THE LIQUIDITY PREMIUM AND THE SOLIDITY PREMIUM

by

Susan Woodward

UCLA

UCLA Department of Economics

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The liquidity premium and the solidity premium are aspects of the term structure of interest rates as related to interest-rate expectations under uncertainty. The liquidity premium is defined as the difference between the forward short-term interest rate, implicit in the current term structure of short-term and long-term rates, and the expected future short-term interest rate. Such an interest-rate differential arises, roughly speaking, in circumstances where the more "liquid" short-term bonds would otherwise (i.e., if there were no differential) tend to be preferred by investors. Analogously, a solidity premium -- defined as the difference between the forward short-term discount and the expected future short-term discount -- arises when the more "solid" long-term bonds would otherwise tend to be preferred. (These somewhat imprecise statements will be amended in what follows.)

The liquidity premium and the solidity premium (which are not simple inverses one of the other) are both determined by the interaction of interest rates and incomes at earlier and later dates. I will show how endowments, productive opportunities, preferences and beliefs, and the arrival of information shape this interaction. The analysis here advances upon some or all earlier treatment of this topic in the following main respects: (1) It is a general equilibrium treatment in which prices and interest rates are all endogenous variables; to postulate, as some of the earlier works on liquidity do, arbitrary patterns of interest-rate variations or anticipations gives seriously misleading results. (2) My analysis considers endowment risk at both near-future and far-future dates. (3) The effects of information arriving at different dates, and concerning endowments at different dates, are clarified. (4) I will distinguish the problem of the term structure from that of liquidity/solidity. (5) And, finally, I will show why and how determination of the solidity premium must be integrated with the more familiar liquidity premium to gain a full understanding.

I. INTRODUCTION AND PREVIEW

Dealing, for simplicity, with a three-date model (dates 0, 1, and 2 years from the present) the current short-term annual interest rate ${}_0r_1$ is defined in:

$$\frac{{}_0P_1}{{}_0P_0} \equiv \frac{1}{1+{}_0r_1} \tag{1}$$

Here ${}_0P_1$ is the price quoted today (date-0) of a unit claim to income to be received next year (at date-1). ${}_0P_0$ is the price of income today; I will ordinarily take current income of any date as the numeraire commodity for prices quoted at that date, so that ${}_0P_0 \equiv 1$. The interest rate denoted ${}_0r_1$ is then the rate quoted at date-0, today, for discounting one-year future claims into their current (present-value) equivalent.

Analogously, the current long-term (2-year) interest rate ${}_0R_2$ is defined in:

$$\frac{{}_0P_2}{{}_0P_0} \equiv \frac{1}{(1+{}_0R_2)^2} \tag{2}$$

Here ${}_0P_2$ is the price quoted today of a claim to income to be received at date-2, and ${}_0R_2$ is the corresponding long-term interest rate for discounting (with proper allowance for annual compounding) such claims into their present-value equivalent. But the price ratio on the L.H.S. of equation (2) can also be written in another way that serves to define the forward short-term rate ${}_0r_2$:

$$\frac{{}_0P_2}{{}_0P_0} \equiv \frac{1}{(1+{}_0R_2)^2} \equiv \frac{1}{(1+{}_0r_1)(1+{}_0r_2)} \tag{3}$$

The notation ${}_0r_2$ indicates that this interest rate, though short-term in that it discounts income claims from date-2 back only to the previous year (date-1), is a rate quoted (or, more precisely in this case, implicit in other quoted prices and interest rates) at date-0.

As the economy moves historically through time, at date-1 an actual short-term rate ${}_1r_2$ will come into existence for discounting date-2 claims. This is the future short-term rate, defined in terms of the prices and interest rate quoted at date-1:

$$\frac{{}_1P_2}{{}_1P_1} \equiv \frac{1}{1+{}_1r_2} \tag{4}$$

(Here, since date-1 income becomes the numeraire for all claims quoted at date-1, it must be that ${}_1P_1 \equiv 1$.) Viewed from date-0, however, this actual future short-term rate ${}_1r_2$ will be uncertain. This brings us, finally, to the formal definition of the Liquidity premium 1:

$$L \equiv {}_0r_2 - E({}_1r_2) \tag{5}$$

The liquidity premium is the excess, on average, of the known forward short-term rate of interest over the unknown future short-term rate.

