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FLEXIBILITY AND UNCERTAINTY

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I. INTRODUCTION

The expected utility approach to decision-making under uncertainty takes lotteries as the basic objects of choice. A lottery is "simple" if all risk is resolved at once; it is "compound" if risk is resolved in stages. For example, a lottery is simple if its outcome is determined by the single flip of a fair coin; it is compound if with probability p a coin with probability of heads π_A is used to determine the outcome, and with probability (1-p) a coin with probability of heads π_B is used, where $p\pi_A + (1-p)\pi_B = 1/2$.

The gradual resolution of risk cannot be expressed in terms of simple lotteries. Whenever information is expected about uncertain future events, and the uncertainty is expressed in the language of lotteries, then they must be compound. In the above example, knowing which coin will be used provides information about the final outcome. It is an axiom of expected utility theory that the distinction between simple and compound lotteries is irrelevant, but with a <u>proviso</u>: no action can be taken while the compound lottery unfolds. This paper studies the consequences for economic behaviour of there being opportunities to act while uncertainties are being resolved.

On the various ways of compounding a given simple lottery, we define a partial ordering based on <u>variability of beliefs</u>. One set of beliefs is called more variable than another if it implies that more final risk is resolved at an intermediate stage. The less (more) one expects to learn by the intermediate period, relative to what one knows today, the less (more) variation one anticipates in beliefs about the final outcome. On the current actions, or positions, available for choice, we define an ordering based on <u>flexibility</u>. One position is called more flexible than another if it leaves a larger set of future positions available at any given level of cost. We formulate a simple sequential decision model incorporating these two orderings to suggest the following behavioural principle: The more variable are a decision-maker's beliefs, the more flexible a position he will choose.

As an application of this principle, consider this paraphrase of a newspaper headline: "Decrease in confidence leads to cutback in new orders for capital goods despite fall in short term interest rates." The decrease in confidence is interpreted as an increase in the variability of beliefs -- i.e., the less confident are current beliefs, the greater is the likelihood of substantial revision in the near future. As a consequence, there is a fall in the demand for inflexible positions (commitments to new capital goods) and a rise in the demand for flexible positions (holding short term liquid assets). Decreased confidence created a temporary premium on liquidity and discount on illiquidity. This principle potentially applies whenever (1) there will be opportunities to act after further information is received, and (2) current actions influence the relative attractiveness, or even availability, of different future actions.

An important determinant of preferences over simple lotteries is the decision-maker's attitude toward risk. The demand for flexibility, however, is basically unconnected with risk aversion. To see why, notice that having many rather than few positions available for future choice implies nothing about the variability of final payoffs. One individual might value flexibility because, by appropriately adapting his behaviour to the information received, it permits him to obtain a more nearly certain pecuniary reward; but another might value it because it allows him to make informed higher risk bets at the last moment. The <u>way</u> flexibility is used to exploit forthcoming information may be dictated by attitudes toward risk; but flexible positions are valuable not because they are safe stores of value, but because they are good stores of options.

The ordering of beliefs based on variability and the ordering of choices based on flexibility are defined in the next two sections. In section IV, qualified versions of the proposition relating these two orderings are stated, and a counterexample is provided to show that such qualifications are unavoidable. Section V illustrates the relationship in a simple asset choice problem. Discussion of the work of others is reserved for section VI.

II. COMPARISON OF BELIEFS BASED ON VARIABILITY

The prospect of changing beliefs arises only when there is opportunity for revision before the facts are revealed. A three period time horizon is needed to describe the process. An individual is unsure which "state" s will occur. S is the (finite) set of possible states. A probability distribution on S, denoted by a vector of probabilities $\pi = (\pi_s)$, represents a belief he might hold. In period one the individual has a prior belief $\bar{\pi}$ about the likelihood of various states occurring. In period two he receives an observation y from a (finite) set of possible observations Y, causing him to revise his beliefs to $\pi(y)$. Period three reveals the true state.

To complete the description of beliefs, let $q = (q_y)$ be the individual's estimate of the probabilities of receiving the respective "messages", and Π be the the $|S| \times |Y|$ matrix whose columns are the vectors $\pi(y)$.[1] For the individual's "model" of his own learning to be consistent, $\bar{\pi}$ must equal $\Pi q \equiv \Sigma q_y \pi(y)$, the prior belief must be a message-probability weighted average of his anticipated posterior beliefs. A structure of beliefs and how they will be revised is thus a probability distribution of the random vector $\pi(y) \in \{\pi\colon \Sigma\pi_s = 1, \pi \geq 0\}$. This distribution is embodied in (Π, q), has mean $\bar{\pi}$, and can be termed the individual's "information structure" (Marschak and Miyasawa, 1968).

Information can come from a multitude of sources, both public and private. The message can simply be an earlier realization of the state variable, as when y is the inflation rate in one quarter and s its rate in the next. Or the message and state can be of quite different form: y might be crop plantings in the spring and s quantities harvested in the fall; y the opinion of legal counsel and s the verdict of a jury; y a vector of leading indicators and s national product in the following year. In most situations with risk, events can and do occur that alter expectations about future events.

Consider two probability distributions of beliefs about the same set of possible states, (Π,q) and (Π',q').[2] How can they be compared? One way to compare them is to look at their mean values, $\bar{\pi}$ and $\bar{\pi}'$. These prior beliefs represent the risk faced by the individual. In some situations comparison of beliefs on the basis of risk makes sense. For example, when states are associated with realizations of a random variable $x(s)$, $\bar{\pi}$ can be regarded as "riskier" than $\bar{\pi}'$ when $\Sigma\bar{\pi}_s\psi(x(s)) \geq \Sigma\bar{\pi}'_s\psi(x(s))$ for all convex functions ψ (Rothschild and Stiglitz, 1970). For this to be the case, the expected value of the random variable must be the same for both prior beliefs. It captures the notion that the outcomes are more "spread out" according to one set of beliefs. Without the added structure provided by $x(s)$ there is no natural way to order the risks embodied in $\bar{\pi}$ and $\bar{\pi}'$.

However we can, without reference to the structure of S, compare the amount of change in beliefs anticipated in (Π,q) and (Π',q'). We shall call one distribution of beliefs more variable than another, denoted $(\Pi,q) \gtrsim (\Pi',q')$, when[3]

$$\text{(I)} \quad \sum_{y\in Y} q_y\phi(\pi(y)) \geq \sum_{y'\in Y'} q'_{y'}\phi(\pi(y')) \quad \text{for all convex functions } \phi.$$

This captures the notion that beliefs $\pi(y)$ are more "spread out" than $\pi(y')$, but also implies that they have the same mean, $\bar{\pi} = \bar{\pi}'$. Beliefs can be ordered in terms of variability only if they describe the same risk.

To understand this ordering of information structures, consider its extremes. Let (Π_*,q_*) be such that each column of Π_* is the same and equal to $\bar{\pi}$. This represents zero variability of beliefs -- beliefs are the same whichever message is received. Depending on the context, this could indicate either that the individual is so confident of his prior belief that further observation cannot alter it, or that the observations are so unrelated to s that nothing can be inferred from them. At the other extreme, let (Π^*, q^*) be such that each element of Π^* is either 0 or 1. This represents maximimum variability of beliefs -- the probability of each state occurring is revised to either 0 or 1 after any observation. The second period messages provide perfect information about the third period state.

The implication of greater variability of beliefs is only apparent in the context of a decision problem. Let B be a finite set of actions and u(b,s) be a payoff function defined on B × S. It can be shown that (I) is equivalent to saying that (Π, q) is always <u>more valuable</u> than (Π', q'), in the sense that (Bohnenblust, Shapley and Sherman, 1949)

$$\text{(II)}\quad \sum_y q_y \operatorname*{Max}_{b\varepsilon B} \sum_s \pi_s(y)u(b,s) \geq \sum_{y'} q'_{y'} \operatorname*{Max}_{b\varepsilon B} \sum_s \pi_s(y')u(b,s) \quad \text{for all bounded } u(b,s).$$

An individual's attitude toward greater risk depends on whether he is risk-averse or risk-seeking. But, since the expressions in (II) represent attainable expected payoffs from the viewpoint of period one, <u>all</u> individuals prefer greater variability in beliefs, regardless of their attitude toward risk.

A feeling for why greater variability is more valuable, and a means for identifying it, is provided by the work of Blackwell (1951, 1953). The "experiment" (Π, q) is <u>more informative</u> than (or sufficient for) the experiment (Π', q') if

(III) there exists a non-negative $|Y| \times |Y'|$ matrix M, with columns summing to 1, such that $\Pi' = \Pi M$ and $q = Mq'$.

If (III) is satisfied, one can construct a "black box" that accepts y as inputs and generates ouputs labelled y' that have exactly the same joint distribution with s as the real y': whenever $y \varepsilon Y$ is fed in, $y' \varepsilon Y'$ is sent out with probability $M_{yy'}q'_{y'}/q_y$. This garbling of y might add "noise" through the random element in generating y', or it might obliterate distinctions between inputs by always assigning them the same output (e.g., as when y is a sample and y' is a subsample of it). Marschak and Miyasawa (1968) demonstrate that (III) is equivalent to (I) and (II). Greater variability of beliefs is desirable because it means that messages convey more information about s.

Throughout our discussion we take the observations, Y, and structure of beliefs, (Π, q), as exogenous to the individual. They are not objects of choice, as they would be, for example, if the individual selected the experiment to perform or information service to consult.

In economic contexts there are three distinct sources of increased variability of beliefs. Each can apply even when all information is public. The first source is an improvement in the information content of available messages. Data can be regularly published that was previously not collected; surveys can be based on larger samples; econometric forecasts can become more accurate. Such changes are the analogues of performing "better experiments" in statistical decision theory. Beliefs become more variable because the observations are more compelling.

The second source is a change in the timing of messages that would be received in due course anyway. This requires explicit recognition of the time element in economic decisions. If an action must be chosen in the next month, then the message consists of what can be observed during that period. The movement of an "announcement date" into that month -- say, of an election outcome, policy statement, crop report, (anticipated) resolution of situations abroad -- increases the information content of the month's observations, and hence the variability of beliefs relevant for that particular decision. Notice that announcement dates for periodically reported data "automatically" move into and out of these planning periods with the passage of time. The announcement of the announcement need not be a surprise. Similarly, a change in the decision problem that alters only the time by which an action must be chosen increases (if more time is allowed) or decreases (if less is allowed) the relevant variability of beliefs.

