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The Use of Realized Values as a Proxy for Prior Rational Expectations

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I. Introduction

Virtually all economic processes are governed by agents' expectations about the future value of certain key variables. In theoretical studies it is common to assume that these expectations are formed rationally on the basis of all available information and are the expected value of the underlying process which generates these variables. The assumption of fully rational expectations is, however, difficult to implement empirically. As a result, most empirical work employs some proxy for the rational expectations of economic agents. Usually this proxy for rational expectations consists of a model of expectations formation based on some limited set of information such as past values of the time series variable being forecast. In this framework, expectations models may range from simple adaptive expectations to Box-Jenkins forecasting rules which efficiently utilize all information contained in the past history of the series. Use of these limited information models of expectations formation may, however, introduce substantial specification error into the economic model employed.

An alternative approach has been to forsake modeling expectations formation and to use the ultimately realized value of a series as a proxy for prior expectations.[1] The rational for this approach is that agents make unbiased, minimum variance forecasts so that realized values are merely the prior expected value plus a random forecasting error. In this case the specification error involved in modeling rational expectations is merely the forecasting error of rational agents. This approach to implementing rational expectations will be superior to a limited information model of expectations formation if the forecasting error is smaller than the specification error of the model of expectations.

It is the purpose of this paper to demonstrate that this condition is not likely to occur for most economic time series. In section II we consider univariate time series models of the Box-Jenkins variety. For these models, of course, efficient use of the prior history of the series always dominates the use of realized values as a proxy for rational expectations. In addition, simple forecasting rules such as adaptive expectations usually dominate use of the realized value. In section III we consider the use of additional information in the formation of rational expectations. We demonstrate that additional information must reduce the forecast error variance by 1/2 before the realized value dominates a model of expectations based on past values of the series as a proxy for rational expectations. In conclusion, it appears that conditions under which realized values should be used as a proxy for rational expectations are not likely to be found in practice.

II. Univariate Time Series Models

In this section we consider a univariate time series model for a variable Z_t being forecast by economic agents. Given the time series representation, we will investigate the use of different proxies for the expected value of Z_{t+1} held by rational economic agents at time t. Denoting this expectation as Z^e_{t+1}, we will examine the use of Z_{t+1}, Z_t and simple adaptive expectations as proxies for Z^e_{t+1}. Specific results will be generated for time series models which are first-difference stationary.

We assume that Z_t can be represented by the ARIMA (p,d,q) process

$$(1) \qquad \Phi(B)\nabla^d Z_t = \Theta(B)a_t,$$

where a_t is white noise, $\Theta(B)$ is a polynomial $1 - \Theta_1 B - \Theta_2 B^2 - \cdots - \Theta_q B^q$ in the lag operator B, ∇^d is the difference operator $(1 - B)^d$, and $\Phi(B) = 1 - \Phi_1 B - \Phi_2 B^2 - \cdots - \Phi_p B^p$. The current value Z_t can be written either in terms of the current and all prior random shocks a_{t-j} (random shock form) or in terms of a weighted average of all prior levels Z_{t-j} and the current shock (inverted form). The random shock form of the model is

$$(2) \qquad Z_t = a_t + \sum_{j=1}^{\infty} \Psi_j \, a_{t-j} = \Psi(B)a_t,$$

where the Ψ weights can be obtained by equating coefficients of B in $\Phi(B)\nabla^d\Psi(B) = \Theta(B)$. The inverted form of the model can be written as

$$(3) \qquad Z_t - \sum_{j=1}^{\infty} \Pi_j \, Z_{t-j} = \Pi(B)Z_t = a_t,$$

where the Π weights are obtained from $\Phi(B)\nabla^d = \Theta(B)\Pi(B)$.

