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AS A

CHARACTERIZATION OF PERFECTLY COMPETITIVE EQUILIBRIUM

by

Joseph M. Ostroy
University of California at Los Angeles

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Joseph M. Ostroy,** UCLA

In the Walrasian definition of competitive equilibrium, attention is confined to the consistency of individuals' plans on the presumption that each agent regards itself as a price-taker. Thus, Walrasian equilibrium may exist where there is no supporting evidence for that presumption - e.g., in an economy with a small number of agents. The self-imposed limitations of the definition imply that it describes necessary but not sufficient conditions for perfectly competitive equilibrium - i.e., an equilibrium for which the presumption of price-taking is justified. In this paper we propose an alternative definition that brings out the perfectly competitive character of the equilibrium that is implicitly behind the usual interpretation.

The approach adopted here has as a basic premise that the description of perfect competition should be invariant to the number of agents. Contrary to what appears to be the typical view that it is logically tied to the study of economies with large numbers of agents, we find that an analysis of the conditions for perfectly competitive equilibrium with small numbers yields a characterization that is essentially the same as it is for large. What differs between economies with small and large numbers is simply the likelihood of its occurrence.

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Consider a group of agents and an allocation that maximizes the utility of one of them, say j, subject to requirement that the others are made no worse off than they would be by reallocating resources amongst themselves, without j. At such an allocation, j is extracting all the gains from trade he might reasonably expect by, as it were, joining the economy; and, the others may be receiving absolutely no surplus from the presence of j. Suppose an economy exhibits an allocation that simultaneously fulfills this condition for all agents taken one at a time. Such a situation is a special case of what will be called a <u>no-surplus allocation</u> that we propose as an alternative to the Walrasian definition of competitive equilibrium.

Some justification for the proposed definition are the following:

(1) Whether the number of agents is large or small, a no-surplus allocation is a Walrasian equilibrium.

(2) A no-surplus allocation is equivalent to the condition that each agent faces perfectly elastic demand schedules, at Walrasian prices, for the goods it sells.

(3) Although almost all Walrasian allocations in economies with small numbers are not no-surplus, in economies with large numbers and a finite number of commodities almost all Walrasian allocations are.

In this paper we shall demonstrate (1) - (3) for the most elementary but nevertheless representative case of finite exchange economies and for sequences of exchange economies consisting of replicas of a fixed finite economy. A novelty of the demonstration is that for large economies we shall permit the presence of external effects. More general sequences of economies exhibit similar conclusions (See Ostroy [1977]). Related results have been established by Makowski [1977] for economies with production by firms.

The concept of no-surplus is reminiscent of the marginal productivity theory of distribution. With such an allocation each agent may be said to be obtaining its marginal product and since this is true for all agents simultaneously, the total product (allocation) may be said to be entirely exhausted by the sum of the marginal products. (Compare the remarks at the end of Section II, below.) The analogy is pointed out by Makowski and it is shown in Ostroy [1978] that the marginal-productivity-product-exhaustion and no-surplus descriptions of perfect competition are essentially equivalent for economies with a continuum of traders.

In the final section, the no-surplus and core characterizations of perfectly competitive equilibrium are compared. Elsewhere, we shall exhibit the connections with the Shapley value (Shapley [1953]), and the concept of incentive-compatibility of Walrasian equilibria (Roberts and Postlewaite [1976]). Mas-Colell [1978] has shown that the no-surplus condition can be used to abstract the common elements of the Core and Value Convergence Theorems.

I. NOTATION AND PRELIMINARY DEFINITIONS

The concepts and results below are formulated for an exchange economy $\mathcal{E} = \{(S_i, w_i)\}_{i=1}^{i=n}$, where S_i describes the preferences and w_i the initial endowments of a typical agent i. It is assumed throughout that the relevant commodity space is R_+^ℓ so that $w_i \in R_+^\ell$. An allocation is denoted by $X = (x_i)$, $x_i \in R_+^\ell$; and, unless otherwise noted X will also be assumed to satisfy the feasibility condition, $\Sigma(x_i - w_i) = 0$.

For an allocation X, $S_i(X)$ is a subset of R_+^ℓ. If $y \in S_i(X)$, then i would prefer y to x_i given the allocations x_j in X for $j \neq i$. Let $\partial S_i(X)$ be the R_+^ℓ-boundary of $S_i(X)$ - the set of points in the closure of $S_i(X)$ that does not belong to its R_+^ℓ-interior. For each $i = 1,\ldots,n$ and $X = (x_i)$, assumptions on preferences will include (some of) the following:

A_1: $x_i \notin S_i(X)$ and $S_i(X)$ is open in R_+^ℓ

A_2: $x_i \in \partial S_i(X)$

A_3: $p \in R^\ell$, $p \neq 0$, $\inf p[S_i(X) - w_i] = 0$ and $y \in S_i(X)$ implies $p(y - w_i) > 0$.

A_4: $S_i(X)$ is convex

A_5: $X' = (x_i')$ and $x_j' = x_j$ implies $S_j(X') = S_j(X)$

A_6: $\{(y, S_i(X)) : y \in S_i(X)\}$ is open in $R_+^\ell \times (\underbrace{R_+^\ell \times \ldots \times R_+^\ell}_{n})$

Consistent with the interpretation of $S_i(X)$ as indicating strictly preferred elements, A_1 says that the underlying preference ordering is irreflexive and $S_i(X)$ does not contain its boundary. A_2 is a version of local nonsatiation that also implies $S_i(X)$ is not empty. A_3 rules out the possible difficulties that arise from the fact that $S_i(X)$ is merely R_+^{ℓ}-open. If $\inf p[S_i(X) - w_i] = 0$ and $pw_i \neq 0$, it is well known that A_3 is superfluous. However, since the conditions that would guarantee $pw_i \neq 0$ are not otherwise needed for the results below, we simply impose A_3. Note that A_3 obviates the need for any assumptions on w_i or Σw_i being in the interior of R_+^{ℓ}. A_4 is essential for a linear price description of perfectly competitive, Walrasian equilibrium. A_5 says that an agent's preferences are unaffected by any part of the allocation that is not assigned to the agent - i.e., there are no external effects. The continuity condition, A_6, will not be used in any of the formal results but it will be called upon, informally, to extend some conclusions.

In the following we shall be concerned with the influence of any single agent j on the rest of the economy and it will be convenient to adopt the convention in which $(\cdot)^j$ indicates the corresponding sum excluding j. Thus, $S^j(X) = \Sigma_{i \neq j} S_i(X)$, $w^j = \Sigma_{i \neq j} w_i$ and $x^j = \Sigma_{i \neq j} x_i$. Similarly, we shall use $(\cdot)^0$ to indicate that the sum excludes no agent. Thus, $S^0(X) = \Sigma S_i(X)$, $w^0 = \Sigma w_i$ and $x^0 = \Sigma x_i$.

REMARK 1: Since $\Sigma x_i = w^0$, A_2 implies that the condition $w^0 \notin S^0(X)$ is equivalent to $w^0 \in \partial S^0(X)$. With A_5, $w^0 \in \partial S^0(X)$ has the usual interpretation that X is Pareto-optimal - there is no other feasible allocation that would be preferred by all agents. However, without A_5, if $w^0 \in \partial S^0(X)$ no such conclusion is warranted. We can only maintain the rather trivial implication that if any agent is to reach a more preferred allocation without relying on trade with others, total resources must be other than w^0.

II. THE NO-SURPLUS CONDITION

For the following definitions and throughout the remainder of the paper A_1 and A_2 are assumed.

Trade is productive for all the members of $\mathcal{E}$ whenever there exists an allocation $X = (x_i)$ such that each agent can achieve a surplus through trade - i.e., $x_i \in S_i(W)$, where $W = (w_i)$ is the initial allocation. We may say that the surplus produced through trade is maximal when X is Pareto-optimal (PO).

PO: $$w^0 \in \partial S^0(X).$$

Following the logic of marginal productivity theory, can an allocation be found that uniquely imputes the total surplus produced through trade to the separate contributions of the participating agents? To formulate this condition we require that each agent receive neither more nor less than its marginal product.

For an exchange economy, we shall say that at the allocation X no agent is receiving more than it is worth if it satisfies the non-negative surplus (NNS) condition.

NNS: $$w^j \notin S^j(X), \qquad j = 1,\ldots,n.$$

NNS places an upper bound on the extent to which an agent can exploit its monopoly power by saying, in effect, that no seller j can enforce an outcome in which its customers would do better by refusing to deal with j and going elsewhere.