The third source is a change in the status of prior beliefs change in the observations. The revised belief $\pi(y)$ combines the information contained in y with the information on which prior beliefs were based. If the amount of information embodied in $\bar{\pi}$ is, in some sense, large and regarded as relevant for s, then the individual has confidence in his prior beliefs, and is likely to make only small revisions upon observing y. Conversely, if the prior information was limited, or its relevance for s is questionable, then subsequent observations carry more weight in the revision. The same messages cause greater variation in beliefs. An example illustrates the principle. Consider an individual who revises his beliefs in a Bayesian fashion and who starts with a uniform prior on the true probability of heads arising for any untried coin. Suppose he has flipped a particular coin many times and it has come up heads half the time. His posterior distribution on the probability of heads is sharply peaked at one-half. Let y be the outcome of the next flip and s the outcome of the flip after that. His prior belief about s is $\bar{\pi} = (\frac{1}{2},\frac{1}{2})$. Observing y will cause little revision because one flip is small relative to the many preceeding. Suppose, however, that the individual suspects the coin has been switched. The suspicion reduces the relevance of the previous outcomes and he reverts toward the uniform prior. His prior belief about s is still $(\frac{1}{2},\frac{1}{2})$, but observing y induces a greater revision of these probabilities than before. Analogously, suppose an agent's model of his economic environment is formed from past experience and some event occurs that causes him to "lose confidence" in his estimate of the model parameters. The suspected change might be in government policy, in the organisation of markets in which he deals, in the phase of a business cycle, in the structure of the economy as a whole. The change reduces the relevance of past experience and increases the anticipated impact of future events on his beliefs.

Changes is the variability of beliefs, the amount to be learned from the passage of time, can thus arise from changes in the quality of data forthcoming, the timing of observations, and the confidence with which prior beliefs are held. Changes arising from this last source may be most relevant for macroeconomic phenomena since a "loss of confidence" in the economy is likely to be widely experienced. Indeed, decreased confidence in this sense comes close to capturing the notion of increased "uncertainty" used by early economists who maintained a distinction between risk and uncertainty. Realistically, changes in confidence are usually accompanied by changes in risk; but we know of no way to usefully combine the two when comparing information structures. Consequently we focus on the behavioural implications of increased variability of beliefs for given risks.

III. COMPARISON OF ACTIONS BASED ON FLEXIBILITY

Consider an information structure, as in the preceeding section, joined to a sequential decision problem. In period one the individual chooses an initial position. In period two, after observing y, there is another opportunity to act, and he chooses a second period position. In period three s is revealed and the consequence of these actions becomes known. What concerns us is how the prospect of learning in the second period about events to become known in the third influences the individual's first period choice.

Let the consequence for the individual be described by a payoff function, $f(a,b,s)$: a is the first period position, b the second, and s the "state of the world" as of the final period (sequence of events beyond the agent's control). The assumed objective is maximization of expected payoff. The payoff may be measured in units of wealth, utility or whatever is appropriate for the particular problem.

Flexibility is a property of initial positions. It refers to the cost, or possibility, of moving to various second period positions. To rank positions by their flexibility some part of the total payoff $f(a,b,s)$ must be imputed to the <u>move</u> from a to b, as distinct from having <u>been in</u> positions a and b. The payoff must be decomposed into a form

$$f(a,b,s) = r(a,s) + u(b,s) - c(a,b,s)$$

where $r(a,s)$ is the direct return on the first period action, $u(b,s)$ is the return on the second period action, and $c(a,b,s)$ is the cost of "switching" from a to b. Greater flexibility will be associated with lower switching costs.

In some decision problems there is a natural decomposition of f. For example, when a and b are portfolios of assets, r(a,s) and u(b,s) can be the portfolio yields over the two time intervals, including dividends, interest and capital gains, and c(a,b,s) can be the cost of liquidating those assets in a that are not in b, including commissions, penalties and bid-ask spreads. But, since the decomposition of f can be arbitrary (e.g., one can always set c = -f, r = 0, u = 0), some requirements must be imposed on c if the meaning of flexibility is not to be equally arbitrary.

Switching costs should capture the notion that, although it is in general costly to change position, one can always move automatically, without overt action, from a first period position to some second period position. Such moves amount to staying in the same position, and the switching costs involved should be zero. The second position that follows most naturally from an initial position depends on the context. The association may be obvious when the first and second period alternatives have the same form: If the positions are portfolios of assets, staying in the same position means holding the same portfolio; if the positions are the presence or absence of hydroelectric development on a particular river, staying in the same position means leaving the river in its existing state of development; if the positions are acceptance or rejection of job offers in the process of search, staying in the same position means staying in one's previous employment status. The association of initial positions with most natural second positions is more difficult when the first and second period alternatives are quite different. If the initial choice is the technology to install in a fixed plant, and the second is the output level at which the plant is operated, one might associate with each technology an output level where its cost advantage over other technologies is greatest (i.e., the earlier choice is least "regretted" given the later choice), and call producing at that level "staying in the same position".

A relabelling of alternatives along these lines permits us to regard the two positions as coming from the same set, and to require that $c(a,a,s) = 0$ for all a and s. Moves that are technologically impossible are assigned arbitrarily large $c(a,b,s)$. Positions that are impossible to occupy in one period or the other are assigned sufficiently negative $r(a,s)$ or $u(b,s)$ to be irrelevant to a rational agent.

Suppose that a relabelling of alternatives and an assignment of switching costs satisfying $c(a,a,s) = 0$ and $c(a,b,s) \geq 0$, for all a,b,s, can be found. Define position a to be <u>more flexible than</u> position a', denoted by $a \succsim_F a'$, if

$$\text{(F)} \qquad c(a',b,s) \geq c(a,b,s) \qquad \text{for all } s \text{ and } b \neq a'.$$

It costs as least as much to switch from a' to another position as from a. One extreme of this ordering is a <u>perfectly flexible position</u>, a^*, for which $c(a^*,b,s) = 0$ for all b and s. The other extreme is an <u>economically irreversible position</u>, a_*, for which $c(a_*,b,s) \geq u(b,s) - u(a_*,s)$ for all s and $b \neq a_*$ (i.e., it never pays to switch out of a_*, no matter which state occurs).[4] Equivalently, a is more flexible than a' if, for every upper bound on switching costs, the set of <u>alternative</u> positions attainable from a contains the set attainable from a'.

This concept of flexibility has two limitations. First, it only partially orders initial positions. Some pairs, or even all pairs, may be unranked by flexibility. The concept's usefulness depends on the decision problem. Second, the rankings may hinge on an arbitrary imputation of switching costs, when no natural assignment is available. The imputation will not affect the optimal decision strategy. But it will affect any rationalization of the strategy in terms of flexibility.[5]

The sources of flexibility suggest where the concept might provide a useful perspective. The mere act of waiting, of postponing commitment to irreversible actions, increases flexibility if the actions can also be undertaken in the future; acquiring a tool of versatile design provides more flexibility than a tool of specialized design; carrying larger inventories expands the range of deliveries possible on short notice; running parallel development programs increases the number of technologies available for future choice. Flexibility provided by these means does not hinge on relationships between agents.

Opportunities for interaction between agents, both in markets and in contractual arrangements, create further sources of flexibility. One specialized tool can be transformed into another through sale and purchase; an asset can be liquidated before maturity through sale to another individual. Such transformations are not costless, of course. Transacting uses real resources, and the price realized from sale may differ from the price paid for purchase of the same good. Costs of the latter sort vary from good to good, depending on their marketability, and from individual to individual, depending on their market expertise and haste to complete the transaction. These differences in liquidity provide a basis for ranking assets according to the flexibility they confer. Flexibility is also an element in contracts between agents. The flexibility "variable" (from the viewpoint of one party) could be the expiry date of a job offer, the quantity of goods covered by a delivery option, or the range of circumstances (either explicit in the agreement or implicit in legal precedent) in which a contractor is relieved of his obligation.

IV. RELATION BETWEEN THE VALUE OF FLEXIBILITY AND THE VARIABILITY OF BELIEFS

The three period sequential decision problem thus gives rise to two partial orderings: One ranks information structures according to variability of beliefs, indicating the amount to be learned from future observation; the other ranks initial positions according to flexibility, indicating the range of alternatives left open at any given level of switching cost. We now explore the relation between these two orderings: To what extent is choosing greater initial flexibility the rational response to increased variability of beliefs?

Why should any such relationship be expected? A connection arises since the prospect of learning enhances the value of flexibility. If there is no prospect of learning in the second period, then the agent knows with certainty what probabilistic beliefs he will hold, and can predict with certainty what position he will choose. Whether the initial position permits a wide or narrow range of other positions to be reached is irrelevant. But if there is something to be learned in the second period, then the agent is uncertain of the beliefs he will hold at that time, and is uncertain which position will then appear most appropriate. Without flexibility, either high switching costs must be incurred or the opportunity to profit from the information must be foregone. The more the agent expects to learn (the more uncertain he is of his future beliefs), the more attractive are initial positions which keep potentially relevant options open.

Exactly how does the prospective information interact with the alternatives available to enhance expected payoff? Suppose the agent commits himself to a first period position a -- and hence to an expected first period return of $\sum_s \bar{\pi}_s r(a,s)$ -- and wishes to evaluate his prospective second period return.

Also suppose, for the moment, that a is perfectly flexible so that switching costs may be ignored. Figure 1 depicts how the expected second period return is determined when there are just two ultimate states and two possible messages. Each point on the horizontal axis, which has length one, represents a probability distribution over the two states. The distance from a point to the right end of the interval is the probability that state 1 occurs; the distance to the left end is the probability that state 2 occurs. The agent's prior belief about s is $\overline{\pi}$; his beliefs conditional on message 1 (which favors state 1) and 2 being received are $\pi(1)$ and $\pi(2)$ respectively. The payoffs in state 1 for the second period actions b_1, b_2, b_3 are indicated on the left vertical axis; the payoffs in state 2 on the right. The expected payoff to taking each position as a function of the probability distribution π is given by the height of the straight line joining the position's payoff in state 1 to its payoff in state 2. The convex upper boundary of these lines is the maximum expected second period return as a function of beliefs about s. When message 1 is received, the expected payoff is greatest in position b_1, and equals u_1^*; when message 2 is received, position b_3 is best, and the expected payoff is u_2^*. Since the agent's prior belief is a message-probability weighted average of his conditional beliefs , $\overline{\pi} = q_1\pi(1) + q_2\pi(2)$, and since his expected payoff from the viewpoint of the first period is the identically weighted average of his conditional expected payoffs, $u^* = q_1u_1^* + q_2u_2^*$, this prior expected payoff is simply the height of the straight line joining $(\pi(1),u_1^*)$ to $(\pi(2),u_2^*)$ above the point $\overline{\pi}$. If no messages were anticipated (i.e., if the second position had to be chosen on the basis of beliefs $\overline{\pi}$), then b_2 would have been the optimal action, and u_0^* the expected payoff. The <u>value</u> of the prospective information is thus $u^* - u_0^*$. Since the maximum expected payoff as a function of π is convex, and since $\overline{\pi}$ is a convex combination of the $\pi(y)$, the value of prospective information is always non-negative.