Given this representation of Z_t, the minimum mean square error forecasting rule is

$$Z^e_{t+1} = \Psi_1 a_t + \Psi_2 a_{t-1} + \Psi_3 a_{t-2} + \cdots \tag{4}$$

or in inverted form

$$Z^e_{t+1} = \Pi_1 Z_t + \Pi_2 Z_{t-1} + \Pi_3 Z_{t-2} + \cdots \tag{5}$$

In the univariate model, equation (4) or (5) would represent rational expectations formation so that Z^e_{t+1} serves as a standard of comparison for the various expectations proxies.

If Z_{t+1} is assumed to be a proxy for Z^e_{t+1} the error e_1 in modeling expectations which results is just the forecasting error of equation (4). That is, $e_1 = Z_{t+1} - Z^e_{t+1} = a_{t+1}$ so that Var $(e_1) = \sigma^2_a$. If Z_t is used as a proxy for Z^e_{t+1}, the modeling error becomes $e_2 = Z_t - Z^e_{t+1}$ so that

$$\text{Var }(e_2) = \sigma^2_a \sum_{j=o}^{\infty} (\Psi_j - \Psi_{j+1})^2. \tag{6}$$

The use of Z_t as a proxy for Z^e_{t+1} will dominate the use of Z_{t+1} for processes in which the summation in equation (6) is less than unity.

The final proxy for Z^e_{t+1} to be considered is the adaptive expectations model.

$$\bar{Z}^e_{t+1} = \lambda(Z_t + (1 - \lambda)Z_{t-1} + (1 - \lambda)^2 Z_{t-2} + \cdots \ ) \tag{7}$$

which differs from the rational forecasting model to the extent that the geometrically declining weights $\lambda(1 - \lambda)^j$ differ from the Π_j. By using equation (2) to eliminate the Z_{t-j}, $\bar{Z}^e_{t+1}$ can be written in random shock form

(8) $$\bar{Z}^e_{t+1} = \sum_{j=o}^{\infty} C_j a_{t-j} ,$$

where $C_o = \lambda$ and $C_j = \lambda \sum_{k=o}^{j} \Psi_k (1 - \lambda)^{j-k}$. The adaptive rule gives a modeling error $e_3 = \bar{Z}^e_{t+1} - Z^e_{t+1}$ so that

(9) $$\text{Var } (e_3) = \sigma_a^2 \sum_{j=o}^{\infty} (C_j - \Psi_{j+1})^2$$

the adaptive rule will dominate Z_{t+1} as a proxy for rational expectations if the summation in equation (9) is less than unity. The adaptive rule will always dominate the use of Z_t since Z_t is merely $\bar{Z}^e_{t+1}$ with the constraint that λ equals unity.

To proceed further it will be necessary to consider specific ARIMA processes in order to define the Ψ_j. Since first differencing is sufficient to induce stationarity in a wide range of actual time series, we will confine our attention to the ARIMA (1,1,1) model

(10) $$(1 - \Phi B)(1 - B)\ Z_t = (1 - \Theta B) a_t$$

where Φ and Θ are restricted to lie between ±1 to insure stationarity and invertability of the differenced series. With $\Phi = 0$ the process is a random walk with error for which Muth has demonstrated the optimality of adaptive expectations ($Z^e_{t+1} = \bar{Z}^e_{t+1}$ with $\lambda = 1 - \Theta$). With both Φ and Θ equal to zero the process is a random walk for which $Z^e_{t+1} = Z_t$ ($\bar{Z}^e_{t+1}$ with $\lambda = 1$) is the rational forecasting rule.