At an allocation X no agent is receiving less than it is worth if it satisfies the non-positive surplus (NPS) condition.

NPS: $$w^j \in \text{cl}\, S^j(X), \qquad j = 1,\ldots,n \quad (\text{cl} \equiv \text{closure}).$$

Otherwise, if $w^j \notin \text{cl}\, S^j(X)$, j is contributing a positive surplus and can therefore claim to be receiving less than it is worth to the rest of the economy. (Unless there are no external effects, A_5, the NNS and NPS conditions may not reflect their marginal productivity interpretations in finite economies. See Remark 2, below.)

An allocation X satisfies NNS and NPS if and only if $w^j \in \partial S^j(X)$, $j = 1,\ldots,n$, where $\partial S^j(X)$ the R^ℓ_+-boundary of $S^j(X)$. To continue the analogy with marginal productivity theory, if each agent is paid its marginal product, will the sum of the payments just exhaust the total product? For an exchange economy, the question is whether there exists a <u>no-surplus</u> (NS) allocation - i.e., an X such that,

NS: $$w^m \in \partial S^m(X), \qquad m = 0,1,\ldots,n.$$

We shall also define a <u>no-surplus economy</u> as an $\mathcal{E}$ such that

NS Economy: $w^j \notin S^j(X)$, $j = 1,\ldots,m$ implies $w^m \in \partial S^m(X)$, $m = 0,1,\ldots,n.$

In an NS economy, a sufficient condition for an allocation to be NS is that it be NNS. Since NNS is also necessary for NS, the defining characteristic of an NS economy is that the NNS and NS conditions are equivalent. (Note that if X is an NS allocation for $\mathcal{E}$ this does not imply the stronger condition that $\mathcal{E}$ is itself NS.)

The geometrical properties that flow from the convexity hypothesis, A_4, can be exploited to give additional perspective on the concept of NS, especially its relation to Walrasian equilibrum.

Let $P = \{p \in R^{\ell} : \|p\| = 1\}$, where $\|\cdot\|$ denotes the Euclidean norm, be the set of vectors that are unit normals to hyperplanes in R^{ℓ}. Define

$$Q^m(X) = \{p \in P : \inf p[S^m(X) - w^m] \geq 0\}, \quad m = 0,1,\ldots,n.$$

If $w^m \notin S^m(X)$, then $Q^m(X) \neq \emptyset$ and conversely, if A_3 is assumed, $Q^m(X) \neq \emptyset$ will be shown to imply $w^m \notin S^m(X)$. This will be used to give a description of X as an NS allocation in terms of the sets $Q^m(X)$.

THEOREM 1: <u>Let</u> $\mathcal{E}$ <u>satisfy</u> $A_1 - A_4$. <u>Then</u> X <u>is an</u> NS <u>allocation if and only if</u> $Q^m(X) \neq \emptyset$, $m = 0,1,\ldots,n$, <u>and</u>

(i) $Q^0(X) \subset Q^j(X)$, $j = 1,\ldots,n$

(ii) $p \in Q^m(X)$ <u>implies</u> $\inf p[S^m(X) - w^m] = 0$, $m = 0,1,\ldots,n$.

PROOF: If $Q^m(X)$, $m = 0,1,\ldots,n$ satisfy (i) and (ii) and $p \in Q^0(X)$, then $\inf p[S^0(X) - w^0] = \inf p[S^j(X) - w^j]$, $j = 1,\ldots,n$. By A_4,

(1) $$\inf p[S^0(X) - w^0] = \inf p[S^j(X) - w^j] + \inf p[S_j(X) - w_j].$$

Therefore, $\inf p[S_j(X) - w_j] = 0$ and by A_3 if $y \in S_j(X)$, $p(y - w_j) > 0$. Thus, for any $p \in Q^0(X)$, if $y \in S^m(X)$, $p(y - w^m) > 0$, $m = 0,1,\ldots,n$. This and (ii) imply that $w^m \notin S^m(X)$, all m.

To show that $w^m \in \partial S^m(X)$ it suffices to show that $w^m \in \text{cl } S^m(X)$. Suppose the contrary and let $0 \neq \|z^m\| = \inf\|\text{cl } S^m(X) - w^m\|$. By A_4 and a standard argument used in the proof of the Separation Theorem, we may set $p = (\|z^m\|)^{-1} z^m \in P$ and obtain the conclusion that $p \in Q^m(X)$. But $0 \neq pz^m = \|z^m\| \leq \inf p[S^m(X) - w^m]$, contradicting (ii).

If X is NS, the Separation Theorem implies that $Q^m(X) \neq \emptyset$, $m = 0,1,\ldots,n$. Let $p \in Q^0(X)$. Since $w^m \in \text{cl } S^m(X)$, $\inf p[S^m(X) - w^m] \leq 0$.

Again by A_4 and (1), above, we may conclude that $\inf p[S^j(X) - w^j] = 0$, $j = 1,\ldots,m$. This establishes (i).

If $p \in Q^m(X)$ and $\inf p[S^m(X) - w^m] > 0$, then $S^m(X)$ can be strictly separated from w^m which contradicts the fact that $w^m \in \partial S^m(X)$. This establishes (ii).

It has been demonstrated by Mas-Colell [1974] and Shafer and Sonnenschein [1975] that only those properties of preferences embodied in $A_1 - A_4$ and A_6 are required for the existence of Walrasian equilibrum. Following this lead, as well as a similar framework used by Vind [1964], the concept of NS is also formulated so as to depend only on the geometrical character of the preference mapping $S_i(X)$. Since these restrictions do not necessarily include completeness or transitivity, preferences need not be numerically representable. Nevertheless, it is useful to interpret NS when $\mathcal{E} = \{(u_i, w_i)\}_{i=1}^{i=n}$, where $u_i : R_+^\ell \to R$ is assumed to be a continuous function representing the underlying complete and transitive preference ordering. With A_5, $S_i(X)$ is derived from u_i as $\{y : u_i(y) > u_i(x_i)\}$.

With numerically representable preferences, the allocation $X = (x_i)$ produces the utility vector $(u_1(x_1),\ldots,u_n(x_n))$. Let

$$V^m = \{v = (v_1,\ldots,v_n) : v_i \leq u_i(x_i),\ i \neq m \text{ and } w^m \in S^m(X)\}$$

for $m = 0,1,\ldots,n$. When $m = 0$, V^m represents n-vectors of utilities that can be exceeded by all the members of the economy and when $m = 1,\ldots,n$, by groups consisting of all but one member. Let ∂V^m be the boundary of V^m. Arrow and Hahn [1971] show that without loss of generality we may take $\partial V^0 = \{v : v_i \geq 0,\ \Sigma v_i = 1\}$.

Define

$$\bar{u}_j = 1 - \min_{\substack{v \in \partial V^0 \\ v \in V^j}} \sum_{i \neq j} v_i = \max_{\substack{v \in \partial V^0 \\ v \in V^j}} \left(\sum_{i=1}^{i=n} v_i - \sum_{i \neq j} v_i \right).$$

The scalar $\bar{u}_j$ represents an upper bound, in terms of utilities, on the value added by j. It is obtained as the maximum that can be extracted from others subject to the condition that they not be made worse off by dealing with j. It is a utility measure of j's marginal product.

From the construction, it is clear that $\Sigma \bar{u}_j \geq 1 = \sup_{v \in V^0} \Sigma v_j$. When $\Sigma \bar{u}_j > 1$, $\mathcal{E}$ exhibits increasing returns in the sense that if each agent were paid its marginal product, the total product would be more than exhausted. (The vector $(\bar{u}_1,\ldots,\bar{u}_n) \notin \text{cl } V^0$.) However, when $\Sigma \bar{u}_j = 1$, we may say that $\mathcal{E}$ exhibits "locally" constant returns in the sense that it is possible to pay each agent its marginal product.

When $\Sigma \bar{u}_j = 1$, there is an $X = (x_i)$ such that $\bar{u}_j = u_j(x_j)$. Call such an X an allocation exhibiting <u>product-exhaustion</u>. Note that product-exhaustion is an ordinal property of X. By construction $\bar{u} = (\bar{u}_1,\ldots,\bar{u}_n) \in \partial V^m$ and therefore $w^m \in \partial S^m(X)$, $m = 0,1,\ldots,n$. Thus X exhibits product exhaustion <u>only if</u> it is an NS allocation. But, the converse does not hold. If, however, it is also assumed that for each $j = 1,\ldots,n$ there are scalars v^j such that $\partial V^j = \{v : \Sigma_{i \neq j} v_i = v^j\}$ it is readily demonstrated that X <u>exhibits product-exhaustion if and only if it is</u> NS. The additional requirements on ∂V^j amount to a transferable utility assumption for the one-fewer sets of agents at least.

