THE FAMILY AS AN INCOMPLETE ANNUITIES MARKET

by

Laurence J. Kotlikoff
University of California, Los Angeles
and the National Bureau of Economic Research
and

Avia Spivak
Ben Gurion University

THE FAMILY AS AN INCOMPLETE ANNUITIES MARKET

by

Laurence J. Kotlikoff
University of California, Los Angeles
and the National Bureau of Economic Research

and

Avia Spivak
Ben Gurion University

The institution of the family provides individuals with risk sharing opportunities which may not otherwise be available. Within the family there is a degree of trust and a level of information which alleviates three key problems in the provision of insurance by markets open to the general public, namely moral hazard, adverse selection, and deception. In addition, provision of insurance within the family may entail smaller transaction costs than arise in the purchase of insurance on the open market. There are a number of important risks for which the "public" market problems of moral hazard, adverse selection, and deception are especially severe. The risk of loss of job or earnings because of changes in the pattern of demand or partial disability is one example. Here the ability of the "public" market to determine the extent to which the individual contributed to his earnings loss or is simply lying about his backache is highly questionable. Other examples are the risk of bankruptcy and the default risk on personal loans. Many family practices in dealing with these types of risks can be explained as implicit insurance contracts made _ex ante_ by completely selfish family members. Love and affection may be important for the enforcement of these implicit contracts, but these need not be their sole or even chief

determinant. Healthy brother A's support for disabled brother B may simply be the _quid pro quo_ for brother B's past implicit promise to support A if A had become disabled instead of B.

The informational advantages of pooling risk within families must be set against the inability of families to provide complete insurance because of the small size of the risk pooling group. The theory of insurance invokes the law of large numbers to demonstrate that a sufficiently large group of people can capture all the gains from risk pooling; the theory does not tell us how far a small group of people can go in pooling risk. The rate at which diminishing returns to additional people sets in in risk pooling appear to be an unexplored topic in the economic literature

Understanding how well family risk sharing arrangements substitute for market risk sharing arrangements is important not only for explaining the excess insurance demand facing public markets given the existence of families, but also for understanding the economic incentives for marriage and family formation when complete and fair public insurance markets do not exist.

This paper is concerned with the provision within the family of insurance against the risk of running out of consumption resources because of greater than average longevity. For a single individual the problem is how to eat a pie over time when one does not know how long one will continue to live. Too much consumption when young may mean relative poverty later on if one lives "too long"; alternatively, if one is excessively frugal when young, there is the risk of dying without ever having satisfied one's hunger. A complete annuity market permits an individual to hedge this uncertainty of the date of death by exchanging his initial pie for a stream of pie slices that continue as long as the individual survives. This

paper demonstrates that implicit contractual arrangements within marriage and the family can substitute to a large extent for the purchase of annuities in public markets. Risk sharing arrangements within the family effectively constitute an incomplete annuities market. Our analysis indicates that these arrangements even in small families can substitute by more than 70% for complete annuities. Given the adverse selection problem and transactions costs in public annuity markets, contracting within an incomplete family annuity market may well be preferred to purchasing annuities in the more complete public market.[1] In the absence of organized public markets in annuities, these risk sharing arrangements provide powerful economic incentives for marriage and family formation.

Throughout the paper individuals are assumed to be completely selfish, i.e., they obtain utility only from their own consumption. One implication of this approach is that voluntary transfers from children to parents or bequests and gifts from parents to children need have nothing at all to do with altruistic feelings; rather, they may simply reflect risk sharing behavior of completely selfish individuals. However, some level of mutual trust and honesty is required since elements of these arrangements are not legally enforceable.

This paper is divided into four sections. The first section describes optimal consumption behavior for a single individual both in the presence of and in the absence of fair annuity markets. We discuss the income and substitution effects involved in the provision of fair annuities and calculate the welfare gains from access to a complete and fair annuity market for the case of the iso-elastic utility function.

Section two develops the theoretical argument for Pareto efficient implicit family annuity contracts and explores potential welfare gains arising from these

arrangements. We present a proof that this implicit family contracting converges to a complete annuities market as the number of family members increases. Although the complexity of the calculations precluded analysis of large families, we do present quantitative results for families of two and three persons. Comparing the two person outcome with that for three persons indicates the degree of diminishing returns to the size of the risk sharing pool. The analysis considers cases in which family members both do and do not have identical survival probabilities (are of similar ages). This framework permits us to ask whether marriage between individuals with similar survival probabilities is more efficient than marriage between individuals with dis-similar survival probabilities.

Optimal family annuity contracts involve agreements on the consumption path of each family member as well as a commitment on the part of each member to name the other members as sole heirs in his estate. Section three discusses the problems of enforcing both aspects of these agreements. In particular, we contrast the Pareto efficient family annuity contracts with the equilibrium that arises when each family member cheats on the other in the sense of consuming in excess of the optimal family plan. Section four summarizes the paper and suggests areas for future research.

I. A Single Person's Consumption Plans
With and Without Fair Annuities [2]

In the absence of an annuity market a single individual's consumption choice problem is to maximize his expected utility (1) from current and future consumption subject to the budget constraint, (2):

(1) $$EU = \Sigma_{t=0}^{D} P_t \, U(C_t)$$

(2) $$\Sigma_{t=0}^{D} C_t R^{-t} = W_o$$

The P_t of equation (1) are probabilities of surviving from age zero through age t. P_o equals one. D is the maximum longevity. For simplicity, we assume the utility function is separable in consumption (C_t) over time. In (2) R, the discount factor, is one plus the interest rate. W_o is the initial wealth of the individual; we ignore possible future earnings streams.

The budget constraint written in equation (2) is identical to the budget constraint that would arise in a certainty world in which individuals never died before age D. While individuals will, on the average, die prior to age D, equation (2) reflects the non-zero probability that an individual will live through age D, i.e., equation (2) is the relevant budget constraint because the individual may actually live through age D, in which case his realized present value of consumption cannot exceed his budget.

Let us now assume that the single person is free to purchase actuarially fair annuities in a complete public annuities market. The budget constraint in this case is:

$$(3) \qquad \Sigma_{t=o}^{D} \; P_t \; C_t \; R^{-t} = W_o$$

In contrast with (2), (3) requires only an equality between the expected present value of consumption and initial wealth. The single individual now chooses his optimal consumption path by maximizing (1) subject to (3); he then exchanges his initial wealth W_o with the insurance company in return for its promise to pay out the C_t stream as long as the person continues to live.

The $P_t R^{-t}$'s in (3) may be thought of as prices. Since each of the P_t's in (3), except P_o which equals unity, is less than one, the consumption choice in the case of a fair annuity market is equivalent to the consumption choice without an annuity market, but with lower prices of future consumption.

Obviously, access to a fair annuity market increases utility by expanding the budget frontier; it also alters the optimal consumption path because of the income and substitution effects resulting from the lower prices of future consumption.

The iso-elastic utility function (4) is convenient for assessing the potential gains from access to a fair, public annuities market as well as the gains from family annuity arrangements.

$$(4) \qquad EU = \Sigma_{t=0}^{D} \; P_t \; \frac{c_t^{1-\gamma}}{1-\gamma} \; \alpha^t$$

In (4) γ is the constant relative risk aversion parameter, and α is the time preference parameter. By considering different values of γ we indicate for this family of utility functions how the gains from annuities and family arrangements depend on the specification of tastes.