With process (10), equation (6) gives Var $(e_2) = \sigma_a^2 (\Phi - \Theta)^2 / (1 - \Phi^2)$ so that Z_t dominates Z_{t+1} as a proxy for agents rational expectations if

$(\Phi - \Theta)^2 < 1 - \Phi^2$. Values of Φ and Θ which satisfy this condition lie within the dotted lines in Figure 1. The region within the dashed lines indicates values of Φ and Θ for which adaptive expectations $\bar{Z}^e_{t+1}$ gives a smaller specification error than does Z_{t+1}.[2] We see that $\bar{Z}^e_{t+1}$ is a better proxy for Z^e_{t+1} over almost the entire range of admissable parameter values. Similar results were obtained with other first difference, stationary processes. The adaptive model is also a good approximation of the best forecasting rule for a wide range of Φ and Θ as illustrated in Figure 2. The region within the two sets of lines indicates parameter values for which the adaptive model has a specification error of only $0.2\sigma_a^2$ and $0.1\sigma_a^2$ when compared to the best forecasting rule of equation (5).

In this section we have seen that there is little justification for the use of Z_{t+1} as a proxy for agents' expectations when dealing with univariate time series. The use of Z_{t+1} always involves a specification error of σ_a^2 when compared with a rational forecast which makes efficient use of information contained in the current and prior values of Z. In addition, the use of Z_{t+1} cannot be justified on the basis of simplicity because the simple adaptive model involves smaller specification error over almost the entire range of parameter values.

III. Use of Additional Information

The use of Z_{t+1} as a proxy for expectations at time t might be justified if additional information beyond current and past Z_t were used by economic agents in forming expectations. This would be the case if expectations were formed rationally and if Z_t were jointly determined by its own history as well as another variable X. The simplest model of this type would be

$$(11) \qquad Z_t = X_{t-1} + \Pi_1 Z_{t-1} + \Pi_2 Z_{t-2} + \Pi_3 Z_{t-3} + \cdots + a_t ,$$

where we assume that X_{t-1} is random and orthogonal to past Z. For this model, the best forecast would be

$$(12) \qquad Z^e_{t+1} = X_t + \Pi_1 Z_t + \Pi_2 Z_{t-1} + \Pi_3 Z_{t-2} + \cdots$$

which has a forecast error variance of σ^2_a.

If expectations are modeled by using Z_{t+1} as a proxy for Z^e_{t+1}, the specification error is $e_1 = Z_{t+1} - Z^e_{t+1} = a_{t+1}$ so that Var $(e_1) = \sigma^2_a$. No matter what the form of the model generating rational expectations, the specification error involved in using Z_{t+1} for Z^e_{t+1} is always equal to the forecast error. If expectations are modeled by making efficient use of only information contained in the past history of Z_t we would obtain

$$(13) \qquad \bar{Z}^e_{t+1} = \bar{\Pi}_1 Z_t + \bar{\Pi}_2 Z_{t-1} + \bar{\Pi}_3 Z_{t-2} + \cdots$$

The assumption that X and Z are orthogonal means that $\bar{\Pi}_j = \Pi_j$ so that equation (13) has a forecast error of $\sigma^2_a + \sigma^2_x$.[3] If $\bar{Z}^e_{t+1}$ is used as a proxy for Z^e_{t+1} we obtain a specification error of $e_2 = \bar{Z}^e_{t+1} - Z^e_{t+1} = -X_t$ so that Var $(e_2) = \sigma^2_x$. Therefore, Z_{t+1} is a better proxy for agents' rational expectations than the best autoregressive model if $\sigma^2_x > \sigma^2_a$. If we examine the forecast error of the rational and autoregressive models, the condition that $\sigma^2_x > \sigma^2_a$ is equivalent to requiring that additional information reduce the forecast error variance by at least 1/2. Stated another way, if additional information is added to a regression of Z_t on its own past values it must reduce the residual variance by at least 1/2 to justify the use of Z_{t+1} as a proxy for rational expectations. This condition holds not only for the simple model of equation (11) but also for models in which the additional information enters in a general distributed lag. Given the lack of strong relationships among economic variables found in various tests for causality (e.g., Pierce), this condition is unlikely to be satisfied.