With ordinal preferences, product-exhaustion is too strong a requirement for characterizing perfectly competitive equilibrum. For example, even economies with an indefinitely large number of small-scale agents would not admit allocations satisfying product-exhaustion when there are multiple

Walrasian equilibria. The concept of NS is therefore proposed as an ordinal extension of the idea of product-exhaustion. In fact, once transferable utility is dropped, even ordinal utility appears to be a detour on the way to a definition in terms of the preference mappings, $S_i(X)$.

III. WALRASIAN AND PERFECTLY COMPETITIVE EQUILIBRIUM

A Walrasian equilibrum (WE) for $\mathcal{E}$ is a pair (X,p), where X is an allocation and $p \in P$, such that for each i,

WE: $$p(x_i - w_i) = 0 \text{ and } y \in S_i(X) \text{ implies } p(y - w_i) > 0.$$

It is useful to give an alternative description.

PROPOSITION 1: <u>Let</u> $\mathcal{E}$ <u>satisfy</u> $A_1 - A_3$. <u>Then</u> (X,p) <u>is a</u> WE <u>if and only if</u> $p \in \bigcap_{j=1}^{j=n} Q^j(X)$.

PROOF. If (X,p) is a WE, $\inf p[S_i(X) - w_i] \geq 0$. This yields by summation $\inf p[S^j(X) - w^j] \geq 0$ and therefore $p \in Q^j(X)$.

If $p \in \bigcap Q^j(X)$, $\inf p[S^j(X) - w^j] \geq 0$. By A_2, $px_i \geq \inf p\, S_i(X)$ and therefore $px^j \geq \inf p\, S^j(X)$. Thus, $p(x^j - w^j) = p(w_j - x_j) \geq \inf p[S^j(X) - w^j] \geq 0$, for all $j = 1,\ldots,n$. This implies $p(x_j - w_j) = \inf p[S_j(X) - w_j] = 0$, all j. The remaining part of the definition of WE follows from A_3.

Inspection of Theorem 1 and Proposition 1 yield

COROLLARY 1.1: <u>Let</u> $\mathcal{E}$ <u>satisfy</u> $A_1 - A_4$. <u>If</u> X <u>is an</u> NS <u>allocation, there is a</u> $p \in \mathbf{P}$ <u>such that</u> (X,p) <u>is a</u> WE.

The following gives a sufficient condition for X to be an NS allocation.

COROLLARY 1.2: Let $\mathcal{E}$ satisfy $A_1 - A_3$. If X is such that $Q^0(X) \neq \emptyset$ and $Q^j(X) = Q^0(X)$, $j = 1,\ldots,n$, then X is NS.

PROOF: From Proposition 1, if $p \in \cap Q^j(X)$, (X,p) is a WE. Since $\bigcap_j Q^j(X) = Q^j(X) = Q^0(X)$ and $p \in \cap Q^j(X)$ implies $\inf p[S^j(X) - w^j] = 0$, we have $p \in \cap Q^j(X)$ implies $\inf p[S^m(X) - w^m] = 0$, $m = 0,1,\ldots,n$. Therefore by Theorem 1, X is NS.

To show that the above sufficient condition is practically necessary for NS, we demonstrate

COROLLARY 1.3: Let $\mathcal{E}$ satisfy $A_1 - A_4$ and let X be an allocation such that the set of net trade vectors $\{(x_i - w_i)\}$ is not contained in a subspace of dimension less than $(\ell-1)$. Then X is NS if and only if there is a $p \in P$ such that $Q^0(X) = Q^j(X) = \{p\}$, $j = 1,\ldots,n$.

PROOF. By Corollary 1.2, if $Q^0(X) = Q^j(X)$, X is NS.

If X is NS, by Corollary 1.1, there is at least one $p \in P$ such that $p(x_i - w_i) = 0$, $i = 1,\ldots,n$. Since the dimension of the span of $\{(x_i - w_i)\}$ is not less than $(\ell-1)$ and $p \in P$ is in its orthogonal complement the dimension is not more than $(\ell-1)$ and the dimension of the orthogonal complement is therefore 1.

Comparing the definition of X as a WE allocation to the definition of an NS allocation, the former simply requires that $\bigcap_j Q^j(X) \neq \emptyset$ while the latter demands that, in "most" cases, $Q^j(X) = Q^0(X)$. Assuming that $Q^0(X) = \{p\}$, if $Q^j(X) \neq \{p\}$, there are valuations of resources for the "one-fewer" sets of agents that are not common to the valuation of resources by the economy as a whole. The definition of WE resolves the issue of resource valuation by imposing the condition that only those elements of each $Q^j(X)$ that are common to $Q^0(X)$ are acceptable. This is nothing other than the well-known restriction that individual agents must take prices as given. However, with an NS allocation, we shall show that such a restriction is superfluous. This will be accomplished by constructing a definition of equilibrium in which competitive self-interest dictates price-taking behavior and that the definition is equivalent to the NS condition.

Informally, demands are perfectly elastic at the price vector p when the attempt by any agent to set prices for the goods it supplies at levels higher than p results in the loss of all sales <u>and</u> markets for the goods supplied by others clear without any adjustment in their prices. To make possible this experiment, each agent must have control over the prices of the goods it supplies.

Normally, the commodity space, R^{ℓ}_{+}, would be chosen so that the number of goods, ℓ, is as small as possible. It aggregates into one all those goods for which individuals have identical, constant marginal rates at substitution. The resulting interpretation is that each commodity represents a market with the presumption that on any market one and only one price will be charged. Upon such a foundation the concept of WE is applied to determine market-clearing prices. Here we do not take the market with its single price for granted. Rather, the purpose is to open up the model to price-setting by individual agents so as to allow "markets" to emerge as a conclusion.

We shall assume that commodities are disaggregated to bring the specification of initial endowments into the appropriate form of a <u>personalized commodity space</u> $\mathcal{E}$ in which each agent is the unique supplier of its own goods. Formally, commodities are in personalized form if

$$w_i, w_j \in R_+^{\ell}, \quad i \neq j, \text{ implies } w_i w_j = 0.$$

For any agent j, we may divide the ℓ (disaggregated) commodities into those that could be supplied by j and those that could not. For $p \in R^{\ell}$, let p_j be the vector of prices of the former and p^j the prices of the latter. For any p and j, $p = (p^j, p_j)$, let $q(p;j)$ denote any vector $q = (q^j, q_j) \in R^{\ell}$ such that $q^j = p^j$ and $q_j \geq p_j$ and $q_j \neq p_j$. At $q(p;j)$ the price of at least one of j's goods is higher than at p while the prices of the remaining goods, that j cannot control, are the same as in p.

The pair (X,p) is a <u>perfectly determinate price equilibrium</u> (PD) for $\mathcal{E}$ if for each $j = 1,\ldots,n$ and any $i \neq j$, if y satisfies

PD_1: $\quad q(p;j)(y - w_i) = 0, \; (q(p;j) - p)y \neq 0, \;$ then $\; y \notin \text{cl } S_i(X)$

and, there exists y_i^j, $i \neq j$, such that

PD_2: $\quad q(p;j)(y_i^j - w_i) = 0, \; y_i^j \in \text{cl } S_i(X) \;$ and $\; \sum_{i \neq j} y_i^j = w^j.$

To interpret, the opportunities available to i are strictly smaller when they are defined by the price vector $q(p;j)$ rather than p since the price of all goods other than j's are the same but j's prices are higher. However, PD states that if i is a purchaser from j, $(q(p;j) - p)y \neq 0$, i has not made the best use of his smaller opportunity set since y is not in $\text{cl } S_i(X)$, while if i had refused to deal with j, then trading at the same prices, he could have found buyers and sellers willing to make exchanges leading to y_i^j in $\text{cl } S_i(X)$. If preferences were transitive and continuous, y_i^j would be strictly preferred to y_i.

We shall regard the existence of (X,p) satisfying PD as synonomous with a price equilibrium exhibitng perfectly elastic demands.

THEOREM 2: Let $\mathcal{E}$ satisfy $A_1 - A_5$ and assume initial endowments are personalized. Then X is an NS allocation if and only if there is a $p \in P$ such that (X,p) is PD.