In the no-annuities case maximization of (4) subject to (2) leads to the consumption plan (5):

$$(5) \qquad c_t = \frac{W_o \; (R\alpha)^{t/\gamma} \; P_t^{1/\gamma}}{\Sigma_{j=0}^{D} \; R^{j\frac{(\gamma-1)}{\gamma}} \; \alpha^{j/\gamma} \; P_j^{1/\gamma}}$$

In the case of fair annuities, maximizing (4) subject to (3) leads to:

$$(6) \qquad c_t = \frac{W_o \; (R\alpha)^{t/\gamma}}{\Sigma_{j=0}^{D} \; R^{j\frac{(\gamma-1)}{\gamma}} \; \alpha^{j/\gamma} \; P_j}$$

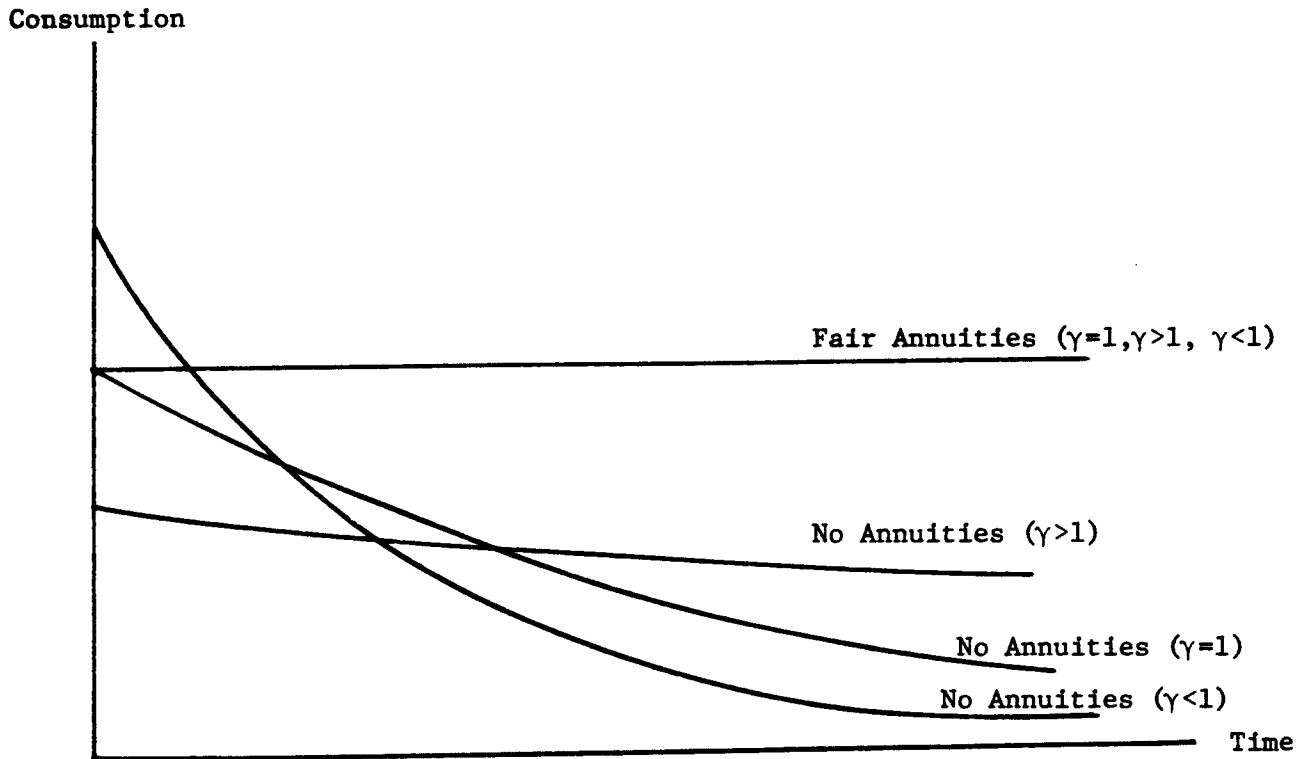In figure I we compare (5) and (6) for the case $R = \alpha = 1$;

Consumption

FIGURE I. Consumption Paths With and Without Fair Annuities

The ability to trade in a fair annuities market may raise or lower

initial consumption depending on whether γ is less than or greater than

unity. Intuitively, the higher is the degree of risk aversion, γ, the

greater is the concern for running out of money because of excessive long-

evity and, hence, the lower is initial consumption. At γ equal to infinity,

(5) dictates equal consumption in each period.

Plugging (5) or (6) into (4), we arrive at two indirect utility functions

for the no-annuity and annuity cases with initial wealth, the interest rate, and

survival probabilities as arguments. These functions are presented in equations

(7) and (8) respectively.

$$(7) \qquad H_0(W_0) = \frac{1}{1-\gamma} W_0^{1-\gamma} [\sum_{j=0}^{D} \alpha^{j/\gamma} R^{j(1-\gamma)} P_j^{1/\gamma}]^\gamma$$

$$(8) \qquad V_0^*(W_0) = \frac{1}{1-\gamma} W_0^{1-\gamma} [\sum_{j=0}^{D} \alpha^{j/\gamma} R^{j(1-\gamma)/\gamma} P_j]^\gamma$$

We measure the increase in utility resulting from access to fair annuities in terms of dollars. From (9) we solve for the value of M which represents the percentage increment in a single person's initial wealth required, in the absence of an annuity market, to leave him as well off as he would be with no additional wealth, but with access to an annuities market.

$$(9) \qquad H_0(MW_0) = V_0^*(W_0)$$

For the iso-elastic utility function this calculation turns out to be independent of the initial level of wealth. Table I reports values of M for different ages and levels of risk aversion using both male and female survival probabilities. The survival probabilities used in this and all subsequent calculations are actuarial estimates from the Social Security Administration.[3] Maximum longevity is taken to be 120 throughout the paper.

Table I indicates that the utility gain measured in dollars from access to an annuities market can be quite large. For a relative risk aversion parameter value of .75 the gain to a 55 year old male is equivalent to a 46.90 percent increase in his initial wealth. The utility gain is age dependent; for γ = .75, the 30 year old male's gain is 24.46 percent, while the 90 year old male's gain is 99.81 percent. This reflects the low probabilities of death when young; annuities are less important to young people because a large fraction of their lifetime utility from consumption is fairly certain due to their lower mortality probabilities in the immediate future. Higher levels of risk aversion naturally increase the gains from access to an annui-

I:

Percentage Increase in Initial Wealth Required to Obtain

Fair Annuities Utility Level*

Age	Relative Risk Aversion (γ)	Males	Females
30	.75	24.5	18.5
55	.75	46.9	34.4
75	.75	71.2	63.0
90	.75	99.8	100.2
30	1.25	30.3	22.7
55	1.25	59.2	43.4
75	1.25	97.0	85.3
90	1.25	152.6	152.9
30	1.75	34.7	26.1
55	1.75	68.9	50.7
75	1.75	119.1	104.6
90	1.75	199.1	199.4

*Throughout table α = .99 and R = 1.01

ties market. The male-female differences in the table reflect the higher male age specific mortality rates. The calculation is somewhat sensitive to the choice of α and R. Raising the interest rate to 5 percent while holding α constant increases the age 55 wealth equivalent factor from 46.90 to 55.57 for the case of γ = .75. The 90 year old wealth equivalent is increased from 99.81 to 115.34.

Income, Substitution Effects, and Unintended Bequests

Without access to an annuity market a single non-altruistic individual will always die prior to consuming all his wealth and, accordingly, make involuntary bequests. The level of these unintended bequests can be quite large. Using (5) we calculated the consumption path as well as the corresponding wealth path for the no-annuity case. Multiplying the probability of dying at each age times the wealth at each age and discounting back to the initial age gives the present expected value of unintended bequests. For γ = .75, R = 1.01 and α = .99, the present expected value of unintended bequests represents 24.47 percent of initial wealth for a single male age 55. This number means that a 55 year old male with no annuity market will, on average, fail to consume about one quarter of his wealth because he is risk averse. Increasing the risk aversion coefficient to 1.75 raises the ratio of present unintended bequests to initial wealth to .3583. These large, unintended bequests occur despite a fairly rapid rate of consumption. In table II we present the optimal age consumption and age wealth paths for a 55 year old single male with no access to annuities and initial wealth of $100,000. Even for high values of risk aversion, current mortality probabilities dictate a fairly rapid rate of consumption. For γ = 1.75 a single male surviving to age 85 consumes less than one third of his age 55 consumption level.