Had the initial position, a, not been perfectly flexible, the payoff to each second period position, $u(b,s)$, need only be replaced by its payoff net of switching costs, $u(b,s) - c(a,b,s)$, to similarly determine the expected second period return and value of information. This return, of course, would vary with the choice of initial position. To extract positive value from the information, the initial choice must keep open at least two viable second period options. Any increase in the number of second period positions available, or decrease in switching costs, can only raise the maximized payoff for each π and hence the expected second period return.

In Figure 2, the information structure (Π,q) embodies more variable beliefs than does (Π',q'). Each structure involves two observations and the same prior belief $\bar{\pi}$. The lower graphs depict the probability densities for the random variables $\pi(y)$ and $\pi(y')$; the "balances" show how the density of $\pi(y)$ can be obtained from that of $\pi(y')$ by a pair of "mean-preserving spreads" (cf. Rothschild and Stiglitz, 1970). Values for all parameters are specified, including the matrix M such that $\Pi' = \Pi M$ and $q = Mq'$ as required by definition (III) of section II. The spreading out of conditional beliefs increases expected returns from $u^{*'}$ to u^{*}. The increase follows from the convexity of $\max_{b\in D} \Sigma\pi_s u(b,s)$ in π, and not from properties of D or $u(b,s)$.

Greater variability of beliefs and greater flexibility are thus both desirable when costlessly available. But flexibility is a property of initial positions and also affects expected first period returns. The rational agent trades off the cost of greater flexibility, in the form of lower first period returns, with its benefits, in the form of higher expected second period returns. Let us consider the optimal decision strategy in more detail.

An optimal strategy for the sequential decision problem consists of a first period position, a, and a set of second period positions, $\{b_y\}$, to be taken depending on the observation $y \in Y$ received, which maximize the expected total payoff. All positions are chosen from the same set D. The expected payoff so obtained can be expressed recursively using the maximum principle of dynamic programming:

$$(1)\quad J(\Pi,q) = \max_{\substack{a\in D\\ \{b_y\}\in D^Y}} \sum_y \sum_s q_y \pi_s(y) f(a,b,s) = \max_{a\in D} \sum_y q_y \max_{b\in D} \sum_s \pi_s(y) f(a,b,s)\,.$$

Decomposing f(a,b,s) into r(a,s) + u(b,s) - c(a,b,s), utilizing $\bar{\pi}_s = \sum_y q_y \pi_s(y)$, and separating terms gives

$$\begin{aligned}(2)\quad J(\Pi,q) &= \max_a\Big[\ \sum_s \bar{\pi}_s r(a,s) + \sum_y q_y \max_b \sum_s \pi_s(y)(u(b,s) - c(a,b,s))\ \Big]\\ &= \max_a\Big[\ \bar{r}(a) + \sum_y q_y\, v(a;\pi(y)\ \Big]\\ &= \max_a\Big[\ \bar{r}(a) + V(a;\Pi,q)\ \Big]\,.\end{aligned}$$

The expected first period return to a position, $\bar{r}(a)$, depends only on the prior belief $\bar{\pi}$. Since $\bar{\pi}$ must be the same for all information structures comparable in terms of variability, the (opportunity) cost of flexibility is independent of what the individual expects to learn. The function $v(a;\pi)$ is the maximum expected second period return, net of switching costs, for a given belief and initial position. It is convex in π and its graph was the convex upper boundary in Figures 1 and 2. $V(a;\Pi,q)$ is the expected second period return to taking initial position a with information structure (Π,q) -- i.e., the expected value of $v(a;\pi)$. The _value_ of any position is its total return $\bar{r}(a) + V(a;\Pi,q)$. The _optimal initial position_ is the one with the highest value.

The convexity of $v(a;\pi)$ in π implies that the value of all initial positions increase with the variability of beliefs. But for a rise in variability to cause a rational shift toward more flexibility, this increase must be greater for the more flexible positions. The relationship required is the following:

(3) If $(\Pi,q) \gtrsim (\Pi',q')$ and $a \gtrsim_F a'$,

then $V(a;\Pi,q) - V(a;\Pi',q') \geq V(a';\Pi,q) - V(a';\Pi',q')$.

To put it another way, any "increment" in variability of beliefs must raise the value of any "increment" in flexibility.[6]

If (3) is true for all $a \gtrsim_F a'$, then an increase in the variability of beliefs, since it leaves $\bar{r}(a)$ unchanged, raises the value of any flexible position relative to any less flexible position, and hence can only move the optimal initial position in the direction of greater flexibility. To get some sense of what this requires, when there are no further restrictions on (Π,q) and (Π',q'), inequality (3) can be written as

$$(4)\quad \sum_y q_y[v(a;\pi(y)) - v(a';\pi(y))] \geq \sum_{y'} q'_{y'}[v(a;\pi(y')) - v(a';\pi(y'))] \; .$$

A sufficient (and indeed necessary condition if (3) is to hold for all $(\Pi,q) \gtrsim (\Pi',q')$) condition for (3) is that the difference $v(a;\pi) - v(a';\pi)$ be convex in π for $a \gtrsim_F a'$. An example at the end of this section shows that this condition is not always met, however, and hence that the prospect that more will be learned can cause less flexible positions to be chosen. The propositions below, by restricting information structures and payoffs in various ways, describe circumstances in which the desired relation between the two orderings can be established. Although our basic conjecture is not true in general, it does hold for certain broad classes of decision problems.

The first two propositions emphasize that for some payoff structures the amount to be learned in the future has no effect on the initial choice. Proofs of all propositions are in the appendix.

PROPOSITION 1: When all positions are perfectly flexible, the optimal initial position is that which offers the highest expected first period return (determined by prior beliefs alone).

PROPOSITION 2: When all positions are economically irreversible, the optimal initial position depends only on prior beliefs $\bar{\pi}$.

Propositions 1 and 2 imply that the same initial position would be chosen for any two information structures $(\Pi, q) \succsim (\Pi', q')$.[8]

PROPOSITION 3: An increase in the variability of beliefs raises the value of any position relative to any economically irreversible position.

This proposition implies that the prospect of more information can induce an agent to change from an irreversible initial position to one that is at least partially flexible, but never the other way around. Moreover, the disadvantage of irreversible positions increases monotonically with the degree of uncertainty about future beliefs.

Capital formation decisions are frequently economically irreversible. One can be "locked in" to positions that are later regretted. Proposition 3 thus suggests an inverse relationship between investment and "lack of confidence" in beliefs. The next two propositions provide counterparts to proposition 3 at the opposite end of the flexibility spectrum: increases in the amount to be learned by waiting enhance the relative value of perfectly flexible positions. Since holding liquid assets, particularly money, provides the greatest flexibility in many situations, these results suggest a direct relationship between the demand for liquidity and the expectation that beliefs will change.

PROPOSITION 4: Anticipating <u>some</u> change in beliefs, as opposed to <u>none</u>, raises the value of any <u>perfectly flexible position</u> relative to that position $\tilde{b}$ which offers the highest expected second period return on the basis of prior beliefs alone.

Position $\tilde{b}$ is the one that maximizes $\sum_s \bar{\pi}_s u(b,s)$ -- that is, the position that would be chosen now if first period returns and the prospect of learning were absent. No restriction is placed on the flexibility of $\tilde{b}$; indeed, the costs of switching out of it may be quite minor.

Since proposition 4 says nothing about positions other than $\tilde{b}$ and a* (perfectly flexible) that might be chosen, it applies most usefully to situations where the choice is between going ahead with what seems best at the moment versus waiting for more information. For example, consider a firm that contemplates producing a good for sale in period three, that is uncertain of its demand, and that expects no further information before the good is produced and sold. It must choose either to go ahead with production now ($\tilde{b}$) or to defer the decision to some future date (a*), realizing that planning can be costlessly resumed (postponement is perfectly flexible) but that losses result if production plans are aborted. Current estimates of demand suggest that production is profitable (i.e., $\Sigma\bar{\pi}_s u(\tilde{b},s) \geq \Sigma\bar{\pi}_s u(a^*,s)$), and there is some opportunity cost to postponing commitment (i.e., $\bar{r}(\tilde{b}) > \bar{r}(a^*)$) -- perhaps because material costs will rise, perhaps because time is a factor in the production process. Clearly the firm should commit to production immediately. But suppose, now, that the firm hears of a consumer survey (or government policy announcement), the outcome of which will be known in period two and could change the perceived profitability of production. Proposition 4 implies that this prospect can induce the firm to rationally postpone production plans until the information is revealed. The significance of 4 lies in the fact that irreversibility was not essential for a relationship between learning and the value of flexibility.

Proposition 5 strengthens the results of 4 in certain directions. By requiring the cost of "undoing" $\tilde{b}$ to be independent of both s and the position switched to, and by requiring that beliefs become more variable in a particular way, a monotonic relationship is obtained between the amount to be learned and the advantage of perfect flexibility over $\tilde{b}$. We shall say that beliefs (Π,q) are a <u>star-shaped spreading</u> of beliefs (Π',q'), denoted $(\Pi,q) \gtrsim_S (\Pi',q')$, if

(5) $Y = Y'$, $q = q'$, $\Pi q = \bar{\pi} = \Pi' q'$, and there exists a set of numbers $0 \leq \lambda_y \leq 1$ such that $\pi(y') = \lambda_y \pi(y) + (1 - \lambda_y)\bar{\pi}$ for each $y = y' \in Y$.

The ordering $\gtrsim_S$ implies $\gtrsim$, but not the converse.

PROPOSITION 5: Let switching costs satisfy $c(\tilde{b},b,s) = c(\tilde{b})$ for all $b \neq \tilde{b}$. Then a star-shaped spreading of beliefs raises the value of any <u>perfectly flexible position</u> relative to that position $\tilde{b}$ which offers the highest expected second period return on the basis of prior beliefs alone.