The rationale for calling L a "liquidity premium" may not be immediately obvious. The basic idea is that an investor who has committed himself at date-0 to receive a forward return ${}_0r_2$ -- normally by investing in a long-term security whose overall return ${}_0R_2$ is a kind of average of ${}_0r_1$ and ${}_0r_2$, as indicated in equation (3) -- has tied his hands with regard to possibly taking advantage of fluctuations up or down in the future interest rate ${}_1r_2$ that will become available at date-1. Put another way, for an investor to make only a short-term commitment at date=0, in return for a yield ${}_0r_1$, leave him in a more "flexible" position at date-1. Note also that the individual will ordinarily anticipate having better information as to his personal situation, as well as to the objective market opportunities, at date-1 in comparison with date-0. The liquidity premium is thus, it is commonly asserted, normally positive to compensate the individual for loss of flexibility consequent upon undertaking long-term commitments.

To interpret the solidity premium it is illuminating to think in terms of bond "discount" as the term is employed (for example) in trading of U.S. Treasury Bills. (The bonds I deal with in this model are, for simplicity, all pure discount bonds, i.e., there are no interim interest payments.) The discount on a current short-term bond is simply:

$$\text{Current Discount} = {}_0d_1 \equiv \frac{1}{1+{}_0r_1} - 1 = \frac{{}_0P_1}{{}_0P_0} - 1 \tag{6}$$

And the discount on the forward one-year bond implicit in the market for short-term and long-term bonds is:

$$\text{Forward Discount} = {}_0d_2 \equiv \frac{1}{1+{}_0r_2} - 1 = \frac{{}_0P_2}{{}_0P_1} - 1 \tag{7}$$

Once again, as time unfolds, an actual short-term discount will come into existence at date-1, defined by:

$$\text{Future Discount} = {}_1d_2 \equiv \frac{1}{1+{}_1r_2} - 1 = \frac{{}_1P_2}{{}_1P_1} - 1 \tag{8}$$

Note that discounts are normally negative magnitudes as interest rates are normally positive. The forward d and r, and the corresponding price ratios, are related by:

$$\frac{{}_0P_2}{{}_0P_1} \equiv 1 + {}_0d_2 \equiv \frac{1}{1+{}_0r_2} \equiv \frac{{}_0P_1}{{}_0P_2} \tag{9}$$

Analogous relations hold, of course, for the current and future d and r and respective price ratios.

This brings us to the formal definition of the solidity premium, S:

$$S \equiv {}_0d_2 - E({}_1d_2) = \frac{1}{1+{}_0r_2} - E(\frac{1}{1+{}_1r_2}) \tag{10}$$

or

$$S \equiv \frac{{}_0P_2}{{}_0P_1} - E(\frac{{}_1P_2}{{}_1P_1}) \tag{11}$$

Note that L can also be expressed in terms of price-ratios:

$$L = \frac{{}_0P_1}{{}_0P_2} - E(\frac{{}_1P_1}{{}_1P_2}) \tag{12}$$

The rationale for calling S a "solidity premium" may also not be immediately obvious. The basic idea is that an investor who has committed himself at date-0 to receive ${}_0r_2$ -- by purchasing a long-term bond or forward contract -- is certain of what he will receive at date-2. If it is on the average more expensive to purchase date-2 consumption by buying at date-0 rather than by waiting until date-1 and then buying a short-term bond, i.e., if $\frac{{}_0P_2}{{}_0P_1} > E(\frac{{}_1P_2}{{}_1P_2})$, the investor has paid a "solidity premium" for the privilege of knowing his date-2 income at date-0.

Yet S is neither the negative nor the reciprocal of L as can be shown by a simple example. Suppose the <u>forward</u> short-term rate of interest were 10%, and let the two possible, equally probable future short-term rates be 0% and 20%. Evidently, here the expected future short-term rate equals the forward short-term rate, and L = 0. What about S? Since the forward short-term interest rate ${}_0r_2$ = 10%, the forward discount ${}_0d_2$ is 1/1.1 - 1 ≐ - .091.