II:

Age Consumption and Age Wealth Paths

for Single Male with No Annuities

Age	Risk Aversion	Consumption	Wealth
55	.75	6825	$100,000
65	.75	4720	47,415
75	.75	2250	13,830
85	.75	395	1,420
95	.75	10	30
55	1.25	5465	100,000
65	1.25	4395	57,200
75	1.25	2810	23,675
85	1.25	990	5,165
95	1.25	110	475
55	1.75	4795	100,000
65	1.75	4100	62,680
75	1.75	2980	30,680
85	1.75	1415	9,505
95	1.75	295	1,690

$R = 1.01$ $\alpha = .99$

The homotheticity property of the iso-elastic utility function permits us to use the ratio of present unintended bequests to initial wealth in separating the utility gains from fair annuities into income and substitution effects.

Suppose a fair insurance company approached a single age 55 year old (γ = .75) male and offered to pay him 24.47 percent of his initial wealth in exchange for his naming the insurance company as his heir on his will. The single male would take the 24.47 percent gain and, because of homotheticity, consume it according to his original no-annuity consumption path. This additional wealth would give rise to an additional .2447 times .2447 present expected bequest. Letting the insurance company also pay for this second round expected bequest and letting the process continue until convergence, the insurance company ends up paying 32.40 = .2447/1-.2447 percent of the single individual's initial wealth. This 32.40 percent figure represents the utility gain from the pure income effect. In this scenario the individual continues to consume at the no-annuity set of prices. Since the total gain from being able to purchase fair annuities and thus face lower prices for future consumption is 46.90 percent, the income effect represents 69.08 percent and the substitution effect 30.92 percent of the total gain.

II. The Family as an Incomplete Annuities Market

Decisions by family members concerning consumption expenditures and inter-family transfers may reflect implicit, although incomplete, annuity contracts. For example, when two individuals get married they generally agree to pool their resources while both marriage partners are alive, and also to name each other as the major, if not the sole, beneficiary in their wills. For each partner the risk of living "too long" is somewhat hedged; if one partner lives to be very old, there is a high probability that his (or her) spouse had already

died leaving him (her) a bequest to help finance his (her) consumption.
While each spouse will gain just from the exchanging of wills, they can
further increase their expected utilities by agreeing on a joint consump-
tion path taking into account each spouse's expected bequest to the other.
The importance of joint consumption planning is highlighted in the case of
an implicit contract between a parent and a child. Here the parent impli-
citly promises to name the child in his will in exchange for the child's
implicit promise to care for the parent if the parent lives too long. Al-
though the child may have zero probability of dying while the parent is still
alive, both can gain because the child agrees to share consumption resources
with the parent.

Mutually advantageous arrangements between family members do not require
either equal consumption by each family member or equal initial endowments.
If an old male marries a very young female and each enter the marriage with
the same dowry, the young female can compensate the old male for his higher
expected bequests by consuming less than the male while they are both alive.
Alternatively, the old male and the young female could consume equally while
married; in this case the young female could compensate the old male for his
higher mortality probabilities by entering the marriage with a larger dowry.
The set of Pareto efficient family contracts is derived by maximizing a weighted
sum of each family member's own expected utility from consumption subject to
the constraint that consumption of surviving family members not exceed their
own resources plus inheritances received from deceased family members.

This view of bequest and consumption arrangements within marriage as an
incomplete annuity market becomes intuitive when one contemplates increasing
the number of members in the family. To simplify the issue, let us assume

that all individuals within the family have identical survival probabilities, and that they enter this multi-person family with identical resources. In the limit as the family, or perhaps "tribe" is a better word, gets large, the consumption path of an individual within the tribe converges to the path a single individual would choose in a complete and actuarially fair annuities market. While we prove this proposition for the multi-period case in appendix A, a two period model can exhibit the main idea.

Since all family members are identical, the family maximizes the expected utility of a representative family member's consumption in period zero, C_0, and consumption in period one, C_1. Let $N+1$ be the number of family members and $i+1$ be the number of members who survive to period one; the two period budget constraint is then:

$$(10) \qquad C_1 = (W_0 - C_0)R\left(\frac{N+1}{i+1}\right).$$

The term $R\left(\frac{N+1}{i+1}\right)$ is the random intertemporal rate of return. Each person in the family consumes C_0 of his W_0 endowment in period zero. For a period one survivor the return on savings reflects the interest rate plus his share of the bequests of those who die. Assuming that each member has an identical probability, P, of surviving to period one, the expected value of $\frac{N+1}{i+1}$ is $\frac{1}{P} - \frac{(1-P)^{N+1}}{P}$. The expected return is lower (the price of consumption in period one is higher) here than in the annuity case because of the possibility that all family members die simultaneously. As n goes to infinity this expected value converges to $\frac{1}{P}$. The variance of this return also converges to zero, implying that the family's consumption choices C_0 and C_1 converge to the choices made in a perfect annuity market.

Quantitative Analysis of Family Risk Pooling

In the case of two family members the frontier of efficient marriage
contracts is obtained as the solution to the following dynamic programming
problem:

$$(11) \quad V_{t-1}(W_{t-1}) = \max[u^H(C^H_{t-1}) + u^S(C^S_{t-1}) + \alpha P_{t/t-1} Q_{t/t-1} V_t(W_t)$$
$$W_t, C^H_{t-1}, C^S_{t-1} \geq 0, \quad t=T,\ldots,1$$

$$+ \alpha P_{t/t-1}(1-Q_{t/t-1}) H_t(W_t) + \theta \alpha Q_{t/t-1}(1-P_{t/t-1}) S_t(W_t)]$$

$$(12) \quad W_t/R + C^H_{t-1} + C^S_{t-1} = W_{t-1}$$

where
$$V_T(W_T) = \max_{C^H_T, C^S_T} u^H(C^H_T) + \theta u^S(C^S_T)$$

In (11) $V_t(W_t)$ is the period t maximum weighted expected utility of the
two family members with joint wealth W_t. In the expression the letters H and
S denote the two family members. C^H_t and C^S_t are the consumptions of the two
u^H and u^S are their utility functions and $P_{t/t-1}$ and $Q_{t/t-1}$ are their respec-
tive period t survival probabilities conditional upon surviving through period
t-1. $H_t(W_t)$ and $S_t(W_t)$ are the maximum expected utilities for each member if
he or she alone survives to period t. These expressions are obtained from
(7) by replacing W_0 and W_t and applying the appropriate probabilities. θ is
the differential weight applied to member S's expected utility.

The first two terms on the right hand side of (11) represent utility from
certain period t-1 consumption. The third term is the family's expected period

t utility multiplied by the probability that both members survive to period t. The last two terms represent expected utilities when one member dies and the other survives.

Appendix B presents an algorithm which we programmed on the computer to solve (11). The algorithm for solving the three family member maximization problems is available from the authors.

Marriage as an Annuity Contract

The solution to (11) permits us to contrast consumption paths and utility levels of married people with those of single persons, assuming throughout that there is no public annuities market. Both spouses are assumed to have identical iso-elastic utility functions in the sense of the same degrees of risk aversion and rates of time preference.

The shape of consumption paths for married couples while they are both alive may differ from that of single individuals for two reasons. First, even if the two spouses have identical survival probabilities, the reduction in risk within the marriage rate acts like a reduction in the price of future consumption. If the relative risk aversion parameter γ exceeds (is less than) one, the identical survival probabilities marriage profile will start above (below) the single person's profile. For γ equal to unity the profiles are identical (see Appendix). If we drew these consumption profiles for married persons in figure 1 they would start between the no annuity and complete annuity profiles.

The second reason why the consumption profiles for married people may differ from that of single people is possible differences in spousal survival probabilities. Higher survival probabilities act like lower rates of time preferences. When an old man marries a young female the slope of the optimal marriage consumption profile reflects the survival probabilities of both the

husband and the young wife. The two spouses compromise with respect to the
rate at which they should eat up their joint wealth while they are both alive.
The old husband would prefer to eat up the wealth more rapidly, and the young
wife would prefer to consume at a slower rate. The formula for each spouse's
consumption when married takes both spouses' survival probabilities into
account as well as the relative spousal utility weights.