The argument against the use of Z_{t+1} as a proxy for agents expectations is strengthened if expectations are economically rational in the sense of Feige and Pearce. That is, agents may economize on the use of information if the value of improved prediction from sophisticated forecasting rules is outweighed by the cost of making these forecasts. Economic rationality may take the form of efficiently utilizing only the information contained in past values of the series being forecast. In this case, we have the results of section II where Z_{t+1} is always dominated by Box-Jenkins forecasting rules and usually dominated by the simple adaptive rule. An alternative view of economic rationality is that agents use a simple forecasting rule like adaptive expectations which approximates the best autoregressive rule. In this context there is also no justification for the use of Z_{t+1} as a proxy for agents' expectations.

IV Summary

In this paper we have examined the use of realized values of a series as a proxy for prior expectations of rational economic agents. We first considered a univariate time series model which was first-difference stationary. In this case, the use of Z_{t+1} as a proxy for Z^e_{t+1} is always dominated by the best autoregressive forecasting rule and is usually dominated by simple adaptive expectations. We next considered a model in which information beyond the own past values would improve the forecasts of Z_{t+1}. We saw that the additional information must reduce the forecast error variance by 1/2 before Z_{t+1} dominates the best autoregressive model as a proxy for rational expectations. Recent tests for causality indicate that this condition is unlikely to be met in practice. The argument against the use of Z_{t+1} as a proxy for prior expectations is further strengthened if agents economize on the use of information when making forecasts.