The future discount ${}_1d_2$ is either $1/1 - 1 = 0$ or $1/1.2 - 1 \doteq -.167$, or on average approximately $-.083$. Thus $S = {}_0d_2 - E({}_1d_2) \doteq -.008$ is negative, even though $L = 0$. (Indeed, we shall see shortly that apart from degenerate cases, if $L = 0$ then S must be negative -- while if $S = 0$, L must be negative.)

I could "continuize" L and S by defining them in terms of continuously compounded interest rates and discounts. In the continuous version it can be shown that indeed, $L = -S$.[1] But this trick cannot be used on the analysis of wheat/corn prices or on franc/dollar currency prices where the analog of the L versus S problem also arises. Therefore, the analysis in terms of discrete time has the more general implications and I shall proceed discretely.

In Section II following, several theories of the liquidity premium will be surveyed briefly. Section III provides the basic correct analysis of the liquidity premium and solidity premium in terms of modern general-equilibrium contingent-claim theory, leading to a number of propositions -- some familiar, some less so -- of economic significance. Section IV considers further how intertemporal productive transformations, and in particular the possibility of simple <u>storage</u>, affect the liquidity and solidity premiums. Section V turns to the role of <u>information</u>, and examines the implications of alternative patterns of information availability. The concluding section provides a summary, and points toward additional unresolved problems.

II. THEORIES OF THE LIQUIDITY PREMIUM

In this section I will not attempt to provide a thorough confirmation or refutation of previous theories, but merely to put the present paper in the context of preceding discussions.

The first theory to be mentioned, though not historically the first to be proposed is the "Pure Expectations" hypothesis of Meiselman [1962]. This hypothesis asserts that the forward interest rate ${}_0r_2$ must be equal to the expected realized future rate $E({}_1r_2)$ -- so that the liquidity premium L defined in (5) is zero. I will show that the Pure Expectations hypothesis does arise in some special cases, but that the only systematic case, risk neutrality, has additional implications which are quite implausible.

The best-known theory which leads to a positive liquidity premium was put forward by Keynes [1930] and Hicks [1946]. Their theory assumes risk-aversion. In particular, these authors emphasize the variability of future short-term interest rates and investors' consequent uncertainty about the capital values of long-term securities. From (2) and (3) above, the price ratio of long-term relative to short-term maturities quoted at date-0 is the known magnitude ${}_0P_2/{}_0P_1 = 1/(1+{}_0r_2)$. At date-1, the corresponding ratio will be ${}_1P_2/{}_1P_1 = 1/(1+{}_1r_2)$, whose magnitude is uncertain at date-0. Also, as ${}_1P_1$ is necessarily equal to unity, all the uncertainty as to the interest rate ${}_1r_2$ is loaded onto the value of the longer-term claim ${}_1P_2$. (In more general T-date model, it of course remains true that the capital values of longer-term maturities will be more sensitive functions of the interest rate than the capital values of short-terms, even if the latter are not entirely free of risk.) The upshot is that, since long-term securities are in this sense riskier, a premium must (allegedly) be paid to their holders. That is, ${}_0P_2$ will be relatively low (long-term securities will be a relative bargain) and thus ${}_0R_2$ and ${}_0r_2$ relatively high (long-term yields, and therefore forward rates of interest, will be attractively large).

The Keynes-Hicks development has been criticized on the grounds that it overemphasizes aversion to "calital-value risk" as opposed to "income risk." For example, someone concerned only to lock in a future flow of income yield would simply make a long-term investment commitment today, and then be entirely unconcerned about possible interest-rate fluctuations leading to changes in capital values at intermediate dates. More precisely, we can say that the Keynes-Hicks argument would be valid if bondholders were risk-averse only as to consumption in the near future (at t=1 in our model). Challenging this assumption, Modigliani and Sutch [1966] advanced their "Preferred Habitat" theory. Allowing for subjective variations in motives to save or to draw income at different dates (due, for example, to life-cycle considerations), they show that risk-aversion as such does not in general lead investors to prefer short-term investments. Rather, each individual will have a preferred maturity habitat, and will be concerned with risk at that date. At the market-wide level, Modigliani and Sutch conclude, nothing general can be said about the sign of the liquidity premium L.