In table III we report the gains from marriage between two individuals
who have identical survival probabilities, identical initial endowments,
and are weighted equally in the marriage contract. We report
marriage gains using both male and female survival probabilities.
The marriage gain is calculated as the percentage increase in a single
person's initial wealth needed to make him (her) as well as he (she) would be
in the marriage. The table also reports the marriage gain as a fraction
of the Table I total gain from complete and fair annuities. These utility
gains are measured in terms of their dollar equivalents. Since utility is
concave in wealth the dollar gain from marriage as a fraction of the dollar
gain from fair annuities is smaller than the actual utility gain from
marriage as a fraction of the utility gain from fair annuities. Table II
also reports this latter fraction which indicates how much of the utility
gain from annuities is captured by marriage.[4]

The figures in table III indicate that marriage can offer substantial
risk pooling opportunities. For a fifty-five year old man using male
survival probabilities, pooling risk through marriage is equivalent to
about a twenty percent increase in his wealth if he had stayed single. The
gains from marriage increase the older one becomes since the risks to be
incurred are much greater as one ages. At age seventy-five for a relative
risk aversion parameter of 1.25 getting married is equivalent to increasing
one's wealth by thirty percent. Death risk pooling through marriage can be

III:

The Annuity Gains from Marriage*

		Male Probabilities			Female Probabilities		
Age	Risk Aversion	Percent Dollar Marriage Gain	Dollar Marriage Gain/Dollar Annuity Gain	Utility Marriage Gain/Utility Annuity Gain	Percent Dollar Marriage Gain	Dollar Marriage Gain/Dollar Annuity Gain	Utility Marriage Gain/Utility Annuity Gain
30	.75	11.7	.478	.499	9.3	.504	.519
55	.75	20.0	.425	.461	15.6	.454	.482
75	.75	25.4	.357	.405	23.5	.373	.417
90	.75	28.2	.282	.339	28.3	.282	.338
30	1.25	13.0	.429	.470	10.2	.447	.479
55	1.25	22.3	.376	.446	17.4	.401	.456
75	1.25	30.1	.310	.419	27.6	.324	.414
90	1.25	37.1	.243	.367	37.2	.243	.367
30	1.75	13.6	.393	.456	10.6	.408	.457
55	1.75	23.5	.341	.451	18.5	.364	.450
75	1.75	33.2	.279	.449	30.5	.292	.436
90	1.75	43.0	.216	.420	43.0	.216	.420

*R equals 1.01 and α equals .99

quite important even at young ages. The table reports gains from 11.7 to 13.6 percent at age thirty using the male probabilities.

Marriage can also close much of the utility gap between no annuities and complete annuities. For example, for a fifty-five year old with risk aversion of .75, the marriage gain measured in dollars represents 42.5 percent of the full annuities gain measured in dollars. In terms of utility, marriage substitutes 46.10 percent for complete and fair annuities. Interestingly, marriage appears to be a somewhat better substitute for fair annuities at younger ages. In addition, there appears to be an interaction between age and the degree of risk aversion making marriage a better substitute for fair annuities at young ages when risk aversion is low and at old ages when risk aversion is high.

Three Person Polygamy and Diminishing Returns in Risk Pooling

We next consider the case of three individuals with identical survival probabilities and initial endowments who agree to maximize the sum of their utilities subject to the constraint that realized consumption can not exceed their pooled resources. The utility maximization is similar to (11); it takes into account each survival contingency, i.e., that either all three, two, one, or none of the individuals will be alive and may inherit at each point in the future.

Table IV records the utility gains from three person polygamy (three sisters would do just as nicely). Over a wide range of ages and parameter values three people appear to be capable of capturing about sixty percent of the gains from fair annuities. While the complexity of the calculations precluded considering a four person arrangement, we can conjecture using tables III and IV how well four people would do together. For example, for

IV:

The Annuity Gains from Three Person Polygomy

Age	Risk Aversion	Percent Dollar Gain	Polygomy Utility Gain/ Fair Annuity Utility Gain
30	.75	15.85	.665
55	.75	28.04	.632
75	.75	37.16	.571
90	.75	43.10	.496
30	1.25	18.05	.635
55	1.25	32.08	.612
75	1.25	45.18	.571
90	1.25	58.17	.524
30	1.75	19.33	.619
55	1.75	34.51	.613
75	1.75	50.74	.596
90	1.75	68.68	.579

Table uses male probabilities, $R = 1.01$, $\alpha = .99$

a fifty year old male with risk aversion of .75, adding one marriage partner

is equivalent to a twenty percent increase in his wealth had he remained

single. From table III the marginal dollar gain from adding a third person is

8.04 percent. If the marginal dollar gain fell at a constant rate in this range,

the fourth person would add 8.04 X $\frac{8.04}{20.0}$ = 3.23 percent.[5] By adding

3.23 to 28.04, we can roughly calculate the extent to which four people can

close the utility gap. This procedure suggests that four people can sub-

stitute by seventy percent for a fair annuities market.

Diminishing returns to risk pooling appears, then, to set in at a

fairly rapid rate. For this example, two people can substitute by 46 percent,

three people by 63 percent, and four people by over 70 percent for full in-

surance.

Is Marrying People of Similar Ages More Efficient?

Suppose you have to decide how to pair up four people, two who are old

and two who are young. Assume that the two old people have identical survival

probabilities, that the two young people have identical survival probabilities,

and that each of the four have the same initial wealth. Is it more efficient

to marry the old people together and the young people together than it is

to mix ages?

There appear to be two competing arguments involved here. On the

one hand marrying the old with the young minimizes society's expected loss

of resources arising when both marriage partners die early.[6] To see this,

assume that old people have a probability p of dying in the next period and

that young people have a probability q of dying in the next period. If we

marry the old with the old and the young with the young, there is a probability

p^2 that both the old people will die and a probability q^2 that both the young

will die in the next period. If both couples were bringing the same resources w into the second period, society's expected loss of resources equals $(p^2 + q^2)w$. If, however, we married the old with the young, the expected resource loss from the old spouse and the young spouse dying simultaneously is $2pqw$. Now $(p^2 + q^2 - 2pg)w = (p - q)^2 w$ and, hence, the expected loss in resources to society is greater from marrying the old with the old and the young with the young. Intuitively, marrying two ninety year olds together and two twenty year olds together leaves a large chance that both ninety year olds will die in the immediate future; any resources which they have failed to consume will be lost to the twenty year olds who, on average, will still be alive. This expected loss of resources to society (by society we mean the four people), will be greater the greater the degree of risk aversion, since high levels of risk aversion will lead our hypothetical ninety year olds to postpone consumption.

The contervailing argument against mixed age marriages is that mixing ages involves greater risk to one of the two partners; the utility cost of this greater risk may exceed the utility gain from the increase in expected resources arising in mixed marriages. To see this consider an old-young marriage in which the young person promises to consume less than the old person in the state of nature in which both spouses survive. Suppose that this promise to the old person of higher consumption in the "both survive" state is large enough to exactly compensate the old person for the loss in expected utility from the state in which his spouse dies, but he survives. The old person's expected utility from this latter "bequest" state is lower when he marries someone young, rather than someone old, because the probability of the young person actually dying is smaller. While the old person is by

assumption no worse off in this compensated old-young marriage, the young person could be worse off than had she (he) married someone young. By entering into this compensated old-young marriage, the young person reduces her (his) payoff in the "both survive" state while leaving the payoff in the "bequest" state unchanged. She (he) also increases the probability of the "bequest" state and lowers the probability of the "both survive" state. Although expected consumption for the young spouse rises, the spreading of the payoffs may lower expected utility depending on the young spouse's degree of risk aversion.

We investigated for our iso-elastic utility function potential efficiency gains from mixed marriage between two fifty-five year olds and two thirty year olds, where each individual was risk averse at the .75 level. The fifty-five year olds were given male survival probabilities, while the thirty year olds had female survival probabilities. From table III we know that the two fifty-five year olds would, by marrying, enjoy an increase in expected utility worth twenty percent of their initial wealth. If the two younger people married, their equivalent dollar gain would equal 9.31 percent for each.