A star-shaped spreading of beliefs is not as improbable as it seems at first glance. First, whenever (Π',q') conveys <u>no</u> information, then $(\Pi,q) \gtrsim (\Pi',q')$ implies $(\Pi,q) \gtrsim_S (\Pi',q')$. Second, if $q = q'$ and there are just two possible observations (e.g., the occurrence or not of a particular event), then $\gtrsim$ implies $\gtrsim_S$, regardless of the number of states. Finally, if the observations' usefulness for prediction is contingent on the validity of some theory, which if false renders them valueless (or if there is some chance that the message will not be received, independent of the y sent), then, letting $\lambda = \lambda_y$ be the probability that the theory is true (that the message will be received), any rise in λ increases variability in the required fashion.

The final proposition comes closest in spirit to the general conjecture. It establishes a monotonic relationship between the variability of beliefs and the flexibility of the rational initial position for particular payoff structures.

PROPOSITION 6: Let the payoff structure satisfy the following conditions:

(i) For each $a \in D$ there is a set $D_a \subset D$ such that $c(a,b,s) = \{0 \text{ for } b \in D_a, \infty \text{ for } b \notin D_a\}$. Furthermore, either $D_a \subset D_{a'}$ or $D_a \supset D_{a'}$ for all $a,a' \in D$.

(ii) For each pair $a,d \in D$ there exists $d' \in D_a$ such that, for all π, either $\underset{b \in D_a}{\text{Max}} \Sigma\pi_s u(b,s) \geq \Sigma\pi_s u(d,s)$ or $\underset{b \in D_a}{\text{Max}} \Sigma\pi_s u(b,s) = \Sigma\pi_s u(d',s)$.

Then an increase in the variability of beliefs increases the flexibility of the optimal first period position.

Condition (i) states that each current action leaves open a set of future positions that are costlessly available, all other options being foreclosed. Furthermore, initial positions are completely ordered by set inclusion of the options left open. Since $D_a \supset D_{a'}$ implies $a \gtrsim_F a'$, the flexibility ordering on D is complete. Condition (ii) is less transparent. To interpret, it states that whenever some position d would be preferable to those which are available, D_a, it is always the same position d' in D_a (which may depend on the particular d) that is the best available option; simply knowing that d is preferred to D_a is enough to determine the optimal second period position.

These two conditions are trivially satisfied when there are just two initial positions, one of which is irreversible, which reveals proposition 3 to be a special case of 6. But there is another class of decision problems that meet its requirements. Assume the total payoff has an additive form, $r(a,s) + u(b,s)$, and that the second period choice is the level of a real-valued control variable subject to an inequality constraint determined by the first period choice: $b \leq z(a)$. Intial positions are completely ordered in the sense of condition (i) by their levels of $z(a)$, with $z(a) \geq z(a')$ implying $a \gtrsim_F a'$. Further assume that $u(b,s)$ is concave in b for each s. Then, whenever the expected second period return could be increased by removing the constraint on b, $b = z(a)$ must be the best available choice. Condition (ii) is thus met. In particular applications, the constraint z(a) might signify maximum capacity of an otherwise constant marginal cost plant, number of delivery options acquired, quantity of a natural resource left unused. In such circumstances, an increase in information expected in period two leads a rational decision-maker to choose a less binding constraint on his second period choice.[9]

The perhaps surprising aspect of proposition 6, given the strong notion of flexibility involved in condition (i), is the necessity of an additional condition, such as (ii), to obtain the sought-after relationship. By relaxing (ii) we can construct a situation in which the prospect of more information causes less flexibility to be chosen. Suppose three positions are available, $D = \{b_1, b_2, b_3\}$. Let the second period positions attainable from each be as follows: $D_{b_1} = \{b_1\}$, $D_{b_2} = \{b_1, b_2\}$, $D_{b_3} = \{b_1, b_2, b_3\}$. Condition (i) is met with $b_3 \gtrsim_F b_2 \gtrsim_F b_1$. Figures 3A and 3B depict second period returns for two possible payoff structures. They heavy solid lines indicate $v(b_2;\pi)$, the maximum expected payoff when only options b_1 and b_2 are open in the second period; the dotted lines indicate $v(b_3;\pi)$, when all three options are open. With payoffs as in 3A, condition (ii) is satisfied. Whenever b_3 is the unconstrained best position, b_2 is the best choice between b_1 and b_2. But in 3B, condition (ii) is not satisfied. Knowing only that b_3 is optimal, one cannot say which of b_1 and b_2 is the next best alternative. The difference $v(b_3;\pi) - v(b_2;\pi)$, that appeared in inequality (4) and indicates the value of having the additional option open , is graphed in the lower part of each Figure. In 3B it is not convex. Two information structures satisfying $(\Pi,q) \gtrsim (\Pi',q')$ are represented in 3B. As we move from the less to the more variable set of beliefs the expected value of having the additional option open vanishes. That is, increasing the amount the decision-maker expects to learn <u>decreases</u> the attractiveness of b_3 relative to b_2 as an initial position (inequality (3) is reversed).

V. LIQUIDITY AS FLEXIBILITY

Markets provide flexibility by allowing assets to be transformed, through sale and purchase, into other assets. In a monetary economy these transformations are effected in two states: the initial asset is exchanged for money, the money is exchanged for the desired good or asset. The liquidity (saleability) of an asset describes the ease, or "costlessness", with which the first stage is accomplished. In such a context money is the most liquid asset since costs associated with the first stage are avoided completely. This section illustrates the relation between flexibility and the prospect of information in a sequential asset choice problem, interpreting the demand for money as a desire for flexibility.

Suppose an individual must choose between three non-diversified portfolios: M, A_1, A_2 (money, asset 1, asset 2). In period one he chooses which asset to hold until period two; in period two, after further information is received, he chooses which asset to hold until period three. There are two ultimate states that can occur, $S = \{s_1, s_2\}$, and two observations that can be received, $Y = \{y_1, y_2\}$. Let the payoff structure be as follows. The return on portfolio M is 0 in both periods with certainty. In period one A_1 and A_2 both return $\bar{r} > 0$ with certainty; in period two they yield 1 and 0 respectively when s_1 occurs (which favors asset 1), 0 and 1 respectively when s_2 occurs. No costs are incurred if the individual switches from M in the first period to either A_1 or A_2 in the second, or if he continues to hold the same portfolio as before. But a "liquidation cost" of $\bar{c} > 0$ is incurred if he switches from either A_1 or A_2 to a different portfolio in the second period. Thus M is <u>more flexible</u> than both A_1 and A_2 in the sense of section III. The total payoff for each sequence of actions and state is given in Table 1.

Initial position (a)	Second position (b)	Payoff $f(a,b,s)$ state 1	state 2
A_1	A_1	$\bar{r} + 1$	$\bar{r}$
	M	$\bar{r} - \bar{c}$	$\bar{r} - \bar{c}$
	A_2	$\bar{r} - \bar{c}$	$\bar{r} - \bar{c} + 1$
M	A_1	1	0
	M	0	0
	A_2	0	1
A_2	A_1	$\bar{r} - \bar{c} + 1$	$\bar{r} - \bar{c}$
	M	$\bar{r} - \bar{c}$	$\bar{r} - \bar{c}$
	A_2	$\bar{r}$	$\bar{r} + 1$

TABLE 1

Let the information structure (Π,q) be described parametrically by

$$(5)\quad \Pi = [\pi_s(y)] = \begin{bmatrix} \rho + \alpha(1-\rho) & \alpha(1-\rho) \\ 1 - \rho - \alpha(1-\rho) & 1 - \alpha(1-\rho) \end{bmatrix}, \qquad q = \begin{bmatrix} \alpha \\ 1-\alpha \end{bmatrix}$$

where $0 \leq \alpha \leq 1$ and $0 \leq \rho \leq 1$. For this class of information structures, $\bar{\pi} = \Pi q = q$. Thus the prior probability that state s_1 will occur and the probability that message y_1 will be received are the same and equal to α. Parameter ρ is the correlation coefficient between y and s, viewing them as random variables taking on values of 1 or 2. Raising ρ for given α increases the variability of beliefs, both in the sense of $\gtrsim$ and of $\gtrsim_S$. When $\rho = 1$, y conveys perfect information about s; when $\rho = 0$, nothing about s can be inferred from y.

Assume the individual is risk neutral and wishes to maximize the expected payoff. The optimal strategy yields an expected return

$$(6) \quad J(\Pi,q) = \underset{a\epsilon D}{\text{Max}} \sum_{j=1} q_j \underset{b\epsilon D}{\text{Max}} \sum_{i=1} \pi_i(y_j)\, f(a,b,s_i) \; .$$

In (6), $D = \{M,A_1,A_2\}$ is the set of possible portfolios, and $f(a,b,s_i)$ is the payoff function of Table 1. Solution of the problem is straightforward but lengthy. Figure 4 presents those aspects of the solution which concern us. Regions A_2, M, A_1 (bounded by dotted lines) are the values of and for which it is optimal to initially hold those assets. Region M vanishes if either $\bar{r} > \bar{c}/2$ or $\bar{r} > 1/2$ -- money is never held if its opportunity cost overshadows either the alternatives' switching costs or the maximum second period yield at stake. Holding M can be rational because if A_1 or A_2 is chosen initially, and subsequent observation indicates that the opposite position promises higher expected returns, then either cost $\bar{c}$ is incurred or the agent passes up the opportunity to profit from the information.

Varying the parameters has plausible effects on the demand for money. Reducing $\bar{r}$, the yield on alternative assets, moves outward the vertical boundaries and downward the lower boundaries of region M, enlarging the set of beliefs for which money is the optimal first period asset. Raising $\bar{c}$, the illiquidity of alternative assets, has a similar effect. Moving α toward 1/2, increasing prior uncertainty about which asset has the highest yield, can move one into region M but not out of it. Increasing ρ, the information content of y, never causes a switch out of M.

An alternative way to see now anticipated information affects the demand for money is to ask: at what $\bar{r}$ is the decision-maker indifferent between all three assets? Letting $\alpha = 1/2$, so he is indifferent between A_1 and A_2, the three regions intersect at $\rho = 2\bar{r}$. The short term yield the individual is willing to forgo by holding money is thus $\bar{r} = \rho/2$ (up to $\bar{r} = \bar{c}/2$, beyond which it stays constant to keep region M from vanishing). The greater is the information expected in the near term, the higher is the yield required for less liquid assets to be held.