The presumption in favor of a normally rising term structure and consequent liquidity premium was temporarily restored in the analysis of Hirshleifer [1972]. Hirshleifer's model represented an advance upon preceding theories in several main respects: (1) it contains an explicit contingent-claim treatment of uncertainty, and of the effects of information arrival in resolving uncertainty; (2) it connects financial liquidity with the underlying physical liquidity of assets in the economy; and (3) most important, it is a general-equilibrium model in which prices and interest rates do not emerge from thin air but are rather founded upon individuals' endowments, preferences, and opportunities with respect to both time and states of the world. Hirshleifer's model was little more than an example, however, and a number of important theorems will emerge from my more general approach here. In addition, I will show that Hirshleifer somewhat misconceived the role of incoming information, and indeed that his justification for a positive liquidity premium cannot in general be sustained.

III. LIQUIDITY PREMIUM AND SOLIDITY PREMIUM IN PURE EXCHANGE

This section analyzes the forces underlying the liquidity premium and solidity premium, using an explicit contingent-claim model of income uncertainty at near-future (date-1) and far-future (date-2) dates. The present date is assumed free of uncertainty; at date-0 each individual is supposed to have a specific known endowment $\bar{c}_0$ of the real income commodity ("corn"). But at date-1 his endowment will be the probability distribution $(\bar{c}_{11},\ldots,\bar{c}_{1E};\pi_{11},\ldots,\pi_{1E})$, where e = 1,...,E indicates the state of the world at the earlier date and π_{1E} represents the associated probability (assessed at date-0). Similarly, at date -2 the endowment will be $(\bar{c}_{21},\ldots,\bar{c}_{2S};\pi_{21},\ldots,\pi_{2S})$, where S = 1,...,S is the index for states of the world at the later date and π_{2s} is the associated probability in terms of beliefs at date-0. In this section "pure exchange" is assumed: there are no productive opportunities (e.g., storage) for physically transforming income of one date into income of any other date.

Suppose there are complete markets at date-0 for claims to consumption contingent upon states of the world, at any date.[2] Then the current price of a claim to income at date-1 contingent upon state-e can be denoted ${}_0P_{1e}$, and similarly ${}_0P_{2s}$ is the current price of a claim to income at date-2 contingent upon state-s. Certainty claims to income as of any given date can be purchased by buying a full complement of the corresponding contingent claims. Thus:

$$ {}_0P_1 \equiv \sum_e {}_0P_{1e} \tag{13}$$

$$ {}_0P_2 \equiv \sum_s {}_0P_{2s} \tag{14}$$

Once the passage of time reveals the state of the world at date-1, individuals will in general revise their beliefs about the probabilities of the date-2 states. (That is, the advent of state-e is not only an income-event but also generally a information-event.) The revised probabilities can be denoted $\pi_{s.e}$. Then a certainty claim to date-2 consumption, quoted at date-1 after state-e has obtained, can be expressed as:

$$_{1e}P_2 = \sum_s {}_{1e}P_{2s.e} \tag{15}$$

We can define the future short-term interest rate $_{1e}r_2$, contingent upon state-e having been realized at date-1,

$$\frac{_{1e}P_2}{_{1e}P_1} = \frac{1}{1+{}_{1e}r_2} \tag{16}$$

As usual, the denominator on the L.H.S. would be unity, since it is the price of the numeraire commodity current at date-1 after state-e obtains.