Figure 1

the ARIMA (1,1,1) Process

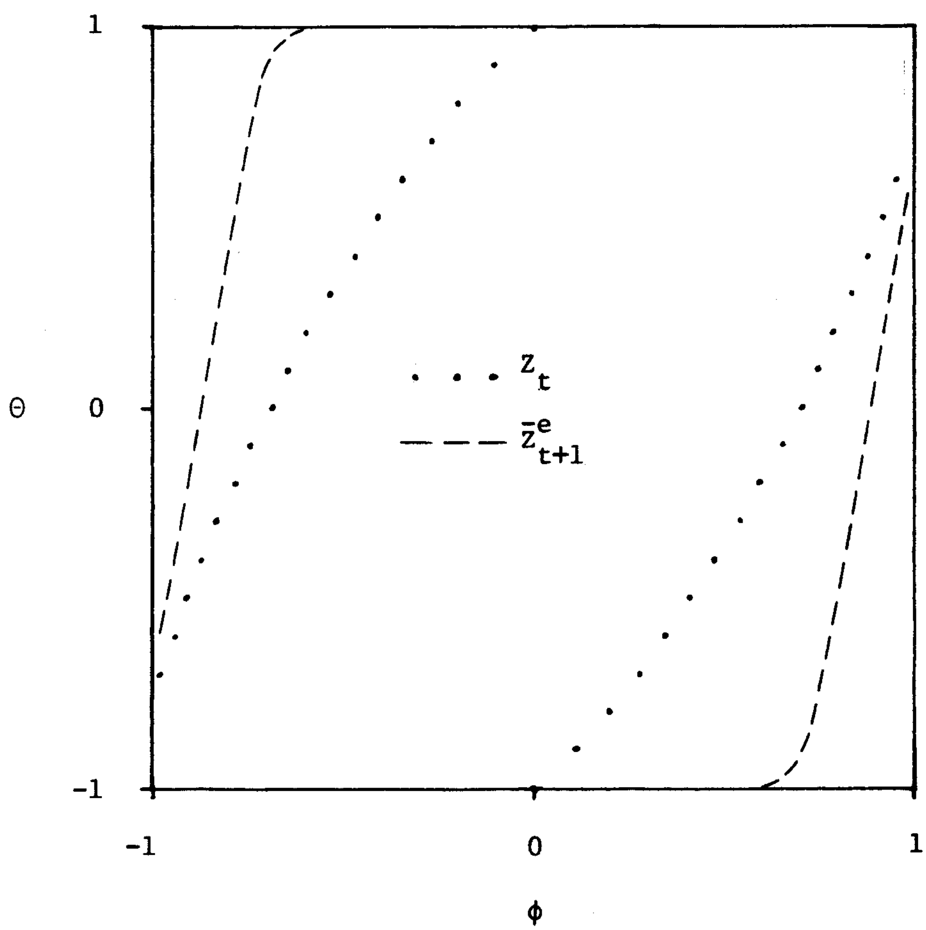
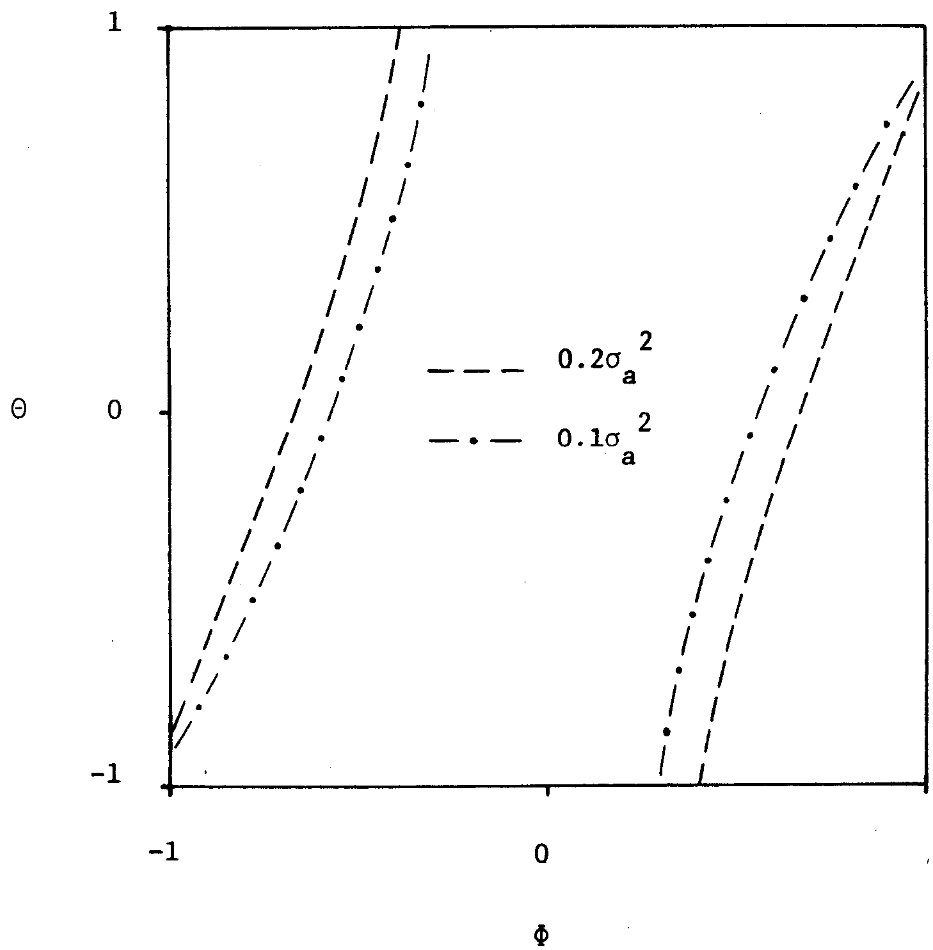


Figure 2

Specification Error for Adaptive Expectations



FOOTNOTES

[1]For example, this approach is implicit in many studies of the term structure of interest rates such as those of McCulloch and Kessel.

[2]Results for the adaptive model were obtained by searching for the value of λ which minimized equation (9).

[3]This point is discussed in greater detail by Shiller. The assumption of orthogonality between X and Z is not crucial but merely serves to simplify the exposition. If the X and Z were not orthogonal, the $\bar{\Pi}$ would adjust so that the specification error of equation (13) was the variance of that part of X which was orthogonal to Z.

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