Although we assumed risk neutrality for this illustration, one can verify that the effect of risk aversion on the value of flexibility is ambiguous. Suppose the agent is extremely risk averse, concerned only with maximizing his minimum possible payoff. If y conveys less than perfect information, $\rho < 1$, then he must hold either A_1 for both periods or A_2; only in that way is he guaranteed at least $\bar{r}$ (see Table 1). Alternatively, if y promises perfect information, $\rho = 1$, then he must hold M initially; only in that way is he guaranteed a return of 1. Since there are points in Figure 4 where M is held although $\rho < 1$, and points where A_1 is held although $\rho = 1$, it is apparent that risk aversion has in one case enhanced and in the other case diminished the value of flexibility.

This example was constructed to distinguish its motive for holding money as much as possible from the motives embodied in existing theories of money demand. Risk was essential, but not risk averse behaviour; differential asset liquidation costs were required, but not compulsory liquidations (to meet, for example, unforeseen "cash requirements"); yields on alternative assets were uncertain, but money was dominated, in terms of both immediate (period one) and future (period two) yields, by all other assets -- none yielded less than 0 in each period. Liquidity has value because it permits profitable exploitation of information not yet received.

Finally, let us point out that with a renaming of the positions M, A_1, A_2, the structure of the example applies to the heterogeneous capital investment problem. Let A_1 and A_2 refer to two different types of capital a firm might acquire, and M refer to acquiring no capital at all (postponing choice to period two). Both investments could be unambiguously profitable, but the firm may be unsure which will be the most profitable. If it expects this uncertainty to be partially resolved by period two, it may rationally reject investing currently in either type of capital. Investment demand falls because of the expectation that more will be learned.

VI. CONNECTIONS WITH EARLIER WORKS

1. Risk and Uncertainty

Our distinction between the risk embodied in beliefs and the variability of those beliefs over time invites comparison with the distinction between risk and uncertainty maintained by some writers. The most well-known juxtaposition of risk and uncertainty is that of Knight (1921). He reserves the term "uncertainty" for those events which cannot be assigned numerical probabilities, and "risk" for those homogeneous, repetitive events whose relative frequencies can be ascertained. The distinction appears to be based on the difference between objectively and subjectively formed estimates, with Knight unwilling to consider numerical probabilities attached to events if there is no statistical basis for their estimation.

Keynes too believed that economic risks involved more than just well defined chances. Propositions and events vary in their "appropriate degree of rational belief." The highest degree is knowledge, or certainty; although that certainty may involve numerical probabilities, such as those assigned to the outcomes of a spin of a roulette wheel known to be fair -- what Knight might have called risk. Keynes view was similar to Knight's in that he did not believe that degrees-of-belief need be numerically scaled; but, unlike Knight, he was concerned with building a theory that involved comparison of degrees-of-belief.[10] Our approach is similar to Keynes' if for no other reason than his concept of degree-of-belief invites interpretation in terms of our ranking based on variability. We would say that the degree-of-belief in a prior distribution over states increases as the variability of (Π,q) decreases, and that the state of perfect certainty or knowledge corresponds to the belief that there is nothing more to learn (i.e., $\pi(y) = \bar{\pi}$ for all y). Furthermore, variability only partially orders information structures and cannot be numerically scaled.

Distinguishing risk from uncertainty in this sense is not new. It is explicit in the terminology of Marschak (1938, 1949), Tintner (1942) and Hart (1942), among others, who use the term uncertainty to describe the prospect of learning. What we have added is the characterization of more informative experiments by Blackwell (1951, 1953) and by Bohnenblust, Shapley and Sherman (1949) (see also Marschak and Miyasawa, 1968; DeGroot, 1962; and Kihlstrom, 1973) to describe changes in uncertainty.

2. Flexibility with and without Uncertainty

The notion of flexibility has arisen in numerous economic contexts. Without risk, flexibility considerations can still be important. Making investment irreversible alters the optimal path of capital accumulation (Arrow, Beckmann and Karlin, 1958; Arrow, 1968, Nerlove and Arrow, 1962); asset liquidation costs influence portfolio choice even when cash needs are perfectly foreseen (Baumol, 1952; Grossman, 1969). That individuals might have a distinct preference for "postponement of choice" in the absence of risk and uncertainty is explored by Koopmans (1964). That preferences for flexibility can be treated axiomatically without reference to probabilities, although they may be equivalent to ones derived from expected utility theory, is demonstrated by Kreps (1979). Marschak and Nelson (1962) remark on the usefulness of flexibility as an economic concept and consider how it might be formalized.

A connection between random changes and the value of flexibility is drawn by Lavington (1921), who provides a superb early discussion of what he terms "the risk arising from the immobility of invested resources." It re-emerges in the context of behaviour of the firm in Kalecki (1937) and in Stigler (1939), who describes one plant as being more flexible than another if it has a flatter average cost curve (this is pursued further by Tisdell, 1968, and Merkhofer, 1975).

The effect of changes in risk on investment has been studied by Smith (1969), Rothschild and Stiglitz (1971), Hartman (1972) and Nickell (1975), among others. These studies support a basically ambiguous relation between risk and investment demand. We focussed on the possible inverse relationship between uncertainty and investment (inflexibility). Although this difference in emphasis reflects, in part, the distinction between changes in risk and changes in uncertainty (risk held constant), it also hinges on the characterization of flexibility in investment decisions. Acquiring additional capital can represent a choice of more flexibility, for example, when it increases plant capacity -- investment today permits the firm to produce more as well as less tomorrow. Regarding investment as one-dimensional variation of a homogeneous capital stock is certainly a possible specification. But if one regards the investment decision under uncertainty as essentially a choice between no investment and various postponable additions to a heterogeneous capital stock, then the illiquidity of specific capital becomes a central consideration, with more investment associated with less flexibility (see remarks at the end of section V).

The connection between flexibility and the prospect of learning is explicit in Hart (1942). He distinguishes risk from uncertainty as we have, and takes the position that, compared to uncertainty, "risk has comparatively little importance in economic analysis." Hart points out that uncertainty can be ignored when all choices are either perfectly flexible or economically irreversible -- our Propositions 1 and 2.[11] Concerning the importance of attitudes toward risk when learning is involved, he states: "... the central problems of uncertainty can be posed and largely solved under the assumption of 'risk neutrality'." Hart also anticipates recent qualifications of the

Simon (1956) - Theil (1957) certainty-equivalence theorem. In the context of environmental preservation, Henry (1974a, 1974b) and Arrow and Fisher (1974) (see also Fisher, Krutilla and Cicchetti, 1972) show that it is sub-optimal to replace probability distributions by their mean values when choosing between irreversible and perfectly flexible alternatives, even though all other requirements for the certainty-equivalence theorem might be fulfilled. Propositions 3 and 4 extend these results.[12]

3. Flexibility and Liquidity

The term "liquidity" has been used to refer both to an asset's certainty of yield, including capital gains, and to the difference between its purchase and sale price, including all transaction costs. Keynes (1930, p.67) leaves some ambiguity when he introduces the term by calling one asset more liquid than another if it is "more certainty realizable at short notice without loss." Makower and Marschak (1938) take care to distinguish an asset's "safety" from its "plasticity", or future saleability, using liquidity to describe the latter property.

Certainty of yield is singled out in the Tobin (1958) - Markowitz (1957) approach to money demand. The title of Tobin's paper aptly expresses the viewpoint: "Liquidity Preference as Behaviour Toward Risk." Flexibility is not an issue since choice is confined to assets free of switching costs.

In his recent contribution to monetary theory, Hicks (1974) outlines another approach, encompassing a broader class of assets that differ in terms of saleability. It represents the application to monetary theory of Hart's framework, and, in comparison with Tobin, could be entitled: "Liquidity Preference as Behaviour Toward Uncertainty." The example of section V fills in the formal details of an illustration sketched by Hicks. A similar connection between emerging information and the demand for liquid (saleable) assets is suggested by Marschak (1949), Goldman (1974, 1978) and Cropper (1976).

In a related paper, not directly concerned with the marketability of assets, Hirshleifer (1972) measures the "illiquidity" of investments by the time required to complete a technologically irreversible process (i.e., the time for the investment to mature), and shows that the prospect of emerging information can explain the lower equilibrium yield on shorter term assets.

Hick's essay, even more than Hart's, suggests the range of macroeconomic phenomena that may be treated with this approach to liquidity preference. He remarks that the separation of determinants of financial and real asset equilibrium, the twin cutting edges of his earlier IS-LM analysis, may need reworking; and that with this more recent approach, in which "the balance sheet must be considered much more generally,... it is desirable for the marginal efficiency of capital and the theory of money to be taken together." Distentangling uncertainty about beliefs from the riskiness of payoff-relevant events, and characterizing economic choices in terms of the flexibility they confer, is a first step in that direction.

APPENDIX

Proofs of Propositions 1-6

The letters a, a', b, b', d, d' denote elements of D, the set of positions available for choice in both periods. For a given prior belief $\bar{\pi}$, $\tilde{b}$ denotes the element of D which maximizes $\sum_s \bar{\pi}_s u(b,s)$. An * superscript on a position indicates that it is perfectly flexible; an $_*$ subscript indicates that it is economically irreversible. An * superscript on information structure parameters, as in (Π^*,q^*), indicates that y conveys perfect information about s; an $_*$ subscript indicates that y conveys no information (i.e., that $\pi(y) = \bar{\pi}$ for all y). $V(a;\Pi,q)$, the expected second period return net of switching costs for a given intial choice, $v(a;\pi)$, the maximum expected second period return for a given belief and intial position, and $\bar{r}(a)$, the expected first period return, are as defined in equation (2) of section IV. The flexibility ordering $\gtrsim_F$ on D is as defined in section III. The information orderings $\gtrsim$ and $\gtrsim_S$ on (Π,q) are defined in (I) of section II and (5) of section IV respectively.

Inequalities used in proving the first four propositions are collected in the following Lemma.

LEMMA: For all $(\Pi,q) \gtrsim (\Pi',q')$,

(L.1) $V(a^*;\Pi,q) \geq V(a;\Pi,q)$ for all a, $a^* \in D$

(L.2) $V(a;\Pi,q) \geq V(a;\Pi',q')$ for all $a \in D$

(L.3) $V(a_*;\Pi,q) = V(a_*;\Pi,q)$ for all $a_* \in D$

(L.4) $V(a^*;\Pi,q) = V(a^{*\prime};\Pi,q)$ for all a^*, $a^{*\prime} \in D$

Proof: (L.1) follows immediately from the definition of V, the non-negativity of switching costs, and the definition of perfect flexibility in section III. (L.2) follows from the definition of V, the convexity of $v(a;\pi)$ in π (it is the maximum of a finite collection of bounded linear functions), and the definition of $\gtrsim$.