PROOF: From Corollary 1.1, if X is NS, there is a $p \in P$ such that (X,p) is a WE. Thus, $pw_i = px_i = \inf p\, S_i(X)$. If $q(p;j)(y - w_i) = 0$ and $(q(p;j) - p)y \neq 0$, then $py < px_i$ and therefore $y \notin \text{cl}\, S_i(X)$ and PD_1 is satisfied. Since $p \in Q^0(X)$ may be chosen so that $p \in Q^j(X)$ and because X is NS, $pw^j = \inf p\, S^j(X) = px^j$. Also by NS there exists $y_i^j \in \text{cl}\, S_i(X)$ such that $\Sigma_{i \neq j}\, y_i^j = w^j$ and therefore for all i and j, $i \neq j$, $py_i^j \geq \inf p\, S_i(X)$ which implies $p(y_i^j - w_i) = 0$ or $q(p;j)(y_i^j - w_i) = 0$, and PD_2 is established.

For the converse it suffices to show that if (X,p) is PD, it is a WE and therefore satisfies PO and NNS. If (X,p) is PD, PO and NNS, PD_2 clearly implies NS.

From PD_2 we have $p(y_i^j - w_i) = 0$, and since $y_i^j \in \text{cl}\, S_i(X)$, $\inf p\, S_i(X) \leq py_i^j$. If $\inf p\, S_i(X) < py_i^j = pw_i$ for some i, then we may choose $q(p;j)$ close to p and y close to x_i such that $q(p;j)(y - w_i) = 0$ and $(q(p;j) - p)y \neq 0$ such that $y \in \text{cl}\, S_i(X)$, contradicting PD_2. Therefore, $pw_i = py_i^j = \inf p\, S_i(X) \leq px_i$ for all i and therefore $pw_i = px_i$. The remainder of the definition of WE follows from A_3..

An implication of Theorem 2 is that PD or NS may be regarded as the definition of perfectly competitive equilibrium. With it we have the conclusion that agents will rationally choose to set as prices for the commodities they supply the ones dictated by WE.

REMARK 2: Without A_5, the absence of external effects, the definition of perfectly competitive equilibrum for finite exchange economies would be deficient. If j set a higher price and sells nothing because his customers go elsewhere, the resulting allocation could be substantially different and, with external effects, preferences could be substantially changed. Thus, even though WE satisfies NS, an agent can possess monopoly power by recognizing the possible adverse effects to customers who might otherwise go elsewhere. This is an added complication due to small numbers. With large numbers external effects do not create a separate problem because the possible disruption to others caused by refusing to deal with any one agent vanishes as the relative size of the agent vanishes.

A trivial illustration of a NS economy occurs whenever the initial endowments for $\mathcal{E}$ are PO. A nontrivial illustration is provided by the following,

EXAMPLE 1: Let the tastes of agent $i = 1,2,3$ be represented by the utility function $u_i(x_i) = u_i(x_{i1}, x_{i2}, x_{i3})$, initial endowments by the matrix W and final allocations by the matrices X_α, where

$$u_1(x_1) = x_{11}(x_{12} + x_{13})$$
$$u_2(x_2) = x_{22}(x_{21} + x_{23});$$
$$u_3(x_3) = x_{33}(x_{31} + x_{32})$$

W = (rows i, columns c):

2	0	0
0	2	0
0	0	2

; X_α = (rows i, columns c):

1	α	$1-\alpha$
$1-\alpha$	1	α
α	$1-\alpha$	1

To show that $\mathcal{E}$ is a NS economy, it must be demonstrated that for any X such that $w^j \notin S^j(X)$, $j = 1,2,3$, $w^m \in \partial S^m(X)$, $m = 0,1,2,3$. Taking $X = X_\alpha$, $0 \leq \alpha \leq 1$, this result is easily verified. Note that these allocations are at the opposite extreme from no-trade since their attainment requires that all agents participate in trade. Of course, because each X_α is an NS allocation and there are only three traders, any pair could also do as well by itself. Each agent's participation in an NS allocation is essential only to itself.

It may also be verified that each X_α is a WE (Corollary 1.1) at prices $p = (r,r,r)$, $r > 0$; and, if any agent j were to raise the price of its good above its value in p, j would sell nothing and excess demands among the other agents would be zero without disturbing the other prices (Theorem 2).

REMARK 3: Note that in Example 1 individual preferences are not strictly convex. (Strict convexity obtains if every line segment connecting two distinct points on $\partial S_i(X)$ lies in $S_i(X)$.) When the commodity space is put in personalized form strict convexity and NS imply that there can be no trade.

At an NS allocation each agent j must be faced with an $S^j(X)$ a portion of whose boundary coincides with the hyperplane $\{y : p(y - w^j) = 0\}$, where p is a WE price vector. If the hyperplane supported $S^j(X)$ but $S^j(X)$ were strictly convex, the only point of intersection would be at $x^j \in \partial S^j(X)$ and therefore $w^j \in \partial S^j(X)$ if and only if $x^j = w^j$.

Since strict convexity is the rule even after commodities are put in personalized form, this observation shows why most finite economies will fail to admit an NS allocation. It also exhibits a geometric parallel with the apparently quite different conclusions for large economies. When the scale of each agent becomes very small, the quantities x^j and w^j will differ very little from w^0 and therefore from each other. In this case, $x^j \in \partial S^j(X)$ will imply $w^j \in \partial S^j(X)$ provided that the boundary of

$S^j(X)$ in the neighborhood of x^j is, at least to a first approximation, linear - i.e., differentiable. Since differentiability of aggregate preferences will be the rule when preferences are convex, this explains why most large economies will exhibit NS.

IV. THE NO-SURPLUS PROPERTY OF WALRASIAN EQUILIBRIA IN REPLICA ECONOMIES

In this section the characterization of perfectly competitive equilibrium established for finite exchange economies is extended to a simple sequence of economies in which the number of agents is indefinitely increasing. It will be demonstrated that, as compared to the finite case, only very mild restrictions are required to ensure that such sequences approach the condition of being a no-surplus economy as the number of agents increases.

By dividing up the economy $\mathcal{E} = \{(S_i, w_i)\}_{i=1}^{i=n}$, another economy may be constructed with a larger number of agents each of whom operates on a smaller scale. Let $\mathcal{E}_k = \{(S_{i_h}, w_{i_h})\}$, $i = 1,\ldots,n$ and $h = 1,\ldots,k$, be such that for each i and h,

$$kw_{i_h} = w_i.$$

For an allocation $X = (x_i)$ for $\mathcal{E}$, define $[X]^k = (x_{i_h})$, $i = 1,\ldots,n$ and $h = 1,\ldots,k$ such that for each i and h,

$$kx_{i_h} = x_i.$$

Since X is a feasible allocation for $\mathcal{E}$ $(\Sigma x_i = \Sigma w_i)$, $[X]^k$ is a feasible allocation for $\mathcal{E}_k$ - i.e.,

$$\sum_i \sum_h x_{i_h} = k \sum_i x_{i_h} = \sum x_i = \sum w_i = k \sum_i w_{i_h} = \sum_i \sum_h w_{i_h}.$$

Attention will be confined to allocations for $\mathcal{E}_k$ that can be written as $[X]^k$ - all agents of the same type i receive the same allocation. Further, the sequences of allocations considered for the sequence of economies $\{\mathcal{E}_k\}$, $k = 1,2,\ldots$ will be restricted to $\{[X]^k\}$ - variations in the allocation as k varies are based entirely on scalar multiplication of the fixed allocation X for $\mathcal{E}$.

To complete the description of $\mathcal{E}_k$, define $S_{i_h}([X]^k)$ such that for each i and h,

$$k\, S_{i_h}([X]^k) = S_i(X).$$

With A_5, $\{\mathcal{E}_k\}$ is a sequence of Edgeworth replica economies that approaches a special case of a continuum of traders economy (Aumann [1964]). Without A_5, the above assumption on preferences preserves the replication hypothesis by requiring that the consequences of external effects vary in proportion to the size of the agents. Twice as many individuals each consuming half as much creates no change in aggregate external effects.

For $k = 1,2,\ldots,$ let $\{j(k)\}$ be a sequence that selects one of the agents of one of the types $j = 1,\ldots,n$ from each $\mathcal{E}_k$. With the above restrictions, $j(k)$ could be set equal to j_k, the last agent of type j in $\mathcal{E}_k$. The total resources of all agents in $\mathcal{E}_k$ except $j(k)$ is,

$$w^{j(k)} = \sum_{i \neq j} \sum_h w_{i_h} + \sum_{h \neq k} w_{j_h} = k \sum_{i \neq j} w_{i_h} + (k-1) w_{j_h} = w^j + \frac{k-1}{k} w_j \ .$$

Analogously, define

$$S^{j(k)}([X]^k) = \sum_{i \neq j} \sum_h S_{i_h}([X]^k) + \sum_{h \neq k} S_{j_h}([X]^k) = k \sum_{i \neq j} S_{i_h}([X]^k)$$
$$+ (k-1) S_{j_h}([X]^k) = S^j(X) + \frac{k-1}{k} S_j(X).$$

The last equality allows us to write $S^{j(k)}([X]^k)$ in the more compressed form $S^{j(k)}(X)$ without any ambiguity.