Marrying the old with the young, assuming an equal consumption (equal weighting) marriage contract, would leave the old person worse off and the young person better off then if they were single. This reflects the fact that while both partners compromise on how fast they should consume their joint wealth, the older person is much more likely to die and therefore bequeath first. This equal consumption marriage gives the young person a 39.7 percent gain, while the old person loses an equivalent of 13.1 percent of his wealth. By weighting the utility of the old spouse more heavily than the utility of the young spouse in the marriage, we consider whether the utility gains to the young spouse are sufficient to compensate the old

spouse and leave both better off than had they married individuals
their own age.

Table V exhibits the old-young utility frontier measured in terms of
the equivalent dollar gains which result from differential weighting of the
old and young in the marriage. When risk aversion equals .75, weights
of 1.7 for the old person and 1 for the young person yield utility levels
for both old and young which exceed those in the old-old, young-young marriages.
The additional gain to the old person from this weighted marriage with the
young person is 3.1 percent; the added gain to the young person is 1.6
percent. The table also indicates the old spouse's share of total family
consumption while both spouses are alive; this share of consumption is deter-
mined by the utility weights.

When the risk aversion parameter is 1.75 a weight of 2.9 applied to
the old person's utility yields outcomes pareto superior to those in the old-
old, young-young marriages. Here the old person's additional gain is .7
percent, while the young person's extra gain is 5.2 percent.

These additional gains from mixed age marriages require, however, a
fairly skewed distribution of consumption within the marriage. In both of
the table V examples, the young-old weightingscheme that dominates old-old,
young-young coupling involve the older spouse consuming about 86 percent
more than the younger spouse while they are both alive. If it is too
costly to negotiate such an arrangement within the marriage, or if the type
of consumption (e.g., housing), within marriage is non-excludable, then equal
consumption marriages of individuals with similar survival probabilities (of
similar ages) will be the rule rather than the exception. Of course,
we have been discussing here marriages in which each spouse has

V:

The Mixed-Marriage Utility Frontier*

Old Person's Utility Weight	Old Person's Consumption Share	Degree of Risk Aversion	Old Person's Percent Dollar Gain	Young Person's Percent Dollar Gain
1	.50	.75	-13.1	39.7
1.1	.53	.75	- 6.7	34.7
1.2	.56	.75	- .1	30.5
1.3	.59	.75	4.8	25.6
1.4	.61	.75	10.0	21.5
1.5	.63	.75	14.7	17.6
1.6	.65	.75	19.1	14.0
1.7	.67	.75	23.1	10.6
1.8	.69	.75	26.9	7.3
1.9	.70	.75	30.4	4.3
2.0	.71	.75	33.7	1.5
1	.50	1.75	- 9.3	42.2
1.3	.54	1.75	- 1.1	36.1
1.6	.57	1.75	5.6	31.0
1.9	.59	1.75	11.0	26.7
2.3	.62	1.75	17.1	21.8
2.6	.63	1.75	20.9	18.7
2.9	.65	1.75	24.2	15.8

*Young person's weight is 1. R = 1.01, α = .99.

the same initial dowry. The old-young marriages can dominate old-old, young-young marriages even under an equal consumption arrangement provided the dowry of the young spouse exceeds that of the old spouse to a sufficient degree.

Incomplete Annuity Arrangements Among Multiple Family Members

In this section we consider incomplete annuity arrangements between two parents and one child and between one parent and two children. As in the case of a marriage between old and young spouses, parents and children can jointly pool the risks of uncertain death.

In table VI we present the gains from implicit annuity contracting within our two types of families. In both cases we assume equal consumption of all family members, but permit the initial wealth of the child or children to vary. All individuals are assumed to have the male survival probabilities; the children are age thirty and the parents age fifty-five. The family dynamic program is presented in the appendix. In each family we maximize the sum of the three individual's expected utility taking all survival contingencies into account. If two out of three family members die simultaneously, the third inherits the remaining family wealth. If one of the three dies first, the other two jointly inherit the remaining family wealth and consume according to the equal weight optimal marriage contract described above.

The numbers in table VI indicate the percentage dollar increment to wealth needed to make a single fifty-five year old or thirty year old with the indicated initial wealth as well off as had he (she) participated in a consumption sharing family arrangement.

VI:

Gains from Incomplete Annuity
Arrangements in the Family*

Initial Wealth of Each Child	Two Parents with One Child		Two Children with One Parent	
	Percent Dollar Gain to Parent	Percent Dollar Gain to Child	Percent Dollar Gain to Parent	Percent Dollar Gain to Child
25,000	14.4	34.2	2.3	24.8
30,000	23.2	20.4	16.9	18.9
35,000	32.0	10.6	31.5	14.6
40,000	40.8	3.2	46.1	11.5

*The Calculations assume equal consumption by all family members.
 Initial wealth of parent or parents is $20,000. R = 1.01, α = .99,
 and γ = .75.

A comparison of table I with table VI indicates how well these small family arrangements can substitute for complete annuity contracts. For example, in the case of two parents with one child, if the child has an initial wealth of 35,000 and the parents have an initial wealth of 20,000, entering into an equal consumption-will swapping arrangement is equivalent for each parent to a 32 percent increase in wealth and for the child to a 10.6 percent increase in wealth. For each parent this arrangement captures 71.2 percent of the utility gain from full annuities; for the child the arrangement substitutes by 45.4 percent for full annuities.

The last two columns of table VI tell a very similar story. Here we are dealing with the case of two children contracting with one parent. When each child contributes $35,000, the gain to the parent is 31.5 percent, while each thirty year old child enjoys a 14.6 percent gain relative to consuming as a single person. The numerical differences in the table for the two different types of families reflect, on the one hand, different monetary contributions of parents relative to children and, on the other hand, differences in the rate at which resources are consumed when all families are alive. Resources are consumed at a slower rate in the two children-one parent case than in the one child-two parent case since each individual's survival probabilities are given equal weight in determining the optimal rate of consumption.

III. Enforcement with and without Altruism

There are two elements in the marriage contracts we have described which raise questions of enforcement. One problem is that a spouse may covertly name a third party as beneficiary on his will in exchange for the same commitment by the third party or simply in exchange for a particular

service. This type of cheating is, however, dangerous. There is always
the possibility that the other spouse will demand inspection of the will,
or, even if a second will is hidden away, that the spouse will somehow become
aware of it. Such disclosure could destroy the marriage, leaving the
cheating spouse with a soiled reputation and unable to engage in additional
implicit annuity contracting with either children or a new spouse. In
addition, the state, to a limited extent, plays a role in enforcing the
initial implicit marriage contract. Thus, the courts will rule invalid a
last minute change in a will which, for example, endows a death bed nurse to the
exclusion of a wife who has lived with and perhaps taken care of the decedent
husband for years.

A second type of cheating is simply that one or both spouses consumes
in excess of the consumption levels dictated by an optimal implicit
marriage contract. While each spouse may correctly believe that he or she
is the beneficiary on the other spouse's will, each may try to take advantage
of the other by increasing his (her) own consumption and thus reducing the
potential bequest available to the other spouse. The ability to observe one's
spouse's consumption within the marriage should effectively internalize
and eliminate this free-rider problem.

This second type of consumption cheating as well as the first type of
cheating, viz., multiple will swapping, may be more problematic for implicit
incomplete annuity agreements between friends or relatives who are physically
separated. The consumption cheating scenario can be modeled as a Nash equili-
brium in which each partner chooses his consumption path taking the
other partner's consumption path and, therefore, potential bequest path as
given. The first order conditions for this nash equilibrium strategy are

available from the authors. Resources are consumed at a faster rate in the Nash equilibrium as each partner fails to consider how his consumption will diminish his expected bequest and thus the expected utility of his partner.

Using male survival probabilities we calculated for two fifty-five year olds the dollar equivalent utility gain relative to being single from engaging in a Nash consumption cheating partnership. The utility gains in the Nash equilibrium proved to be almost identical to those in the more efficient marriage contract. At levels of risk aversion (γ) of .75, 1.25, and 1.75 the percentage dollar increments are respectively 19.9, 22.2, and 23.5. While the rate of consumption is faster in the Nash equilibrium, it is not much faster than in the marriage contract. For example, when risk aversion is .75 and the interest rate and rate of time preference equal one percent, age fifty-five consumption in the Nash equilibrium equals 6.5 percent of age fifty-five wealth; in the marriage contract age fifty-five consumption is 6.4 percent of age fifty-five wealth. Intuitively cheating by over consuming is fine provided one's partner actually dies; but if one's partner survives then the early excessive consumptions will require relative deprivation later on. Apparently this latter consideration dominates the former leaving utility in the cheating equilibrium at essentially the same level as under a marriage contract. These examples suggest that consumption cheating does not represent a substantial impediment to consumption risk sharing arrangements.