(L.3) follows from the fact that economic irreversibility means that $c(a_*,b,s) \geq u(b,s) - u(a_*,s)$ for all b and s. Hence $v(a_*;\pi) = \sum_s \pi_s u(a_*,s)$, implying $V(a_*;\Pi,q) = \sum_s \bar{\pi}_s u(a_*,s)$. Since $(\Pi,q) \gtrsim (\Pi',q')$ implies $\bar{\pi} = \bar{\pi}'$, it follows that $V(a_*;\Pi,q) = V(a_*;\Pi',q')$.

(L.4) is obtained by applying (L.1) to both a* and a*' in turn.

The Propositions are stated here in their mathematical form. The optimal first period position refers to that with the highest value, $\bar{r}(a) + V(a;\Pi,q)$.

PROPOSITION 1: $\bar{r}(a^*) \geq \bar{r}(a^{*\prime})$ implies $\bar{r}(a^*) + V(a^*;\Pi,q) \geq \bar{r}(a^{*\prime}) + V(a^{*\prime};\Pi,q)$ for all (Π,q).

Proof: Immediate from (L.4).

PROPOSITION 2: $\bar{r}(a_*) + V(a_*;\Pi,q) \geq \bar{r}(a_*^{\prime}) + V(a_*^{\prime};\Pi,q)$ implies $\bar{r}(a_*) + V(a_*;\Pi',q') \geq \bar{r}(a_*^{\prime}) + V(a_*^{\prime};\Pi'q')$ whenever $\Pi q = \Pi' q'$.

Proof: From the definition $\bar{r}(a) \equiv \sum_s \bar{\pi}_s r(a,s)$ in (2), and the demonstration that $V(a_*;\Pi,q) = \sum_s \bar{\pi}_s u(a_*,s)$ in the proof of (L.3), it follows that $\bar{r}(a_*) + V(a_*;\Pi,q) = \sum_s \bar{\pi}_s (r(a_*,s) + u(a_*,s)) = \bar{r}(a_*) + V(a_*;\Pi',q')$, where $\bar{\pi} = \Pi q = \Pi' q'$.

To shorten the statements of the remaining propositions, we use the fact that the change in value of a position is the same as the change in $V(a;\Pi,q)$ when the prior belief $\bar{\pi}$ is fixed (i.e., when $(\Pi,q) \gtrsim (\Pi',q')$).

PROPOSITION 3: $(\Pi,q) \gtrsim (\Pi',q')$ implies $V(a;\Pi,q) - V(a;\Pi',q') \geq V(a_*;\Pi,q) - V(a_*;\Pi',q')$.

Proof: The left side of the inequality is non-negative by (L.2). The right side is 0 by (L.3).

PROPOSITION 4: $(\Pi,q) \gtrsim (\Pi_*,q_*)$ implies $V(a^*;\Pi,q) - V(a^*;\Pi_*,q_*) \geq V(b;\Pi,q) - V(b;\Pi_*,q_*)$

Proof: (Π_*,q_*) indicates no variability in beliefs, and $(\Pi,q) \gtrsim (\Pi_*,q_*)$ implies $\Pi q = \Pi_* q_* = \bar{\pi}$. Therefore $\pi_*(y_*) = \bar{\pi}$ for all y_*. The definition of $\tilde{b}$ and fact that $c(a^*,b,s) = 0$ for all b,s then implies $V(a^*;\Pi_*,q_*) = \underset{b\varepsilon D}{\text{Max}} \sum_s \bar{\pi}_s u(b,s) = \sum_s \bar{\pi}_s u(\tilde{b},s)$. Similarly, $V(\tilde{b};\Pi_*,q_*) = \sum_s \bar{\pi}_s u(\tilde{b},s)$ since $c(\tilde{b},\tilde{b},s) = 0$ and $c(\tilde{b},b,s) \geq 0$ for all s, $b \neq \tilde{b}$. Hence $V(a^*;\Pi_*,q_*) = V(\tilde{b};\Pi_*,q_*)$. Combining this with $V(a^*;\Pi,q) \geq V(\tilde{b};\Pi,q)$ from (L.1) gives the claimed inequality.

PROPOSITION 5: $c(\tilde{b},b,s) = c(\tilde{b})$ for s and $b \neq \tilde{b}$, and $(\Pi,q) \gtrsim_S (\Pi',q')$ imply

$$V(a^*;\Pi,q) - V(b;\Pi,q) \geq V(a^*;\Pi',q') - V(b;\Pi',q').$$

Proof: The definition of $v(a;\pi)$ and fact that $c(a^*,b,s) = 0$ for all b,s imply the first equality in

$$\text{(A.1)}\quad v(a^*;\pi) - v(\tilde{b};\pi) = \underset{b}{\text{Max}} \sum_s \pi_s u(b,s) - \underset{b}{\text{Max}} \sum_s \pi_s (u(b,s) - c(\tilde{b},b,s))$$

$$= \text{Min}\ \{\ c(\tilde{b})\ ,\ \underset{b}{\text{Max}} \sum_s \pi_s (u(b,s) - u(\tilde{b},s))\ \}.$$

The second equality follows from $c(\tilde{b},b,s) = \{0 \text{ for } b = \tilde{b},\ c(\tilde{b}) \text{ for } b \neq \tilde{b}\}$. Since $(\Pi,q) \gtrsim_S (\Pi',q')$ requires that $\pi'(y) = \lambda_y \pi(y) + (1-\lambda_y)\bar{\pi}$, where $0 \leq \lambda_y \leq 1$ for each $y \varepsilon Y$,

$$\text{(A.2)}\quad \underset{b}{\text{Max}} \sum_s \pi_s(y)(u(b,s) - u(\tilde{b},s)) \leq \lambda_y \underset{b}{\text{Max}} \sum_s \pi_s(y)(u(b,s) - u(\tilde{b},s)) + (1-\lambda_y) \underset{b}{\text{Max}} \sum_s \bar{\pi}_s (u(b,s) - u(\tilde{b},s))$$

$$\leq \underset{b}{\text{Max}} \sum_s \pi_s(y)(u(b,s) - u(\tilde{b},s)).$$

The first inequality follows from the convexity of $\underset{b}{\text{Max}} \sum_s \pi_s (u(b,s) - u(\tilde{b},s))$ in π, as it is the maximum of a finite collection of linear functions of π. The definition of $\tilde{b}$ implies $\underset{b}{\text{Max}} \sum_s \pi_s (u(b,s) - u(\tilde{b},s)) = 0$; this, together with $\lambda_y \leq 1$ and $\underset{b}{\text{Max}} \sum_s \pi_s(y)(u(b,s) - u(\tilde{b},s)) \geq 0$, gives the second inequality.

Combining (A.1) and (A.2) gives

(A.3) $v(a^*;\pi(y)) - v(\tilde{b};\pi(y)) \geq v(a^*;\pi'(y)) - v(\tilde{b};\pi'(y))$

for each $y \in Y$. Since $\gtrsim_S$ requires that $q_y = q_y'$ for each y, this means that

(A.4) $V(a^*;\Pi,q) - V(\tilde{b};\Pi,q) \geq V(a^*;\Pi',q') - V(\tilde{b};\Pi',q')$.

PROPOSITION 6: (i) For each $a \in D$ there exists $D_a \subset D$ such that $c(a,b,s) = \{0 \text{ for } b \in D_a, \infty \text{ for } b \notin D_a\}$. For all $a, a' \in D$, either $D_a \subset D_{a'}$ or $D_a \supset D_{a'}$.

(ii) For each pair $a, d \in D$ there exists $d' \in D_a$ such that, for all π, either $\underset{b \in D_a}{\mathrm{Max}} \sum_s \pi_s u(b,s) \geq \sum_s \pi_s u(d,s)$ or $\underset{b \in D_a}{\mathrm{Max}} \sum_s \pi_s u(b,s) = \sum_s \pi_s u(d',s)$.

Then $(\Pi,q) \gtrsim (\Pi',q')$ and $a \gtrsim_F a'$ implies

$$V(a;\Pi,q) - V(a';\Pi,q) \geq V(a;\Pi',q') - V(a';\Pi',q').$$

Proof: Condition (i) implies that

(A.5) $v(a;\pi) = \underset{b \in D_a}{\mathrm{Max}} \sum_s \pi_s u(b,s)$.

Hence, for $a \gtrsim_F a'$,

(A.6)
$$\begin{aligned} v(a;\pi) - v(a';\pi) &= \underset{b \in D_a}{\mathrm{Max}} \sum_s \pi_s u(d,s) - \underset{b' \in D_{a'}}{\mathrm{Max}} \sum_s \pi_s u(b',s) \\ &= \underset{d \in D_a}{\mathrm{Max}} \left\{ \underset{b \in \{D_{a'},d\}}{\mathrm{Max}} \sum_s \pi_s u(b,s) - \underset{b' \in D_{a'}}{\mathrm{Max}} \sum_s \pi_s u(b',s) \right\} \\ &= \underset{d \in D_a}{\mathrm{Max}} \left\{ \mathrm{Max}\, \{ 0 , \sum_s \pi_s u(d,s) - \sum_s \pi_s u(d',s) \} \right\}. \end{aligned}$$

The position d' in the last expression is the <u>fixed</u> $d' \in D_a$ asserted to exist for each $d \in D$ in condition (ii) (i.e., $d'(a,d)$). The second and third expressions are equal since $D_a \supset D_{a'}$; the last equality follows from condition (ii). Since the innermost Maximum in the last expression in (A.6) is between two linear functions of π, it is a convex function of π.

The outer Maximum over $d \in D_a$ is thus a maximum of convex functions, and hence $v(a;\pi) - v(a';\pi)$ is convex in π. Therefore $(\Pi,q) \gtrsim (\Pi',q')$, by definition (I) of section II, implies that

$$\begin{aligned}
\text{(A.7)}\quad V(a;\Pi,q) - V(a';\Pi,q) &= \sum_y q_y[v(a;\pi(y)) - v(a';\pi(y))] \\
&\geq \sum_{y'} q'_{y'}[v(a;\pi'(y')) - v(a';\pi'(y'))] \\
&= V(a;\Pi',q') - V(a';\Pi',q').
\end{aligned}$$

Requiring that the flexibility ordering on D be complete just allows us to strengthen the implication of increased variability of (Π,q) from "the optimal initial position is not less flexible" to "the optimal initial position is at least as flexible".