As a direct extension of the finite case, the sequence $\{[X]^k\}$ satisfies the NNS condition for $\{\mathcal{E}_k\}$ if for each selection $\{j(k)\}$, $k = 1,2,\ldots,$

NNS: $$w^{j(k)} \notin S^{j(k)}(X).$$

A similar extension applies to the definition of NPS but this will be ignored because it plays no role. The extension of NS is somewhat more delicate.

For an allocation $[X]^k$ for $\mathcal{E}_k$, define

$$d(\partial S^{j(k)}(X), w^{j(k)}) = \inf \|\partial S^{j(k)}(X) - w^{j(k)}\|.$$

The distance is non-zero if and only if either (1) $w^{j(k)} \notin \mathrm{cl}\ S^{j(k)}(X)$, in which case $j(k)$ is contributing a positive surplus to $[X]^k$, or (2) $w^{j(k)} \in S^{j(k)}(X)$ and $j(k)$ is contributing a negative surplus. We shall say that $\{[X]^k\}$ satisfies the NS condition asymptotically (ANS) if for any $\{j(k)\}$, $k = 1,2,\ldots,$

ANS: $$\lim k\ d(\partial S^{j(k)}(X), w^{j(k)}) = 0.$$

ANS requires not only that the surplus contributed by any one agent, positive or negative, go to zero as k increases, but it must go to zero sufficiently rapidly so that the surplus contributed by any agent to all other agents goes to zero faster than $(k)^{-1}$, a measure of the size of $j(k)$.

We shall say that $\mathcal{E}_\infty$ is an NS economy if for every $\{j(k)\}$,

NS Economy: $$w^{j(k)} \notin S^{j(k)}(X) \text{ implies } \lim k\ d(\partial S^{j(k)}(X), w^{j(k)}) = 0.$$

For the allocation $[X]^k$ for $\mathcal{E}_k$, define

$$Q^{j(k)}([X]^k) = \{p \in P : \inf p[S^{j(k)}(X) - w^{j(k)}] \geq 0\}.$$

There is no ambiguity in writing $Q^{j(k)}([X]^k)$ as $Q^{j(k)}(X)$. With A_4, if $[X]^k$ is NNS for $\mathcal{E}_k$, $Q^{j(k)}(X)$ is non-empty. For a sequence $\{j(k)\}$ in which $j(k) = j$, let $Q^{j(\infty)}(X) = \bigcap_{k=1}^{k=\infty} Q^{j(k)}(X)$.

LEMMA: Let $\mathcal{E}$ satisfy A_1, A_2, and A_4 and let $\{[X]^k\}$ be a sequence of NNS allocations for $\{\mathcal{E}_k\}$. Then if $j(k) = j$, $k = 1,2,\ldots,$

(i) $\{Q^{j(k)}(X)\}$ is a decreasing sequence of non-empty closed subsets of P.

(ii) $Q^{j(\infty)}(X) \subset Q^0(X)$.

PROOF: That $Q^{j(k)}(X)$ is non-empty follows from the hypothesis that $[X]^k$ is NNS, A_4, and the Separation Theorem. To show that $Q^{j(k)}(X)$ is closed, let $p^r \in Q^{j(k)}(X)$ and $\lim p^r = p$. Thus, $\inf p^r[S^{j(k)}(X) - w^{j(k)}] \geq 0$, or $p \in Q^{j(k)}(X)$.

To establish that $\{Q^{j(k)}(X)\}$ is decreasing, it suffices to show that for any $k > 1$ and $j(k) = j(k-1)$, if $p \in Q^{j(k)}(X)$ then $p \in Q^{j(k-1)}(X)$. Letting $p \in Q^{j(k)}(X)$,

$$(1)\quad \inf p[S^{j(k)}(X) - w^{j(k)}] = \inf p[S^j(X) + \frac{k-1}{k} S_j(X) - (w^j + \frac{k-1}{k} w_j)]$$

$$= \inf p[S^j(X) - w^j] + \frac{k-1}{k} \inf p[S_j(X) - w_j] \geq 0.$$

By A_2, $\inf p[S^0(X) - w^0] \leq 0$ and since

$$\inf p[S^0(X) - w^0] = \inf p[S^{j(k)}(X) - w^{j(k)}] + (k)^{-1} \inf p[S_j(X) - w_j],$$

we must have,

$$(2)\qquad \inf p[S_j(X) - w_j] \leq 0.$$

When $0 \geq \alpha = \inf p[S_j(X) - w_j]$, $\frac{k-2}{k-1}\alpha \geq \frac{k-1}{k}\alpha$. Thus, (1) and (2) yield,

$$\inf p[S^j(X) - w^j] + \frac{k-2}{k-1} \inf p[S_j(x) - w_j] \geq 0. \tag{3}$$

But (3) means that $p \in Q^{j(k-1)}(X)$ which demonstrates (i).

To establish (ii), first note that since $Q^{j(\infty)}(X)$ is the intersection of a decreasing sequence of non-empty compact sets, it is non-empty. If $p \in Q^{j(\infty)}(X)$, then the inequality in (1) is satisfied for all k which obviously implies that $\inf p[S^j(X) - w^j] + \inf p[S_j(X) - w_j] = \inf p[S^0(X) - w^0] \geq 0$, or $p \in Q^0(X)$.

REMARK 4: The principal mathematical distinction between $\mathcal{E}$ and $\mathcal{E}_\infty$ is the conclusion in part (ii) of the above lemma. For $\mathcal{E}$ the analogue of (ii) - $Q^j(X) \subset Q^0(X)$ - does not hold even for a WE, except under certain conditions that imply X is NS. For example, if $Q^0(X) = \{p\}$ and $Q^j(X) \subset Q^0(X)$, $j = 1,\ldots,n$, then $Q^j(X) = Q^0(X)$ and by Corollary 1.2, X is NS. As a restriction, $Q^0(X) = \{p\}$ is quite mild, yet that is all we shall need to obtain the conclusion that $\mathcal{E}_\infty$ is NS. (See Corollary 3.3, below.)

For $\mathcal{E}$, Corollaries 1.2 and 1.3 showed that $Q^j(X) = Q^0(X)$, $j = 1,\ldots,n$, is sufficient and practically necessary for X to be NS. In the following extension of Theorem 1 describing ANS in terms of its "dual", it is shown that for $\mathcal{E}_\infty$ there is complete equivalence.

THEOREM 3: _Let_ $\mathcal{E}$ _satisfy_ $A_1 - A_4$. _Then_ $\{[X]^k\}$ _is_ ANS _for_ $\{\mathcal{E}_k\}$ _if and only if_ $Q^{j(\infty)}(X) = Q^0(X)$, $j = 1,\ldots,n$.

PROOF: If $\bigcap_k Q^{j(k)}(X) = Q^0(X)$, then since part (i) of the Lemma shows that $\{Q^{j(k)}(X)\}$ is decreasing we must have $Q^0(X) \subset Q^j(X)$, $j = 1,\ldots,n$. Therefore, Proposition 1 allows us to conclude that for any $p \in Q^0(X)$, $\inf p[S_j(X) - w_j] = p(x_j - w_j) = 0$, $j = 1,\ldots,n$.

Let $\|z^{j(k)}\| = \inf\|\text{cl } S^{j(k)}(X) - w^{j(k)}\|$. If for all $j(k)$, $\|z^{j(k)}\| = 0$, there is nothing more to prove - i.e., $\|z^{j(k)}\| = 0$, $Q^{j(k)}(X) \neq \emptyset$ and A_3 imply that $w^{j(k)} \in \partial S^{j(k)}(X)$. Therefore, assume $\|z^{j(k)}\| \neq 0$. Letting $p^{j(k)} = (\|z^{j(k)}\|)^{-1} z^{j(k)}$, we obtain as in the proof of Theorem 1 that $p^{j(k)} \in Q^{j(k)}(X)$. But

$$0 < k\|z^{j(k)}\| = k\, p^{j(k)} z^{j(k)} \leq k \inf p[S^{j(k)}(X) - w^{j(k)}]$$

$$\leq k\, p^{j(k)}(x^{j(k)} - w^{j(k)}) = p^{j(k)}(w_j - x_j).$$

The last equality follows from the identity,

$$(1)\quad k(x^{j(k)} - w^{j(k)}) = k[x^j + \frac{k-1}{k} x_j - (w^j + \frac{k-1}{k} w_j)] = k(x^j - w^j) + (k-1)(x_j - w_j)$$

$$= (w_j - x_j).$$

Since $p \in Q^0(X)$ implies $p(w_j - x_j) = 0$ and by hypothesis $\inf\|Q^0(X) - p^{j(k)}\| \to 0$, we have the desired conclusion that $k\|z^{j(k)}\| \to 0$.