Aside from the questions of third party will swapping, there appear to be ways of structuring the payments of individuals within the family so as to insure the viability of these implicit contracts. An equal consumption marriage contract between two individuals with the same

survival probabilities and same initial endowments is a good first example. If each spouse maintains control over his own wealth while both spouses are alive and consumes at the same rate as the other spouse, then each will separately have an incentive to continue the contract at every point in time. A similar type of individual control can be maintained in family arrangements; rather than have the parents use up all their resources before the children begin to contribute to their support, the children can contribute each period in return for that period's expected parental bequest. This scenario of parents maintaining control over their wealth until the very end as enforcement leverage over their children may partly explain the limited use of gifts as a tax saving intergenerational transfer device.

Another means of enforcing these implicit contracts is simply altruism. All of our calculations have involved maximizing a weighted sum of individual family member's utilities. If, however, each family member is altruistic towards each other and each weights each family member's utility from con- sumption in the same way, then all family members would unanimously agree on the utility maximand. The calculations we have presented can, therefore, be thought of as resulting from the maximization of an agreed upon altruistic family utility function. Since all family members agree on the maximand, there is no problems of enforcement.

This discussion suggests the empirical difficulties if not impos- sibility of determining whether intergenerational transfers reflect altruism or simply risk mitigating arrangements of essentially selfish individuals in the absence of perfect insurance markets. Distinguishing between the

selfish and altruistic models is fundamental to a number of major economic
questions including the impact of the social security system on national
saving and the effectiveness of fiscal policy.[7]

V.   Summary and Conclusions

This paper has demonstrated that consumption and bequest sharing
arrangements within marriage and larger families can substitute to a large
extent for complete and fair annuity markets. In the absence of such public
markets, individuals have strong economic incentives to establish relation-
ships which provide risk mitigating opportunities. Within marriages and
families there is a degree of trust, information, and love which aids in
the enforcement of risk sharing agreements. Our calculations indicate
that pooling the risk of death can be an important economic incentive
for family formation; the paper also suggests that the current instability
in family arrangements may, to some extent, reflect recent growth in pension
and social security public annuities. The methodological approach of this
paper can be applied to the study of family insurance against other types
of risks. Of chief interest are those types of risks which anonymous public
markets handle very poorly. Disability insurance and insurance against
earnings losses are good examples.

Our approach has been to compare family insurance with perfect insurance.
It would seem worthwhile to compare family insurance with public market in-
surance where the market insurance is subject to adverse selection and moral
hazard problems and family insurance is not. Realistic specification of the
degree of adverse selection and moral hazard may indicate that family insurance
dominates public market insurance even in small families.

FOOTNOTES

[1]The transaction costs we have in mind here include the time costs involved in negotiating individual specific annuity contracts. As we demonstrate in the text, each individual's optimal annuity contract depends on his rate of time preference, his degree of risk aversion, and his survival probabilities. Some individuals may prefer a constant annuity stream; others an increasing or decreasing stream of annuity payments.

[2]Yaari (1965) is the pioneering paper on this subject.

[3]We use the low morality male and female probabilities reported on pages 17 and 19 of the Social Security Administration Actuarial Study No. 62.

[4]This fraction is calculated as $\frac{(1+m)^{1-\gamma} - 1}{(1+a)^{1-\gamma} - 1}$, where m is the fractional wealth equivalent gain from marriage, and a is the fractional wealth equivalent gain from fair annuities.

[5]This is probably a lower bound estimate for the contribution of the fourth person; the marginal dollar gain can't fall at a constant 40 percent rate forever, because if it did the total dollar gains would, in the limit, not sum up to 46.9 percent, the full annuity gain of table 1. Presumably the marginal dollar gain falls at a decreasing rate and 3.23 percent probably underestimates the fourth person's marginal contribution.

[6]We are grateful to Finis Welch for suggesting this line of argument.

[7]See Barro (1974).

References

Barro, Robert J., "Are Government Bonds Net Wealth?" Journal of Political
Economy, 82, No. 6, Nov./Dec. 1974.

Barro, Robert J., and Friedman, James W., "On Uncertain Lifetimes," Journal
of Political Economy, 85, No. 4, Aug. 1977: 843-49.

Feldstein, Martin S., "Social Security, Induced Retirement, and Aggregate
Capital Accumulation," Journal of Political Economy, 82, No. 5, 1974.

Royden, H. L., Real Analysis, Second Edition, Toronto, Canada: MacMillan
Co., 1968.

U.S. Dept. of H.E.W., Social Security Administration, United States Population
Projection for OASDHI Cost Estimates, Actuarial Study, No. 62, Dec.
1966.

APPENDIX A

Proof that Family Annuity Contracting Converges to a Complete Annuities Contract

As we discuss in the text, the family budget constraint, assuming identical family members, converges to the budget constraint that a single individual would face in a complete annuities market. We now demonstrate that as family size increases family annuity contracting leads to a consumption path for each identical family member that converges to the consumption path of an individual with perfect annuities. In addition, each family member's expected utility converges to the expected utility with complete annuities. The proof is based on Jensen's inequality and Fatou's lemma.

For a family with N+1 identical members the optimal consumption of each member at time t-1, C_{t-1}, is obtained as the solution to the following recursive problem:

$$(A1) \qquad V_{t-1}^N(W_{t-1}) = \max u(C_{t-1}) + \alpha q_t \sum_{i=0}^N V_t^i(W_t \frac{N+1}{i+1}) \, q_t^i(1-q_t)^{N-i}$$

subject to:

$$\frac{W_t}{R} + C_{t-1} = W_{t-1}, \qquad 0 \le C_{t-1} \le W_{t-1}$$

and

$$V_T^N(W_T) = u(W_T) \quad \text{for all } N \ge 1$$

In (A1) α is the time preference factor, and q_t is the probability of survival through period t conditional upon surviving through t-1.

Let (C_{t-1}^N, W_t^N) be the optimal solution to (A1). Since we assume strict concavity of u, the solution is unique. Also let (C_{t-1}^*, W_t^*) be the optimal solution to an individual's choice problem in a complete annuities market. This problem may be written recursively as:

(A2) $\qquad V^*_{t-1}(W_{t-1}) = \max u(C_{t-1}) + \alpha q_t \, V^*_t(W_t/q_t)$

subject to:

$$\frac{W_t}{R} + C_{t-1} = W_{t-1}, \qquad 0 \le C_{t-1} \le W_{t-1}$$

and

$$V^*_T(W_T) = u(W_T)$$

It can easily be shown that V^N and V^* inherit the strict concavity, monotonicity, and differentiability of u.

We now state our main result:

__Theorem:__ For all $t = 0, \ldots, T$, $\lim\limits_{N \to \infty} C^N_t = C^*_t$;

$$\lim\limits_{N \to \infty} W^N_t = W^*_t;$$

$$\lim V^N_t(W_t) = V^*_t(W_t) \qquad \text{for all } W_t.$$

The proof of this theorem is arranged in a sequence of one remark and four lemmas.

We define the random variable $X^N = \dfrac{N+1}{i+1}$, where i+1 is the number of survivors out of N+1 people when each has probability p of surviving.

