stochastic component, the risk premium was stochastic, or the actual futures rates were corrupted by observation errors due, for instance, to different reporting times for the published spot and futures data. The properties of our estimator will depend, of course, on the properties of $U_{t+\tau}$. If $U_{t+\tau}$ is normal and independent over t and τ then minimizing SSR will provide maximum likelihood estimates of the model parameters. If $U_{t+\tau}$ is correlated over t and τ then the estimates will still be consistent because of the assumption that the spot rate r_t is exogenous to the process of price formation in the futures market. Partial derivatives required for the estimation were computed numerically as

$$\frac{\partial G(r_t, \dots, \Lambda)}{\partial \Lambda} = \frac{G(r_t, \dots, \Lambda + \Delta\Lambda) - G(r_t, \dots, \Lambda)}{\Delta\Lambda}$$

where Λ denotes any fitting parameter.

²⁰Convergence was quite rapid and usually required less than seven iterations. For the initial estimates we verified that the same minimum was reached after starting with markedly different initial conditions.

²¹An examination of the regression results reveals that the residuals for the six contracts traded at any time are highly correlated. This is illustrated in Figure 2 which contains the theoretical futures premium from the best fit model of Table 2 and the actual futures premium for several selected days. An error in the drift (risk premium) or in α (θ) would lead to consistently high or low prediction errors. While this residual correlation will not produce inconsistent parameter estimates it will create problems with F-tests for various model constraints because

the number of data points is not an indicator of the number of degrees of freedom. Figure 2 makes it clear that the contracts traded at any time convey two pieces of information -- the initial slope of the futures premium and its decay over time. As a result, for the F-tests we will assume that each time point contains two degrees of freedom.

²²If the variance of the permanent shocks to the drift goes to zero, all observed changes in the drift would be attributed to transitory shocks. This would represent a case where individuals were very certain of the long run drift in interest rates. Optimal forecasting would then involve no adjustment of the expected drift ($\lambda_2 = 0$).

²³Two additional specifications of the risk premium were also estimated. The first specification allowed separate risk premiums for the two data intervals while the second allowed the shift coefficient β' to exponentially decay back to unity at a rate estimated from the data. Both specifications produced only a slight change in the model parameters and in the value of SSR.

²⁴We also allowed λ_2' to exponentially converge back to λ_2 at a rate which was estimated from the data. The estimated rate of convergence was close to zero and there was only a slight change in SSR and the model parameters.

Table 1

DATA INTERVAL	r_o^e	d_o^e	λ_1	λ_2	α	β_o	β_1	θ	R^2	SSR
3/76 to 1/78	4.917	0.053	1.201	0.046	0.983	0.158	-0.025	0.996	0.891	45.67
3/76 to 1/77	4.921	0.046	1.187	0.047	0.992	0.116	-0.018	0.994	0.968	7.74
1/77 to 1/78	4.219	0.023	1.164	0.006	0.949	0.107	-0.014	0.993	0.941	10.11
3/76 to 1/77	4.888	0.076	1.344	0.017	0.991	0.0	0.0	0.0	0.963	8.71
1/77 to 1/78	4.340	0.0	1.226	0.0	0.0	0.067	-0.008	0.990	0.937	10.77
3/76 to 1/77 (4 contracts)	4.972	0.045	1.199	0.050	0.997	0.132	-0.021	0.993	0.982	4.22
3/76 to 1/77 $r_{t-t}^e = r_{t-t}$	0.0	0.0	0.0	0.0	0.0	-0.681	0.154	0.961	0.718	66.21

Notes: The F-test for a zero risk premium over the initial data interval gives a test statistic of 3.42 which is distributed as F(3,82). The F-test for zero drift over the second data interval yields a test statistic of 2.05 which is distributed as F(3,94). The 95% and 99% significance levels for both tests are approximately 2.73 and 4.08 respectively. See Footnote 16 for a discussion of the degrees of freedom used for these tests.