FOOTNOTES

[1] $|X|$ denotes the number of elements in a set X.

[2] The sets of possible observations, Y and Y', for the two information structures are left implicit in our notation since it is only the induced probability distribution on the $|S| - 1$ dimensional simplex $\{\pi\}$ which concerns us. Y and Y' may differ in both the number and type of messages they contain.

[3] More correctly, we should say "as least as variable as", since the relation as we have defined it is reflexive. The shorter phrase is used for compactness. We adopt the same convention by using the phrase "more flexible than" instead of "as least as flexible as" in section III.

[4] This term was used by Cummings and Norton (1974, p.1022) in commenting on the work of Fisher, Krutilla and Cicchetti (1972).

[5] There is a further difficulty when there are only two possible positions, say a and a'. Definition (F) implies that both $a \gtrsim_F a'$ and $a' \gtrsim_F a$: Since there is only one alternative to a', namely a, and $c(a',a,s) \geq c(a,a,s) = 0$, it follows that $c(a',b,s) \geq c(a,b,s)$ for all $b \neq a'$, and hence $a \gtrsim_F a'$; reversing the roles of the two positions produces $a' \gtrsim_F a$. With two alternatives, the only possible rankings are between a perfectly flexible position and one that is imperfectly flexible, and between an imperfectly flexible position and one that is irreversible. With three or more alternatives these ambiguities do not arise.

[6] Notice that such a relationship, when valid, also says that any increment in flexibiltiy raises the value of any increment in information. Thus, if the individual's choice was how much information to purchase, with his flexibility being exogenously given to him -- the reverse of the conceptual experiment we are interested in -- we could say that an increase in flexibility induces the rational agent to purchase a larger quantity of information (assuming, that is,

that the total payoff is additive in the price paid for the information and the profit obtained by exploiting it). The relationship between flexibility and information is thus much like that between complementary factors of production.

[7] In light of the partial nature of the orderings involved and our verbal convention of using "more" to indicate "as least as much as", it should be remembered that the phrase, "more variable beliefs imply a more flexible initial position", is technically interpreted to mean, "at least as variable beliefs imply a not less flexible position".

[8] In the circumstances of Proposition 1, but not in those of Proposition 2, the optimal second period position will depend other aspects of (Π,q) than $\bar{\pi}$.

[9] Confining our attention to three period problems does not necessarily limit the general applicability of this section's results. The second period payoff can be interpreted as the next period's value function in an ongoing dynamic program, with b as the decision state variable and s as the "nature" state variable (which includes any exogenously given information that has accumulated as of period three).

[10] Keynes (1936) later replaces the term "degree of rational belief" with "confidence of beliefs".

[11] This observation is also made by Hirshleifer (1972) and Hicks (1974).

[12] Proposition 4 appears to resolve the issue raised by Cummings and Norton (1974) about whether absolute irreversibility was crucial to the Fisher, Krutilla, and Cicchetti (1972) argument.

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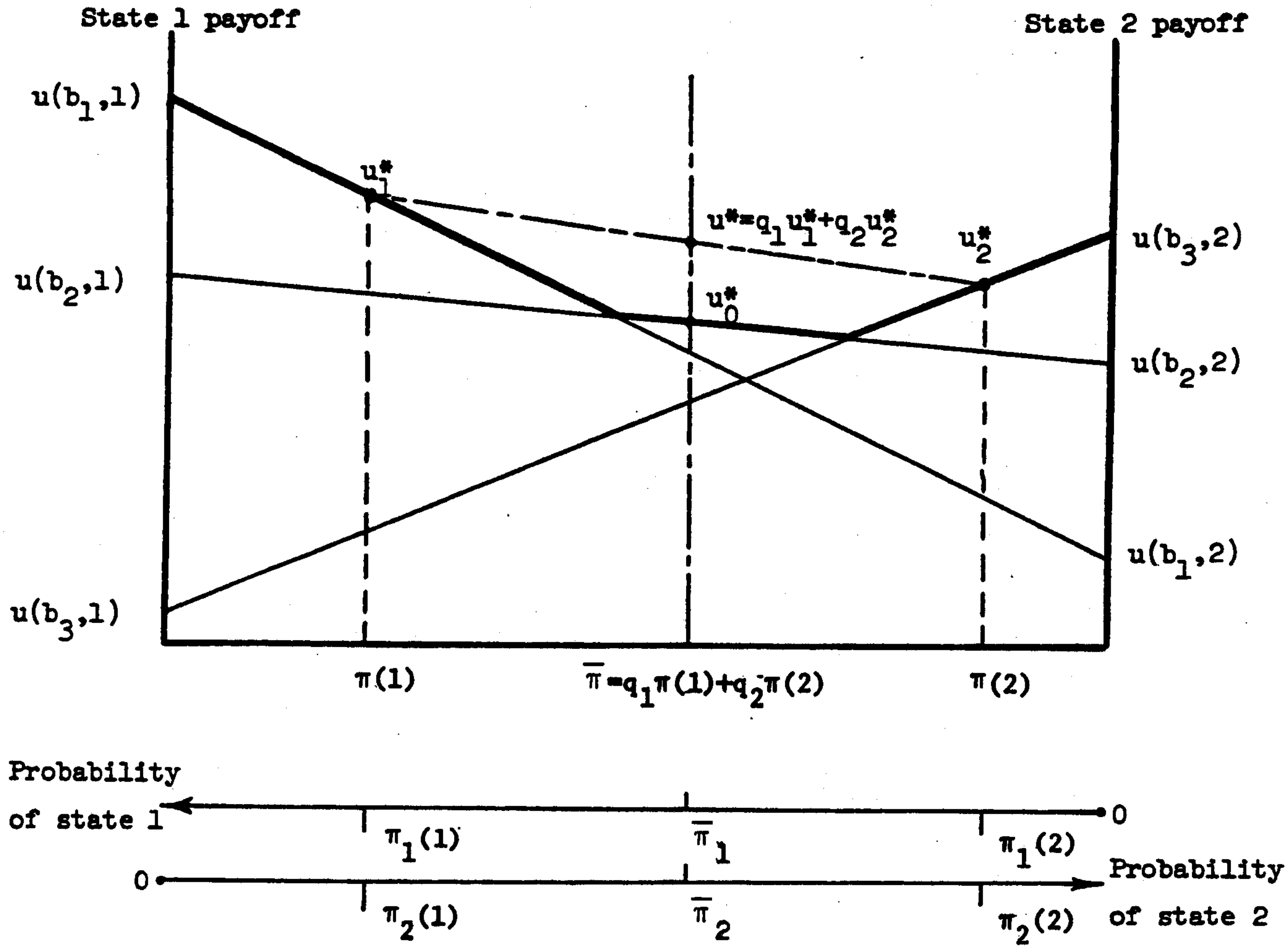
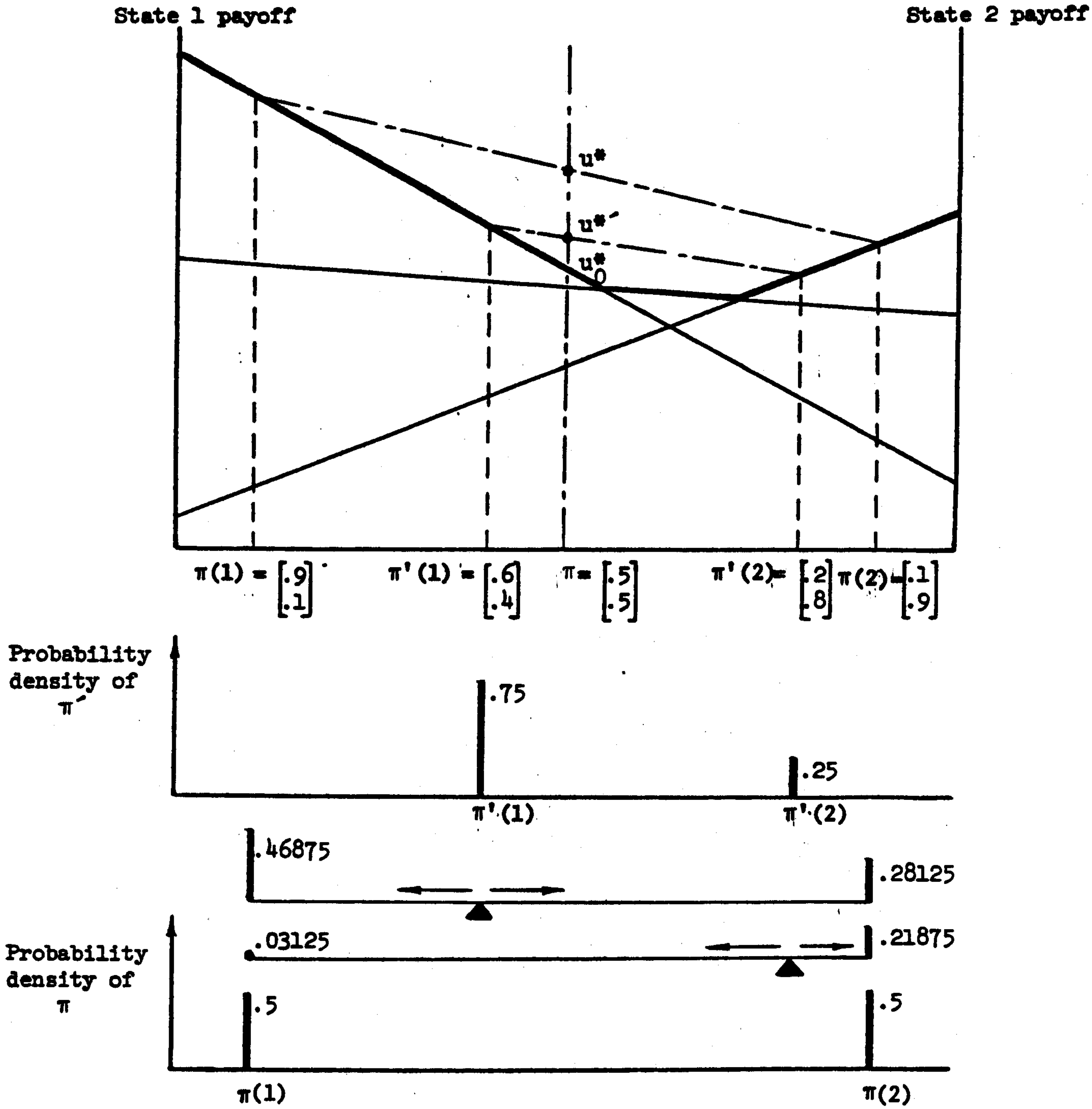