The expected future rate of interest, in terms of probability beliefs at date-0, can then be written

$$E(1+{}_1r_2) = E\left(\frac{_1P_1}{_1P_2}\right) = \sum_e \pi_e \frac{_{1e}P_1}{\sum_s {}_{1e}P_{2s.e}} \tag{17}$$

Repeating equation (5), the liquidity premium L can be expressed in terms of prices of the dated consumptions as:

$$L = \frac{_0P_1}{_0P_2} - E\left(\frac{_1P_1}{_1P_2}\right) \tag{18}$$

And in terms of the underlying bundles of contingent claims as:

$$L = \frac{\sum_e {}_0P_{1e}}{\sum_s {}_0P_{2s}} - \sum_e \pi_e \frac{_{1e}P_1}{\sum_s {}_{1e}P_{2s.e}} \tag{19}$$

To press further, I shall have to say something more about the forces underlying the determination of prices. First of all, I will assume away any differences of beliefs in the economy: everyone agrees as to the probabilities π_e, π_s, and $\pi_{s.e}$. Let every individual make choices under uncertainty so as to maximize expected utility $U = E(v)$, where $v(c_0, c_1, c_2)$ is his "cardinal" preference-scaling function for dated consumption income vectors. In addition to the standard postulate of <u>state</u>-independence of the utility function, I will also be making the simplifying assumption of <u>time</u>-independence: that the v function is separable in the variables c_0, c_1, and c_2. While this limits the generality of our results, nonseparable tastes (i.e., allowing for possible intertemporal preference complementarities) would be a second-order effect that can only be accommodated by rather burdensome notation.[3]

If the social endowments of income are positive in every state, and if everyone assigns infinite negative utility to zero consumption at any date, an interior solution is guaranteed in which each individual holds positive amount of every contingent claim c_{1e} and c_{2s}. Then in equilibrium at date-0 each individual's expected marginal utilities will be proportional to the prices:

$$\frac{v_0}{{}_0P_0} = \frac{\pi_e v_{1e}}{{}_0P_{1e}} = \frac{\pi_s v_{2s}}{{}_0P_{2s}} \tag{20}$$

where $v_0 \equiv \partial U/\partial c_0$, $\pi_{1e} v_{1e} \equiv \partial U/\partial c_1$, and $\pi_{2s} v_{2s} \equiv \partial U/\partial c_2$.[4] This equation reveals, therefore, the relation of prices to preferences and endowments (which together determine the marginal utilities) and to probability beliefs.

Substituting from (20) into (19) leads to:

$$L = \frac{\sum_e \pi_e v_{1e}}{\sum_s \pi_s v_{2s}} - \sum_e \pi_e \frac{v_{1e}}{\sum_s \pi_{s.e} v_{2s}} \tag{21}$$

Or, in a more condensed notation:

$$L = \frac{E(v_1)}{E(v_2)} - E \frac{v_1}{\underset{e}{E}(v_2)} \tag{22}$$

Here $\underset{e}{E}$ indicates the expectation (of date-2 marginal utility) conditional upon state-e at date-1. (Expectations symbolized simply as E are taken with respect to beliefs at date-0.) Of course:

$$E(\underset{e}{E}(v_2)) = \sum_e \pi_e \underset{e}{E}(v_2) = E(v_2) \tag{23}$$

Before going to more complex results involving possibilities of storage over time (physically liquid assets) and/or patterns of information arrival other than the simple advent of some state-e at date-1, we can develop a number of propositions quite directly from the preceding equations.

PROPOSITION #1: If individuals are all risk-neutral, the liquidity premium is zero. (The Pure Expectations hypothesis is valid.)

The preference-scaling function $v(c_0,c_1,c_2)$ is, I have been assuming, additively separable in c_0,c_1, and c_2. If in addition each individual is risk-neutral at each date, marginal utilities v_0,v_{1e}, and v_{2s} will be constants (not necessarily all equal). More specifically, at date-1 all the v_{1e} will have a common value v_1, at date-2 all the v_{2s} will have a common value v_2 and of course v_0 will also be constant. In a representative-individual model, the fixed v_1 and v_2 make L necessarily zero in equation (22). Note, however, that the converse does not hold: L=0 does not in general imply risk-neutrality. As has been shown by LeRoy (1978) the contingent interest rates ${}_{1e}r_2$ will be non-random. (Moreover, since v_1 and v_2 are insensitive to incomes at totals 1 and 2, things become a little trickier if we allow interpersonal differences in magnitudes of the constant (risk-neutral) marginal utilities, for then equation (20) could not hold for everyone. (That is, some or all individuals would go to corner solutions where an inequality condition rather than an equality would govern the ratios of prices to expected marginal utilities.) The overall consequence for L is not in general predictable, but there is no systematic tendency for the liquidity premium to be greater or less than zero.