Table 2

r_0^e	d_0^e	λ_1	λ_2	α	β_0	β_1	θ	λ_2^e	Δd^e	β^e	R^2	SSR
4.878	0.074	1.235	0.017	0.990	0.021	-0.003	0.991	$= \lambda_2$	0.0	6.015	0.949	21.45
4.906	0.049	1.199	0.044	0.990	0.102	-0.015	0.992	0.003	0.026	1.0	0.955	18.79
5.573	0.0	1.115	0.0	0.0	0.047	0.0	0.987	0.0	0.0	1.0	0.721	117.11

Table 3

Constraint	SSR	F
$\theta = \alpha$	18.82	0.29
$\lambda_1 = 1$	19.41	6.01
$\beta_1 = 0$	21.37	24.99
$\alpha = 1$	20.94	20.82
Zero Drift	97.67	254.68

Note: The F-test statistics are distributed as $F(1,182)$ except for the last test which involves three constraints. The 95% and 99% significance levels for these tests are approximately 3.84 and 6.63 respectively.

Figure 2

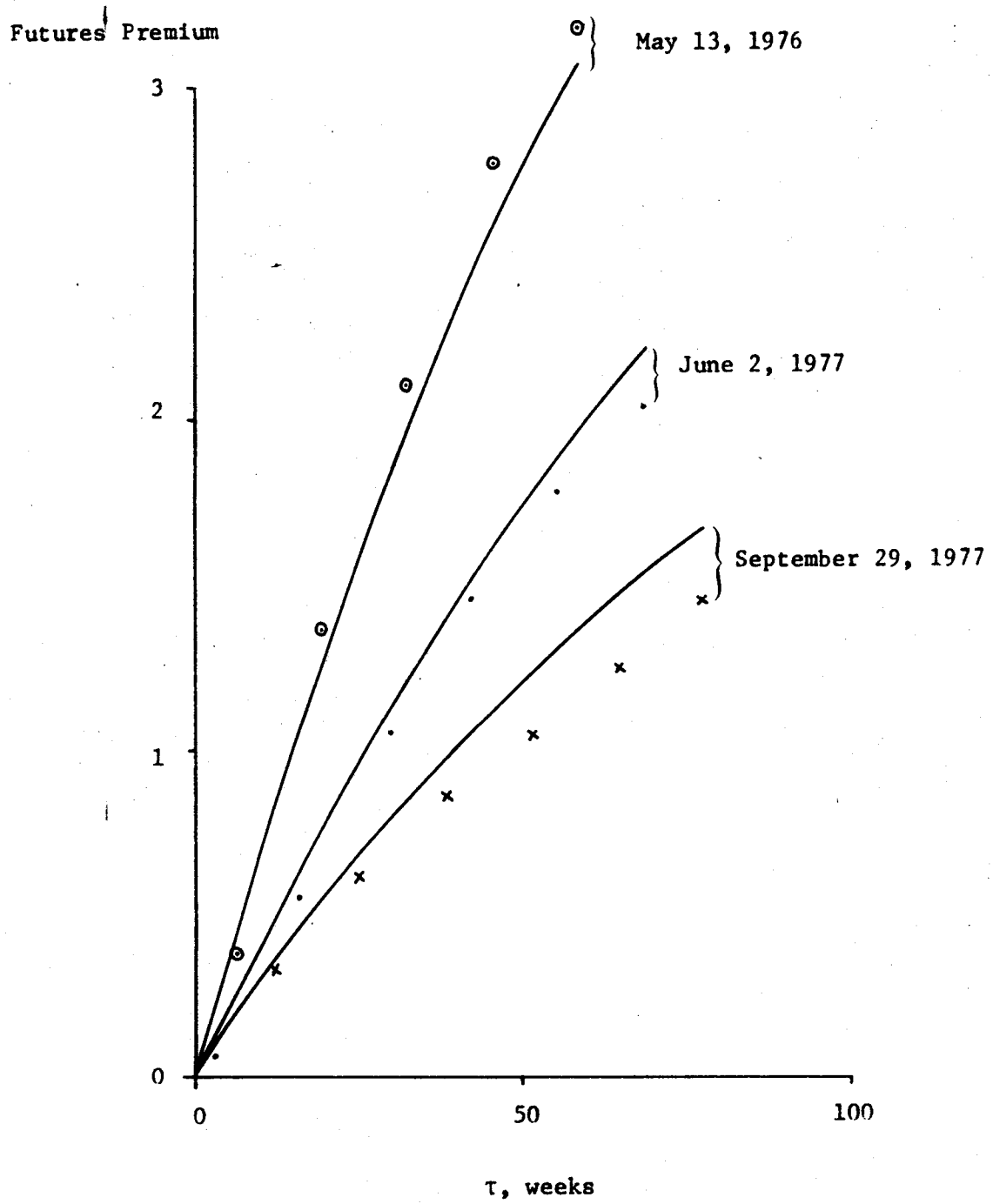


Figure 3

Comparison of Futures and Forward Rates

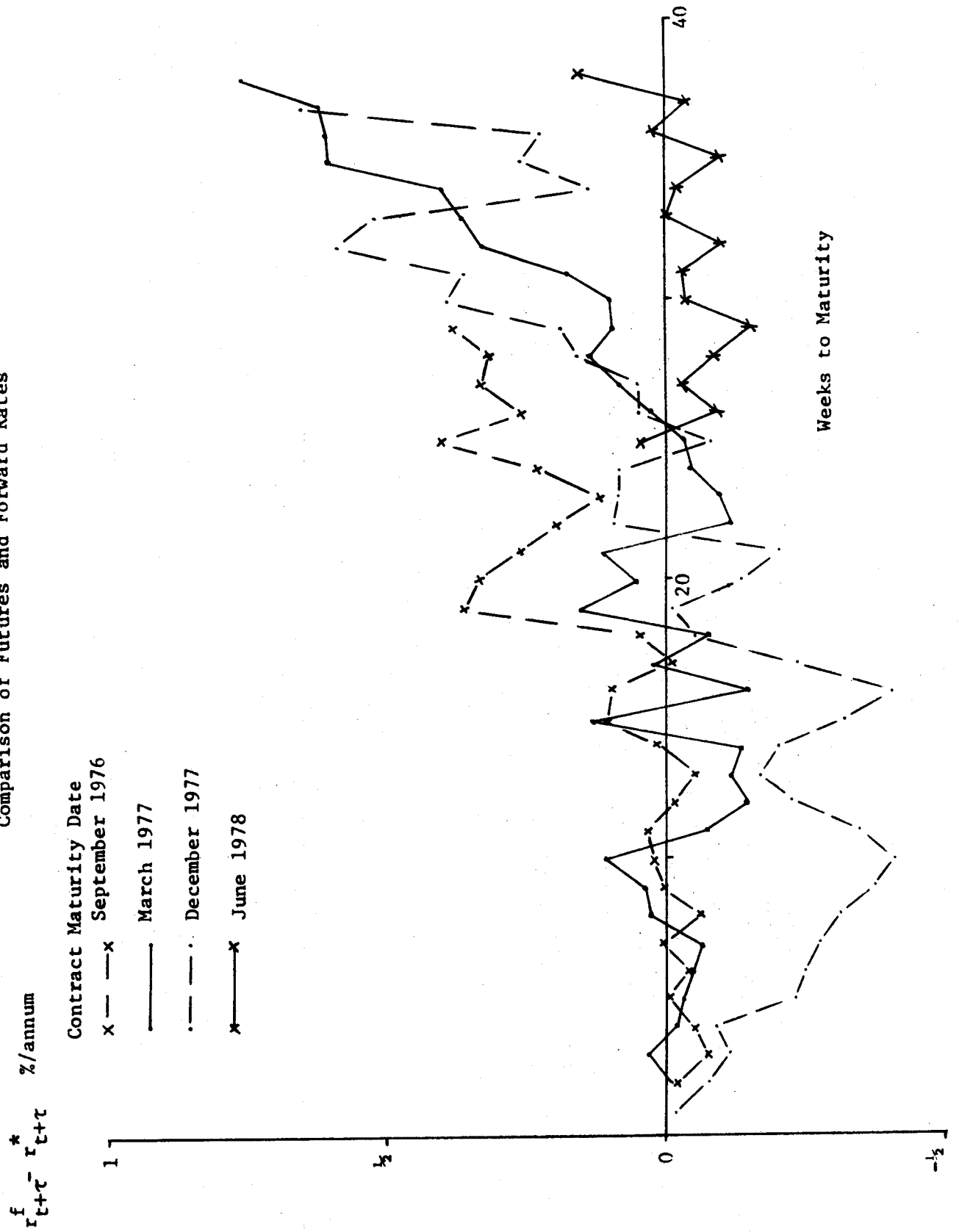


Figure 4

$$\text{Risk Premium } (\beta_0 + \beta_1 r_t) \theta \left(\frac{1 - \theta^\tau}{1 - \theta} \right)$$

Risk Premium