For the converse, we first show that ANS implies $w^0 \notin S^0(X)$ and therefore $Q^0(X) \neq \emptyset$. Suppose the contrary that $w^0 \in S^0(X)$. Then since $w^{j(k)} \to w^0$, by A_1 we have for sufficiently large k that $w^{j(k)} \in S^0(X)$. Now, $S^0(X) = S^{j(k)}(X) + S_{j(k)}(X)$ and for k large, $S_{j(k)}(X)$ intersects any ϵ-neighborhood of the origin. Therefore, for all sufficiently large k, $w^{j(k)} \in S^{j(k)}(X)$, which contradicts the hypothesis that $\{[X]^k\}$ is ANS.

Let $p \in Q^0(X)$. Then A_2 implies,

$$0 = \inf p[S^0(X) - w^0] = \inf p[S^{j(k)}(X) - w^{j(k)}] + (k)^{-1} \inf p[S_j(X) - w_j].$$

Multiplying by k, we obtain

$$0 = k \inf p[S^{j(k)}(X) - w^{j(k)}] + \inf p[S_j(X) - w_j].$$

Since $\{[X]^k\}$ is ANS,

$$\lim \sup\{k \inf p[S^{j(k)}(X) - w^{j(k)}]\} \leq 0.$$

Thus, $\inf p[S_j(X) - w_j] \geq 0$, $j = 1,\ldots,n$, which implies $\inf p[S_j(X) - w_j] = 0$, all j. Therefore,

$$\inf p[S^j(X) - w^j] + \frac{k-1}{k} \inf p[S_j(X) - w_j] \geq 0$$

which means that for all k, $p \in Q^{j(k)}(X)$. This establishes $Q^0(X) \subset Q^{j(\infty)}$. Part (ii) of the Lemma gives the opposite inclusion.

To exhibit the connections with WE, simply extend Proposition 1. Clearly, if (X,p) is a WE for $\mathcal{E}$, $([X]^k,p)$ is a WE for $\mathcal{E}_k$, $k = 1,2,\ldots$; and, with A_4, the converse holds.

PROPOSITION 2: <u>Let</u> $\mathcal{E}$ <u>satisfy</u> $A_1 - A_4$. <u>Then</u> (X,p) <u>is a</u> WE <u>for</u> $\mathcal{E}$ <u>if and only if</u> $p \in \bigcap_j Q^j(\infty)(X)$.

The following extension of Corollary 1.1 is contained in the proof of Theorem 3.

COROLLARY 3.1: <u>Let</u> $\mathcal{E}$ <u>satisfy</u> $A_1 - A_4$. <u>If</u> $\{[X]^k\}$ <u>is</u> ANS, <u>there is a</u> $p \in P$ <u>such that</u> $([X]^k,p)$ <u>is a</u> WE <u>for</u> $\mathcal{E}_k$.

The relation between WE and ANS is summarized by,

COROLLARY 3.2: <u>Let</u> $\mathcal{E}$ <u>satisfy</u> $A_1 - A_4$ <u>and let</u> X <u>be an allocation for</u> $\mathcal{E}$ <u>such that</u> $Q^0(X) = \{p\}$. <u>Then</u> (X,p) <u>is a</u> WE <u>if and only if</u> $\{[X]^k\}$ <u>is</u> ANS.

rather than proceed with the formal details, we shall outline an independent proof that suggests why ANS would imply the perfect determinacy of prices. It also shows, by invoking the continuity assumption on preferences, A_6, that we shall be able to obtain a perfectly competitive equilibrium even in the presence of external effects. (Compare Remark 2.)

If X is an allocation for $\mathcal{E}$, let $Y^k = (y^k_{i_h})$ be an allocation for $\mathcal{E}_k$ defined by

$$y^k_{i_h} = \begin{cases} w_{i_h}, & \text{if } i_h = j_k \\ x_{i_h}, & \text{if } i = j \text{ and } i_h \neq j_k \\ x_{i_h} - (k)^{-1}(x_{i_h} - w_{i_h}), & \text{if } i \neq j. \end{cases}$$

Y^k describes a reallocation away from $[X]^k$ in which everyone refuses to trade with j_k. Those agents of the same type as $j(k)$ obtain the same outcome as in $[X]^k$ while the members of the other types cut back their excess demands by $(k)^{-1}$. Note that $\Sigma_i \Sigma_h y_{i_h} = w^0$, so Y^k is in fact a feasible allocation for $\mathcal{E}_k$.

The total disruption caused by refusing to trade with j_k may be measured by,

$$\|Y^k - [X]^k\| = \sum_i \sum_h \|y^k_{i_h} - x_{i_h}\|.$$

Substituting the definition of Y^k, this equals

$$\sum_{i \neq j} \sum_h \|(x_{i_h} - (k)^{-1}(x_{i_h} - w_{i_h}) - x_{i_h})\| + \|(w_{j_k} - x_{j_k})\|$$

$$= \sum_{i \neq j} \|(w_{i_h} - x_{i_h})\| + \|(w_{j_k} - x_{j_k})\| = (k)^{-1} \sum_i \|(w_i - x_i)\|.$$

Therefore, $\lim \|Y^k - [X]^k\| = 0.$

PROOF: If (X,p) is a WE for $\mathcal{E}$, then $\{[X]^k\}$ is NNS and therefore by the Lemma, $Q^{j(\infty)}(X) \subset Q^0(X)$. Since $Q^0(X) = \{p\}$, $Q^{j(\infty)}(X) = Q^0(X)$ and by Theorem 3, $\{[X]^k\}$ is ANS.

The converse implication is Corollary 3.1.

Conditions for $\mathcal{E}_\infty$ to be an NS economy are given by,

COROLLARY 3.3: <u>Let</u> $\mathcal{E}$ <u>satisfy</u> $A_1 - A_4$ <u>and assume that for any</u> X <u>for which</u> $\{[X]^k\}$ <u>is</u> NNS, $\{(x_i - w_i)\}$ <u>does not lie in a subspace of dimension less than</u> $(\ell-1)$. <u>Then</u> $\mathcal{E}_\infty$ <u>is</u> NS <u>if and only if</u> $Q^0(X) = \{p\}$.

PROOF: If $Q^0(X)$ is not a singleton, the dimensionality assumption on $\{(x_i - w_i)\}$ implies that there is at least one $p \in Q^0(X)$ and one j such that $p(w_j - x_j) < 0$. The identity (1) in the proof of Theorem 3 means that $k\, p[x^{j(k)} - w^{j(k)}] < 0$. By A_2, $k \inf p[S^{j(k)}(X) - w^{j(k)}] \leq k\, p(x^{j(k)} - w^{j(k)})$. Thus, there is a $p \in Q^0(X)$ that does not belong to any $Q^{j(k)}(X)$. But this contradicts the condition $Q^0(X) = Q^{j(\infty)}(X)$ that is established in Theorem 3 as necessary for ANS.

The converse implication follows from the Lemma, part (ii), and Theorem 3.

To complete the extension from $\mathcal{E}$ to $\mathcal{E}_\infty$, an analogue of Theorem 2 should be given demonstrating that ANS is equivalent to a WE with, asymptotically, perfectly determinate prices. Because commodities are personalized, a literal version would require an infinite-dimensional commodity space. While not impossible, such a construction is unnecessary. We need only consider a space of dimension 2ℓ so as to distinguish the goods supplied by any one $j(k)$ and those supplied by others. However, If (X,p) is a WE for $\mathcal{E}$ and $Q^0(X) = \{p\}$, then even if there are external effects, $[X]^k$ is a stable, non-cooperative equilibrium for $\mathcal{E}_k$ when k is large because if any agent were to raise the prices of its commodities, the others could, with impugnity, go elsewhere.