PROPOSITION #2: Maintaining our assumption of intertemporal separability in preference, and assuming away productive intertemporal transformations, if there is no endowment risk at date-2 then the liquidity premium is zero.

If every individual has a constant endowment at date-2, v_2 is simply a constant. Then inspection of equation (22) reveals immediately that L = 0.

PROPOSITION #3: Continuing to assume intertemporal separability in preference and absence of intertemporal productive opportunities, if the endowment distributions at date-1 and date-2 are statistically independent then the liquidity premium is zero.

If the risks at date-2 are independent of those at date-1, the advent of any state at the earlier date conveys no additional information about what might happen at the later date. Then, in equation (22), $\underset{e}{E}(v_2 = E(v_2)$ and so L = 0.

Proposition #3 is a very robust result, as it is independent of degrees of risk-aversion and also of distributions of endowed incomes at either date. One important implication: it is <u>not</u> the case that greater uncertainty about the far future as compared to the near future (greater variance in c_2 than in c_1) tends of itself to produce a positive liquidity premium.

Another implication of our discussion is that "Preferred Habitats" such as have no direct implications for the liquidity premium. Modigliani and

Sutch successfully refuted the Keynes-Hicks argument that the liquidity premium was presumptively positive. Affirmatively, however, they claimed that the liquidity premium would be positive or negative, depending upon the aggregate weights of individuals' desires to save or to consume at different dates. But Preferred Habitats (whatever they may be) can only explain the term structure, not the liquidity premium! In particular, there might be such strong desires for current and near-future consumptions as to generate a sharply rising term-structure -- and yet, if (say) there is statistical independence between risks at dates-1 and -2 then Proposition #3 tells us the liquidity premium must be zero.

I now turn to the solidity premium. Starting from equation (11), the price-ratio version of the definition of S_1 the following development leads to an analog of equation (22):

$$S \equiv \frac{{}_0P_2}{{}_0P_1} - E\left(\frac{{}_1P_2}{{}_1P_1}\right)$$

$$S = \frac{\sum_s {}_0P_{2s}}{\sum_e {}_0P_{1e}} - E\left(\frac{\sum_s {}_{1e}P_{2s.e}}{{}_{1e}P_1}\right) \tag{24}$$

$$= \frac{\sum_s \pi_s v_{2s}}{\sum_e \pi_e v_{1e}} - \sum_e \pi_e \left(\frac{\sum_s \pi_{s.e} v_{2s}}{v_{1e}}\right) \tag{25}$$

$$= \frac{E(v_2)}{E(v_1)} - E\left(\frac{E_e(v_2)}{v_1}\right) \tag{26}$$

Certain very simple relations must hold between the signs of L and S, following from Jensen's Inequality which may be written:

$$E(1/x) \geq 1/E(x), \tag{27}$$

(where the equality holds only for non-random x).

In particular, taking x as the random variable, $\frac{{}_1P_2}{{}_.P_1}$, it can be shown that L and S <u>can never both be positive</u>. (Furthermore, only if $\frac{{}_1P_2}{{}_1P_1}$ is non-random can they both equal zero.)[6] Thus, the possible cases of interest are:

(1) $L < 0, \quad S < 0$

(2) $L \geq 0, \quad S < 0$

(3) $L < 0, \quad S \geq 0$

(4) $L = 0, \quad S = 0$ (the non-random case).

Using these relations, propositions for S analogous to our three preceding propositions for L can easily be derived.

PROPOSITION #1a: If individuals are all risk-neutral, the solidity premium is zero.

Risk-neutrality therefore leads to case (4) above, where L and S are <u>both</u> zero. The other ways this may occur are: 1) if endowments at date-1 and date-2 are both nonrandom, so that v_1 and v_2 are constants, and (2) in a pathological case we shall encounter later where interest rates are nonrandom.