__Remark 1:__ $E(X^N) = \dfrac{1}{P} - \dfrac{(1-P)^{N+1}}{P}$.

__Proof:__ $\qquad E(X^N) = (N+1) \, \Sigma^N_{i=0} \dfrac{1}{1+i} \binom{N}{i} P^i (1-P)^{N-i}$

Using the combinatorial identity $\dfrac{1}{1+i} \binom{N}{i} = \dfrac{1}{N+1} \binom{N+1}{1+i}$, we obtain

$$E(X^N) = \frac{N+1}{N+1} \Sigma^N_{i=0} \binom{N+1}{1+i} P^i (1-P)^{N-i} = \frac{1}{P}[\Sigma^{N+1}_{j=0} \binom{N+1}{j} P^j (1-P)^{N+1-j} - (1-P)^{N+1}] =$$

$$= \frac{1}{P} - \frac{(1-P)^{N+1}}{P}$$

We use recursive induction on t and Remark 1 to prove the following lemma.

<u>Lemma 1</u> For all $N \geq 0$, $V_t^*(W_t) \geq V_t^N(W_t)$.

<u>Proof</u>: For $t = T$ the condition clearly holds. We now prove the condition holds for $t-1$, assuming it holds for t.

$$V_{t-1}^N(W_{t-1}) = u(C_{t-1}^N) + \alpha q_t \Sigma_{i=0}^N V_t^i(W_t^N \frac{N+1}{i+1}) \, q_t^i (1-q_t)^{N-i}$$

$$\leq u(C_{t-1}^N) + \alpha q_t \Sigma_{i=0}^N V_t^*(W_t^N \frac{N+1}{i+1}) \, q_t^i (1-q_t)^{N-i}$$

$$= u(C_{t-1}^N) + \alpha q_t \, E \, V_t^*(W_t^N X^N) \leq u(C_{t-1}^N) + \alpha q_t V_t^*(E \, W_t^N X^N)$$

The first inequality follows from the induction assumption; the second follows from Jensen's inequality. From Remark 1 we know:

$$E \, W_t^N X^N = W_t^N \frac{1}{q_t} \, (1- (1-q_t)^{N+1}) \leq \frac{W_t^N}{q_t} \, .$$

So,

$$V_{t-1}^N(W_{t-1}) \leq u(C_{t-1}^N) + \alpha q_t V_t^*(\frac{W_t^N}{q_t})$$

$$\leq u(C_{t-1}^*) + \alpha q_t V_t^*(\frac{W_t^*}{q_t}) = V_{t-1}^*(W_{t-1}).$$

The last inequality holds because (C_{t-1}^*, W_t^*) are maximizing values of (A2).

<u>Corollary 1</u>: $\lim_{N \to \infty} V_t^N(W_t) \leq V_t^*(W_t)$ for all $t = 0, \ldots, T$, $W_t \geq 0$.

The following lemma establishes a technical property of $\{V^N\}_{N=0}^\infty$.

<u>Lemma 2</u>: $\{V^N\}_{N=0}^\infty$ constitutes an equi-continuous family of functions.[*]

<u>Proof</u>: It is sufficient to show that $\frac{\partial V_t^N}{\partial W_t}(W_t)$ is uniformly bounded by a number $M_t(W_t)$ from above and by a number $m_t(W_t)$ from below. W_t is held

_____

[*]For a discussion of this concept see Royden, p. 177.

constant throughout the proof; the proof is by induction. For t=T, $M_T = m_T = u'$.

To start the induction we note two facts:

(A3)
$$\frac{\partial v^N_{t-1}}{\partial W_{t-1}}(W_{t-1}) = u'(C^N_{t-1})$$

(A4)
$$u'(C^N_{t-1}) = \alpha q_t R \sum^N_{i=0} \frac{\partial v^i_t}{\partial W_t}(W^N_t \frac{N+1}{i+1}) \, q^i_t (1-q_t)^{N-i}$$

(A4) is the first order condition for a maximum of (A1). (A3) is sometimes

referred to as an "envelope theorem" and may be obtained by noting that the

two sides are equal to the same Lagrange multiplier for the constrained

maximization problem (A1).

Since u' is declining and $C^N_{t-1} \le W_{t-1}$, $\frac{\partial v^N_{t-1}}{\partial W_{t-1}} \ge u'(W_{t-1})$

$u'(W_{t-1})$ is, therefore, the uniformly lower bound for all N, $m_{t-1}(W_{t-1})$. To obtain an

upper bound we note that:

$$\frac{\partial v^i_t}{\partial W_t}(W^N_t \frac{N+1}{i+1}) \le \frac{\partial v^i_t}{\partial W_t}(W^N_t) \le M_t(W^N_t).$$

The first inequality holds because the derivative of a concave function is

declining; the second follows from the induction assumption. These expres-

sions and (A4) imply:

$$u'(C^N_{t-1}) \le \alpha q_t \, RM_t(W^N_t).$$

Now, let $\hat{C}_{t-1}$ and $\hat{W}_t$ satisfy $u'(C_{t-1}) = \alpha q_t \, RM_t(\hat{W}_t)$ and the constraint

$\hat{W}_t = (W_{t-1} - \hat{C}_{t-1})R$. Clearly $C^N_{t-1} \ge \hat{C}_{t-1}$; hence $u'(C^N_{t-1}) \le u'(\hat{C}_{t-1})$, and

$u'(\hat{C}_{t-1})$ is the uniform upper bound on $\frac{\partial v^N_{t-1}}{\partial W_{t-1}}$, M_{t-1}.   Q.E.D.

Lemma 3: $\lim_{N \to \infty} v^N_t(W_t) = v^*_t(W_t)$ for all $0 \le t \le T$ and W_t. We again use induc-

tion to prove this proposition. For t=T the condition holds trivially. Let

(C^{Nk}_{t-1}, W^{Nk}_t) be a subsequence of (C^N_{t-1}, W^N_t) converging to $(\bar{C}_{t-1}, \bar{W}_t)$.

This subsequence exists because the sequence is bounded. To facilitate the notation we drop the subscript k.

The key step is to show that:

$$\lim_{N\to\infty} \Sigma_{i=o}^{N} \; V_t^i(\bar{W}_t \frac{N+1}{i+1}) \; q_t^i(1-q_t)^{N-i} \geq V_t^*(\frac{\bar{W}_t}{q_t})$$

We now specify the random variable X^N as a function from the infinite sample space $\Omega = \{0,1\} \times \{0,1\} \times \ldots$ to the Reals by writing $X^N(\omega) = \{\frac{N+1}{i+1}:$ where i is the number of 1's in the first N coordinates of $\omega\}$.

From the Strong Law of Large Numbers, $\lim_{N\to\infty} X^N(\omega) = 1/q_t$ a.e. in Ω. The next step is to define the random variable:

$$y^N(\omega) = \{V^i(W_t^N \; X^N(\omega)): \text{ where i is the number of 1's in the first N}$$
$$\text{coordinates of } \omega\}$$

As $N\to\infty$, $i\to\infty$ a.e. By the induction assumption $V_t^i(\omega) \to V_t^*(\omega)$ for all ,

and also $W_t^N \; X^N(\omega) \to \frac{\bar{W}_t}{q_t}$. From equi-continuity, $V_t^i(W_t^N \; X^N(\omega)) \to V_t^*(\frac{\bar{W}_t}{q_t})$.

So $y^N(\omega) \to V_t^*(\frac{\bar{W}_t}{q_t})$ a.e., and Fatou's Lemma[*] may be applied.

$$\Sigma_{i=o}^{N} \; V_t^i(\bar{W}_t \frac{N+1}{i+1}) \; q_t^i(1-q_t)^{N-1} = \int_{\Omega} y^N(\omega)$$

Hence:

$$\lim_{N\to\infty} \int_{\Omega} y^N(\omega) \geq \int_{\Omega} V_t^*(\frac{\bar{W}_t}{q_t}) = V_t^*(\frac{\bar{W}_t}{q_t}).$$

This is true for every converging subsequence.

From this lemma and Corollary 2 we have the result:

$$\lim_{N\to\infty} V_{t-1}^N(W_{t-1}) = V_{t-1}^*(W_{t-1}) \qquad \text{Q.E.D.}$$

Using the same technique, we conclude by proving:

---

[*]Royden, p. 83.

__Lemma 4__: $C_t^N \rightarrow C_t^*$ $\qquad$ $t = 0,\ldots,T.$

Again let $\{C_{t-1}^N\}$ denote a subsequence converging to $\bar{C}_{t-1}$, and let (C_{t-1}, W_{t-1}) be another feasible consumption plan.