More precisely, when $[X]^k$ is a WE allocation for $\mathcal{E}_k$, $w^{j(k)} \notin S^{j(k)}(X)$. By invoking A_6 and $\lim\|Y^k - [X]^k\| = 0$, we have for all k sufficiently large, $w^{j(k)} \notin S^{j(k)}(Y^k)$, and therefore $Q^{j(k)}(Y^k) \neq \emptyset$. The Lemma may be extended, using A_6 and $\lim\|Y^k - [X]^k\|$, to show that for some k', $\bigcap_{k>k'} Q^{j(k)}(Y^k) \subset Q^0(X)$. Similarly, if $Q^0(X) = \{p\}$ then $\bigcap_{k>k'} Q^{j(k)}(Y^k) = Q^0(X)$ for every $\{j(k)\}$ and the arguments used in the proof of Theorem 3 may be generalized to yield,

$$\lim\{k \inf\|\partial S^{j(k)}(Y^k) - w^{j(k)}\| = 0.$$

We conclude this section with an example illustrating that the condition $Q^{j(\infty)}(X) = Q^0(X)$ in Theorem 3 is necessary for ANS by showing that without it there is non-vanishing monopoly power to small-scale agents. To avoid complications, A_5 is assumed.

EXAMPLE 2: Consider the economy of equal numbers of type 1 and type 2 traders illustrated by the Edgeworth-Bowley box of Figure 1.

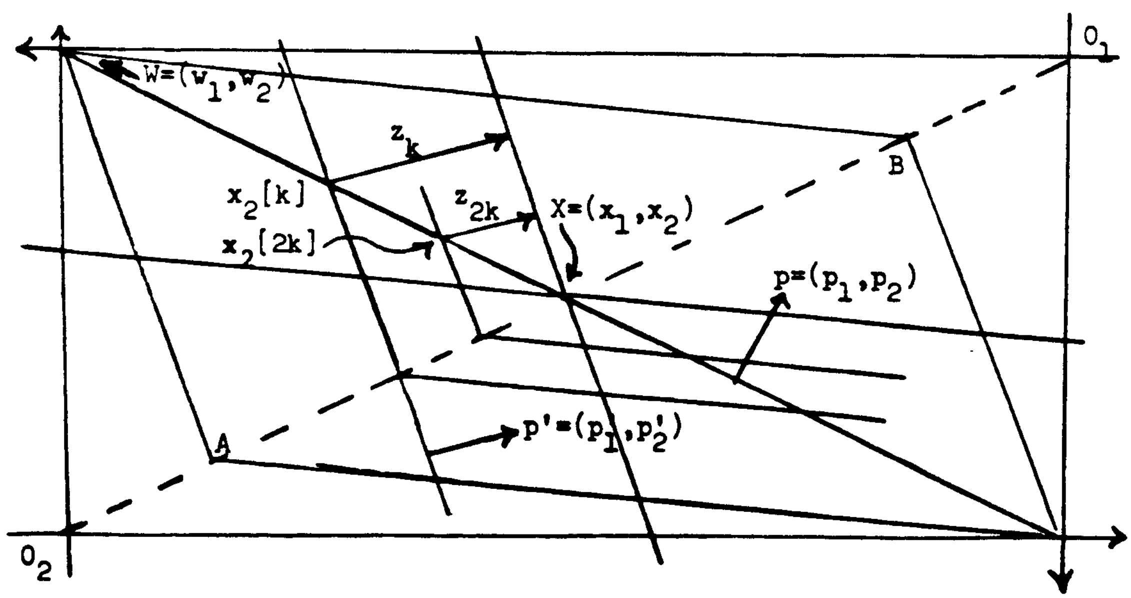

Figure 1

Let $\mathcal{E}_k^{j(k)}$ denote the economy consisting of all agents in $\mathcal{E}_k$ except $j(k)$, $j = 1,2$. No matter what the value of k, the WE allocations for $\mathcal{E}_k$ include all points along the line AB, while the WE allocations for $\mathcal{E}_k^{1(k)}$ or $\mathcal{E}_k^{2(k)}$ include only the points A or B, respectively.

The example is of some historical interest because of its similarity to Edgeworth's master-servant example in which each master has need of only one servant and one servant cannot serve two masters. If the minimum wage at which servants would offer themselves is α and the maximum wage at which masters would accept is β, Edgeworth noted that as long as there are an equal number of masters and servants, then no matter how many there are, the equilibrium wage is indeterminate, lying somehwere between α and β $(\alpha < \beta)$. If there is one more master than servant (or one more servant than master) indeterminacy disappears and the wage rate becomes β (or α). Edgeworth attributed the indeterminacy to the indivisibility of the good "domestic service" and on the basis of his example modified his proposition that large numbers of traders in a market lead to a determinate outcome only when goods are divisible. The self-evident similarity of the master-servant example to Example 2, in which goods are divisible, suggests that some other explanation is called for. What is common to Edgeworth's example and Example 2 is the failure of ANS.

To see this, consider the WE allocation $X = (x_1,x_2)$ in Figure 1 replicated k times for $\mathcal{E}_k$ and consider how close the members of $\mathcal{E}_k^{1(k)}$ can come to doing as well. From the construction of Y^k, above, all of the type 1 traders can share x_1 and all of the type 2's may share $x_2[k] \equiv x_2 - \frac{1}{k}(x_2 - w_2)$. It would then require z_k additional resources to minimize, with respect to Euclidean distance, the amounts of additional resources necessary for all the type 2 agents to be as satisfied as they are in $[X]^k$. If we double k, we may decrease the additional resources to make each type 2 agent as satisfied, but we cannot decrease the amounts by more than one-half -- i.e., $2\|z_{2k}\| = \|z_k\|$. Thus, the influence of any type 1 agent, measured in this way, does not decrease as k increases, and $\{[X]^k\}$ is not ANS.

To exhibit the consequences for the elasticity of demand, assume that other traders are transacting at the WE prices p. Then, the maximum exchange rate a seller of type 1 can set, without losing all his business, does not go to p as k increases. After the type 2 traders have purchased all that the $(k-1)$ type 1 traders are willing to supply at p, the remaining type 1 trader can offer any exchange rate ρ such that $p_1/p_2 < \rho < p_1'/p_2'$ and sell, when k is large, as much of commodity 1 as it likes.

REMARK 4: Let $\delta(A,B)$ denote the Hausdorff distance between two non-empty closed sets in R^{ℓ}. The condition (a) $\bigcap_{k=1}^{k=\infty} Q^{j(k)}(X) = Q^0(X)$ used to characterize NS in Theorem 3 is equivalent to (b) $\lim \delta(Q^{j(k)}(X), Q^0(X)) = 0$. Condition (b) will be shown elsewhere to replace (a) in economies that do not have a finite number of types of agents.

Let $W(\mathcal{E}_k)$ and $W(\mathcal{E}_k^{j(k)})$ be subsets of P denoting the (closed) sets of price vectors corresponding to WE allocations in $\mathcal{E}_k$ and $\mathcal{E}_k^{j(k)}$. Since $\lim \delta(W(\mathcal{E}_k^{j(k)}),W(\mathcal{E}_k))$ does not exist in Example 2, it might be conjectured that this is the source of non-vanishing monopoly power of individual agents and that (c) $\lim \delta(W(\mathcal{E}_k^{j(k)}),W(\mathcal{E}_k)) = 0$ is an alternative to $\lim \delta(Q^{j(k)}(X), Q^0(X)) = 0$ as a characterization of ANS. This conjecture is false in both directions - (b) neither implies nor is implied by (c).

To show that (c) does not suffice for (b), simply perturb the initial endowments in Example 2 so that $W(\mathcal{E}_k) = \{p\}$ for all k but the WE allocation remains at the point where preferences are kinked. It may then be demonstrated that (c) holds. However, as long as preferences are not differentiable at the WE allocation, the same argument exhibiting the failure of ANS in Example 2 leads to the same conclusion here.

To show that (c) is not necessary, consider an Edgeworth-Bowley box example where preferences are differentiable in the interior and there are a continuum of WE prices and allocations for each $\mathcal{E}_k$. Examples may be constructed to show that $W(\mathcal{E}_k^{j(k)})$ does not approach $W(\mathcal{E}_k)$ - i.e. (c) fails. (For a graphical illustration see Bewley [1973], Example 2.) Nevertheless, the smoothness of preferences implies that as k increases the total surplus attributable to any agent at any Walrasian allocation goes to zero. Even though the removal of any one would cause an abrupt change in WE prices, no agent can usefully exploit this. The failure of "markets" to clear when an agent threatens to withdraw his supplies if he does not receive a higher price may be essentially confined to the disappointed demands of the very trader who is asking for more.