State 1 payoff
State 2 payoff
u(b1,1)
u(b2,1)
u(b3,1)
u*1
u*=q1u*1+q2u*2
u*2
u*0
u(b3,2)
u(b2,2)
u(b1,2)
π(1)
π̄=q1π(1)+q2π(2)
π(2)
Probability
of state 1
0
π1(1)
π̄1
π1(2)
0
Probability
of state 2
π2(1)
π̄2
π2(2)

FIGURE 1



$$\Pi = \begin{bmatrix} .9 & .1 \\ .1 & .9 \end{bmatrix} \qquad \Pi' = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \qquad M = \begin{bmatrix} .625 & .125 \\ .375 & .875 \end{bmatrix} \qquad q = \begin{bmatrix} .5 \\ .5 \end{bmatrix} \qquad q' = \begin{bmatrix} .75 \\ .25 \end{bmatrix}$$

FIGURE 2

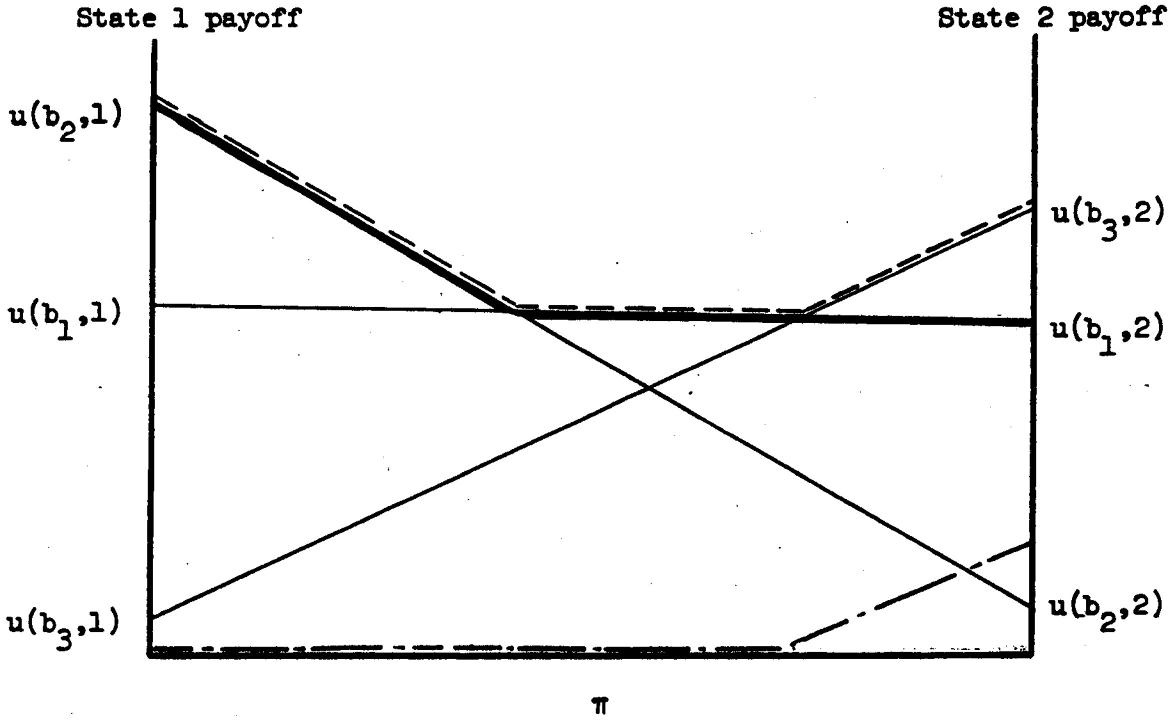


FIGURE 3A

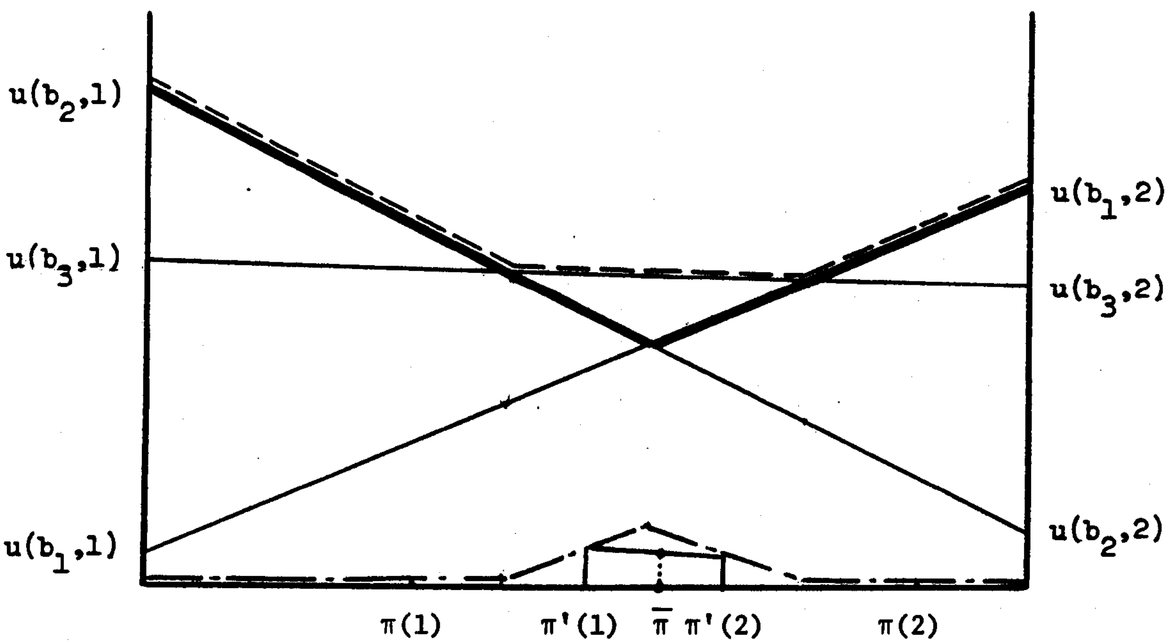


FIGURE 3B

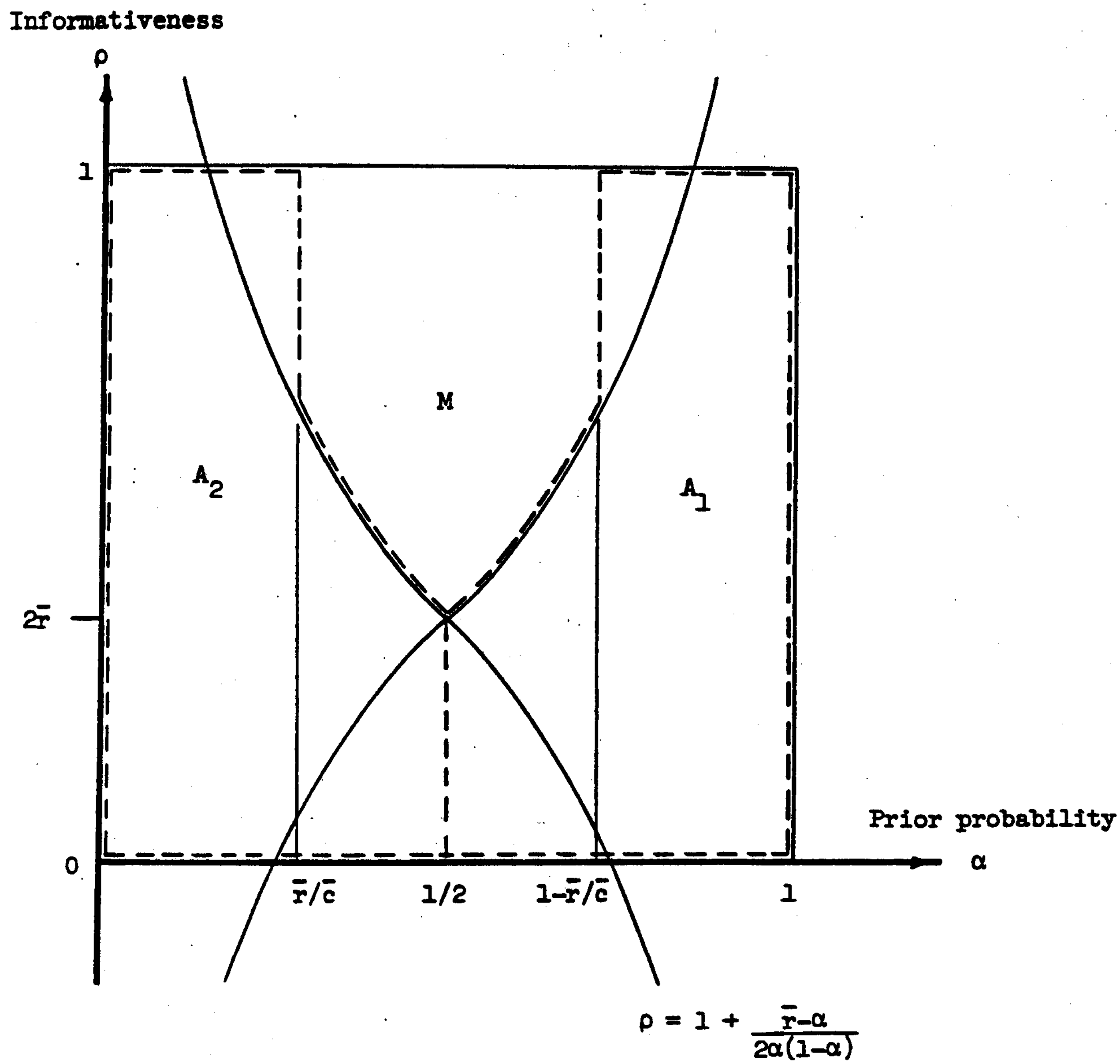

Informativeness
ρ
1
M
A2
A1
2r̄
0
Prior probability
α
r̄/c̄
1/2
1-r̄/c̄
1
ρ = 1 + (r̄-α)/(2α(1-α))

FIGURE 4