For all N, the optimality of (C_{t-1}^N, W_t^N) implies

$$u(C_{t-1}^N) + \alpha q_t \ \Sigma_{i=0}^N \ V_t^i(W_t^N \ \frac{N+1}{i+1}) \ q_t^i (1-q_t)^{N-i} \geq u(C_{t-1}) + \alpha q_t \ \Sigma_{i=0}^N \ (W_t \ \frac{N+1}{i+1}) \ q^i (1-q_t)^{N-i}$$

By repeating the argument of Lemma 3 the two sides tend to:

$$u(\bar{C}_{t-1}) + \alpha q_t \ V_t^*(\frac{\bar{W}_t}{q_t}) \geq u(C_{t-1}) + \alpha q_t \ V_t^*(\frac{W_t}{q_t})$$

Since (C_{t-1}, W_t) was arbitrarily chosen $(\bar{C}_{t-1}, \bar{W}_t)$ solves the maxmimization problem (12). Hence $(\bar{C}_{t-1}, \bar{W}_t) = (C_{t-1}^*, W_t^*)$, since the solution to (12) is unique because of strong concavity. But if every subsequence of C_{t-1}^N converges to C_{t-1}^*, it means that the entire sequence is converging as well.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ Q.E.D.

APPENDIX B

Computational Algorithm For the Two Family Members Dynamic Risk Pooling Problem

This appendix indicates the algorithm used to solve the two family members dynamic programming problem, copied here as B(1). The algorithm for the case of three family members is similar to that for two members and is available from the authors. While we consider the iso-elastic family of utility functions, our algorithm can be applied to any homothetic utility function.

(B1) $\quad V_{t-1}(W_{t-1}) = \max_{W_t, C^H_{t-1}, C^S_{t-1} \geq 0, \; t=T,\ldots,1} [u^H(C^H_{t-1}) + u^S(C^S_{t-1}) + \alpha P_{t/t-1} \, Q_{t/t-1} \, V_t(W_t)$

$\qquad\qquad + \alpha P_{t/t-1}(1-Q_{t/t-1}) \, H_t(W_t) + \theta \alpha Q_{t/t-1}(1-P_{t/t-1}) \, S_t(W_t)]$

subject to:

(B2) $\quad W_t/R + C^H_{t-1} + C^S_{t-1} = W_{t-1}$

Again, the letters H and S correspond to the two family members with respective conditional survival probabilities $P_{t/t-1}$ and $Q_{t/t-1}$. W_t is joint family wealth, and θ is the weighting factor. $H_t(W_t)$ and $S_t(W_t)$ are the expected utility levels for each family member if he alone survives to period t.

Optimal values for C^H_t and C^S_t are found recursively starting at period T and proceeding to period 0. We demonstrate that $V_t(W_t)$ may be written in the form:

(B3) $\quad V_t(W_t) = v_t \dfrac{W_t^{1-\gamma}}{1-\gamma}$ where v_t is a constant. We also show that total

family consumption, C_t, is given by:

(B4) $\qquad C_{t-1} = W_{t-1} \dfrac{v_T^{1/\gamma}}{v_T^{1/\gamma} + (\alpha K_t R)^{1/\gamma} R^{-1}}$

where K_t is another constant. Given total family consumption, consumption of
the two members is:

(B5) $\qquad C_{t-1}^H = \dfrac{C_{t-1}}{1+\theta^{1/\gamma}} , \qquad C_{t-1}^S = C_{t-1} \dfrac{\theta^{1/\gamma}}{1+\theta^{1/\gamma}}$

We demonstrate that K_t is a function of v_t and that v_{t-1} is a function of
K_t. Starting then at the initial value for K_t, K_{T+1}, we can compute v_T;
v_T in turn gives K_T which, in turn gives v_{T-1}. Proceeding in this fashion
to period zero we compute the entire sequence of v_t's and K_t's. These values
can then be used in (B4) to compute the ratio of consumption to wealth at each
period. These ratios together with an initial level of wealth plus (B2)
and (B5) generate the optimal consumption path. The homotheticity of the
utility function permits us to calculate recursively the shape of the con-
sumption path independently of the initial level of wealth.

Starting with period T the (B1) maximization problem is:

$$V_T(W_T) = \max \frac{1}{1-\gamma}(C_T^H)^{1-\gamma} + \theta \frac{1}{1-\gamma} (C_T^S)^{1-\gamma}$$

$$\text{s.t.} \qquad C_T^H + C_T^S \leq W_T, \qquad C_T^H, C_T^S \geq 0.$$

Solving this maximization and computing the indirect utility function for V_T,
we have:

(B6) $\qquad V_T(W_T) = v_T \dfrac{1}{1-\gamma} W_T^{1-\gamma}$, where $v_T = (1+\theta^{1/\gamma})^\gamma$.

(B7) $\qquad C_T^H = W_T \dfrac{1}{1+\theta^{1/\gamma}}, \qquad C_T^S = W_T \dfrac{\theta^{1/\gamma}}{1+\theta^{1/\gamma}}$.

For $t < T$, (B1) for the iso-elastic case is written as:

(B8)
$$V_{t-1}(W_{t-1}) = \max_{C_{t-1}^H, C_{t-1}^S} \frac{1}{1-\gamma} (C_{t-1}^H)^{1-\gamma} + \frac{\theta}{1-\gamma} (C_{t-1}^S)^{1-\gamma}$$

$$+ \alpha \frac{P_t}{P_{t-1}} \frac{Q_t}{Q_{t-1}} v_t \frac{1}{1-\gamma} W_t^{1-\gamma} + \alpha \frac{P_t}{P_{t-1}} (1 - \frac{Q_t}{Q_{t-1}}) h_t \frac{1}{1-\gamma} W_t^{1-\gamma}$$

$$+ \theta\alpha(1- \frac{P_t}{P_{t-1}}) \frac{Q_t}{Q_{t-1}} s_t \frac{1}{1-\gamma} W_t^{1-\gamma}$$

s.t.
$$C_{t-1}^H + C_{t-1}^S + \frac{W_t}{R} = W_{t-1}.$$

In going from (B1) to (B8) we use the fact that $H_t(W_t) = h_t \frac{W_t^{1-\gamma}}{1-\gamma}$ and $S_t(W_t) = s_t \frac{W_t^{1-\gamma}}{1-\gamma}$ for the iso-elastic utility function. The values for h_t and s_t are implicitly defined as the bracketed term in equation (7) in the text with $j = 0$ corresponding to time t and with each family member's survival probabilities from time t substitutes for P_j.

It is easy to see from (B8) that for given total family consumption, C_t^H and C_t^S will always satisfy (B5). Hence we may rewrite (B8) as:

(B9)
$$V_{t-1}(W_{t-1}) = \max_{C_{t-1}} v_T \frac{1}{1-\gamma} C_{t-1}^{1-\gamma} +$$

$$\alpha \frac{W_t^{1-\gamma}}{1-\gamma} [\frac{P_t}{P_{t-1}} \frac{Q_t}{Q_{t-1}} v_t + \frac{P_t}{P_{t-1}} (1 - \frac{Q_t}{Q_{t-1}})h_t + \theta(1- \frac{P_t}{P_{t-1}})\frac{Q_t}{Q_{t-1}} s_t]$$

Denoting the term in brackets by K_t we now have:

(B10)
$$V_{t-1}(W_{t-1}) = \max_{C_{t-1}} v_T \frac{1}{1-\gamma} C_{t-1}^{1-\gamma} + \alpha \frac{1}{1-\gamma} W_t^{1-\gamma} K_t$$

s.t.
$$C_{t-1} + \frac{W_t}{R} = W_{t-1}.$$

Maximizing (B10) and computing the indirect utility functions yields

(B11)
$$v_{t-1} = (v_T^{1/\gamma} + (\alpha R K_t)^{1/\gamma} R^{-1})^\gamma$$

(B12)
$$C_{t-1} = W_{t-1} \frac{v_T^{1/\gamma}}{v_T^{1/\gamma} + (\alpha K_t R)^{1/\gamma} R^{-1}}$$