V. NO-SURPLUS AND CORE EQUIVALENCE AS ALTERNATIVE CHARACTERIZATIONS OF PERFECTLY COMPETITIVE EQUILIBRIUM

The presence of external effects vitiates any comparisons between the core, which is limited to allocations that are truly Pareto-optimal, and NS. (See Remark 1.) Thus, in the following discussion A_5 is assumed. It will be useful to separate the comparisons for $\mathcal{E}$ from those for $\mathcal{E}_\infty$.

$\mathcal{E}$: Let T be any non-empty subset of $I = \{1,\ldots,n\}$, the index set of agents. Define $w_T = \Sigma_{i\in T}\, w_i$, $S^T(X) = \Sigma_{i\in T}\, S_i(X)$ and $Q^T(X) = \{p \in P : \inf p[S^T(X) - w^T] \geq 0\}$

An allocation X is said to be in the core of $\mathcal{E}$ if for all T, $w^T \notin S^T(X)$. With A_1 - A_5, it may be shown that X is in the core if and only if for all T, $Q^T(X) \neq \emptyset$.

Consider the following four conditions:

$$\text{(i)} \quad Q^j(X) = Q^0(X), \quad j = 1,\ldots,n \qquad \text{(NS)}$$

$$\text{(ii)} \quad \bigcap_j Q^j(X) \neq \emptyset \qquad \text{(WE)}$$

$$\text{(iii)} \quad Q^T(X) \neq \emptyset, \text{ all } T \subseteq I \qquad \text{(CORE)}$$

$$\text{(iv)} \quad Q^j(X) \neq \emptyset, \quad j = 1,\ldots,n \qquad \text{(NNS)}$$

Obviously, (i) $\mapsto$ (ii) and (iii) $\mapsto$ (iv), where $\mapsto$ means "implies but is not implied by." Further, it is well-known that (ii) $\mapsto$ (iii).

The core criterion for $\mathcal{E}$ to be perfectly competitive is that (iii) $\Rightarrow$ (ii). The NS criterion is (iv) $\Rightarrow$ (i), assuming the linear independence hypothesis in Corollary 1.3. The relation between the two criteria is summarized by,

PROPOSITION 3: $[(\text{iv}) \Rightarrow (\text{i})] \mapsto [(\text{iii}) \Rightarrow (\text{ii})]$.

The implication $\Rightarrow$ follows from the fact that (i) implies all of the other conditions. The following example demonstrates $\mapsto$.

EXAMPLE 3: Let tastes be defined by the utility functions $u_i(x_i) = u_i(x_{i1}, x_{i2}, x_{i3})$, $i = 1,2,3$ and 4, and let initial and final allocations be given by W and X^*, respectively, where

$$u_1(x_1) = (x_{11}x_{12}x_{13})^{1/3}$$
$$u_2(x_2) = (x_{21}x_{22}x_{23})^{1/3}$$
$$u_3(x_3) = (x_{31}x_{32}x_{33})^{1/3}$$
$$u_4(x_4) = \frac{x_{41} + x_{42} + x_{43}}{3}$$

; W =

i \ c		
3	0	0
0	3	0
0	0	3
1	1	1

; X^* =

i \ c		
1	1	1
1	1	1
1	1	1
1	1	1

With little difficulty, the reader may verify that X^* is a WE allocation at the price vector $p^* = (r,r,r)$, $r > 0$, that yields the utility vector $u^* = (u_1^*,u_2^*,u_3^*,u_4^*) = (1,1,1,1)$.

An outline of the argument that X^* is the only allocation in the core is as follows: The coalition $\overline{123}$ can achieve any (u_1,u_2,u_3) such that $u_1 + u_2 + u_3 = 3$, and 4 can achieve $u_4 = 1$ on its own. It follows that for any $u = (u_1,u_2,u_3,u_4)$ in the core, $u_1 + u_2 + u_3 = 3$ and $u_4 = 1$, and if $u_i < 1$, $i \neq 4$, then because $w^T \in \text{cl } S^T(X^*)$, $T = \overline{i4}$, the coalition $\overline{i4}$ will upset u. This means that $u = (1,1,1,1)$ and this is achievable only by the allocation X^*.

X^* does not satisfy NS -- $w^T \notin \text{cl } S^T(X^*)$ for $T \in \{\overline{124}, \overline{134}, \overline{234}\}$.

The core does not capture the monopoly power of individuals 1,2 or 3. For example, without 1, the WE price vector for the economy $\mathcal{E}^1$ is $p = (2r,r,r)$. In terms of prices, 1's contribution or marginal product may be measured by the decrease in the WE price of the commodity he supplies. From a non-cooperative point of view, there does not appear to be any reason why 1 should surrender all of the surplus represented by this price decrease to the other traders.

Such a finding has already been reported for economies composed of a (non-atomic) continuum of agents and large or atomic agents. See Gabszewicz and Mertens [1971] and Shitovitz [1973]. They constructed classes of economies exhibiting Equivalence $[(iii) \Rightarrow (ii)]$, but the presence of the large traders precludes $[(iv) \Rightarrow (i)]$. Example 3 shows that such results also occur in purely atomic economies.

$\mathcal{E}_\infty$: Let I_k be the index set of agents in $\mathcal{E}_k$. The analogous conditions are quite similar.

(i) $Q^{j(\infty)}(X) = Q^0(X), \quad j = 1,\dots,n$ (NS)

(ii) $\cap_j Q^{j(\infty)}(X) \neq \emptyset$ (WE)

(iii) $Q^T(X)$, all $T \subseteq I_k$, $k = 1,2,\dots$ (CORE)

(iv) $Q^{j(\infty)}(X) \neq \emptyset, \quad j = 1,\dots,n.$ (NNS)

Again, the core criterion for $\mathcal{E}_\infty$ to be perfectly competitive is $[(iii) \Rightarrow (ii)]$ and NS is $[(iv) \Rightarrow (i)]$. Example 2 shows that Proposition 3 also holds for $\mathcal{E}_\infty$.

The result $[(iii) \Rightarrow (ii)]$ was initiated by Edgeworth [1881], revived by Shubik [1959], and put into its current form by Debreu and Scarf [1963]. (Hildenbrand [1974] gives the considerable extensions of this result.) It appears to require a different method of proof than $[(iv) \Rightarrow (i)]$ and it does not hold in quite the same generality. However, when $Q^0(X) = \{p\}$, (i) - (iv) coincide and $[(iii) \Rightarrow (ii)]$ is an immediate implication of the Lemma in Section IV. In this respect, our approach is closely related to the simplication of the Debreu-Scarf result obtained by Hansen [1969].

REMARK 5: In the demonstration of $[(iv) \Rightarrow (i)]$, allocations were restricted to those satisfying equal treatment - allocations of the form $[X]^k$. With the core approach, $[X]^k$ may be derived as a conclusion. This can also be demonstrated via NNS. In fact, the NNS condition may be used to exhibit the analogue of $[(iv) \Rightarrow (i)]$ for sequences of economies in which agents are not drawn from a fixed, finite set of types. This result, in Ostroy [1977], parallels a core convergence theorem of Nishino [1971].

If we adopt a generic point of view, distinctions between the core criterion and NS vanish. For $\mathcal{E}_\infty$, they coincide when $Q^0(X) = \{p\}$ and for $\mathcal{E}$, even though endless cases such as Example 3 may be constructed, the fact remains that within the entire class of finite economies each criterion says that the set of perfectly competitive economies is negligible.

It is only at the conceptual level that differences emerge. With the core criterion, perfectly competitive equilibrium is obtained as the residual outcome after all groups of agents cooperate in the interests of improving upon any given allocation but no group is able to hold together to use its potential monopoly power to extract a more favorable outcome. With NS, just the opposite occurs. Only small groups (individuals) are able to form but they are relied upon to bargain as monopolists for the maximum they can possibly extract. It is remarkable that these two seemingly contradictory approaches should "almost always" yield the same conclusion. The explanation is that the economic forces permitted and precluded by the one are almost always counterbalanced by what is precluded and permitted by the other. A closer look at the discrepancies, such as Example 3 for $\mathcal{E}$, Example 2 for $\mathcal{E}_\infty$ (due to Edgeworth!), or the results of Shitovitz and Gabszewicz and Mertens, show that what is permitted and precluded by the one is not counterbalanced by the other. Which interpretation - the core or NS criterion -

is the preferred description of what will frequently be equivalent mathematical conditions characterizing perfectly competitive equilibrum should be judged on its eventual connections to the theory of imperfect competition.

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