PROPOSITION #2a: If there is no endowment risk at date-2, and preferences are separable and risk-averse (and no production is possible), the solidity premium is negative.

PROPOSITION #3a: With separable and risk-averse preferences and no production, if the endowments at date-1 and date-2 are statistically independent, the solidity premium is negative.

Both Propositions #2a and #3a are derived easily from equation (26). For #2a, $\underset{e}{E}(v_2)$ is simply just v_2 (not random) and for #3a $E(v_2)$ is not dependent on e and hence is simply $\underset{e}{E}(v_2)$. In both cases the term involving v_2

can be factored, leaving the term

$$\left[\frac{1}{E(v_1)} - E\left(\frac{1}{v_1}\right)\right] \tag{28}$$

to determine the sign of S. This bracketed expression, by Jensen's Inequality, is negative if v_1 is non-degenerate.

It is very enlightening to be able to express the liquidity premium and solidity premium in relation to anticipated fluctuations in interest rates and patterns of future income endowments.[7] Equation (22) can be rewritten:

$$L = \frac{1}{E(\underset{e}{E}(v_2))} \left[E(v_1) - E \frac{v_1}{\underset{e}{E}(v_2)} E(\underset{e}{E}(v_2))\right]$$

$$= \frac{1}{E(v_2)} \left\{E\left[\frac{v_1}{\underset{e}{E}(v_2)} \underset{e}{E}(v_2)\right] - \left[E \frac{v_1}{\underset{e}{E}(v_2)}\right] \left[E(\underset{e}{E}(v_2))\right]\right\}$$

And so, finally:

$$L = \frac{1}{E(v_2)} \operatorname{Cov}\left[\frac{v_{1e}}{\underset{e}{E}(v_2)}, \underset{e}{E}(v_2)\right] \tag{29}$$

And, similarly:

$$S = \frac{1}{E(v_1)} \operatorname{Cov}\left[\frac{\underset{e}{E}(v_2)}{v_{1e}}, v_{1e}\right] \tag{30}$$

By analogy with equation (19), after the realization of state-e at date-1 the following holds at an interior solution:

$$\frac{v_{1e}}{{}_{1e}P_1} = \frac{\pi_{s.e} v_{2s}}{{}_{1e}P_{2s.e}} \tag{31}$$

Since ${}_{1e}P_1$ is now the numeraire, the denominator on the L.H.S. is unity. Then:

$$ {}_{1e}P_2 = \sum_s {}_{1e}P_{2s.e} = \sum_s \frac{\pi_{s.e} v_{2s}}{v_{1e}} = \frac{\underset{e}{E}(v_2)}{v_{1e}} \tag{32}$$

Now, using equation (16):

$$ 1 + {}_{1e}r_2 = \frac{1}{{}_{1e}P_2} = \frac{v_{1e}}{\underset{e}{E}(v_2)} \tag{33}$$

We see therefore that the first variable of the covariance in equation (27) represents the interest-rate risk envisaged by individuals in the economy. The second variable of the covariance, the conditional expected marginal utility at date-2, reflects the risk of income fluctuations at the later date. The covariance in equation (29) for L reflects the interaction of interest-rate risk with income fluctuations at the later date.

Similarly, the covariance in equation (30) for S reflects the interaction of discount risk with income fluctuations at the earlier date, since $1 + {}_1\rho_2 = 1/(1+{}_1r_2) = {}_1P_2/{}_1P_1 = \underset{e}{E}(v_2)/v_{1e}$. Thus, both equations (29) and (30) capture the interaction of price risk with income risk, one with income at the earlier date, and the other (with the price inverted) with income at the later date.

Equation (29) can be used to check certain of our previous results as to the determinants of the liquidity premium (maintaining our assumption of time-separable utility). In particular, if there is no income risk at date-2, the covariance is zero since $\underset{e}{E}(v_2)$ is a constant and so L = 0 (Proposition #2). And if date-1 and date-2 incomes are statistically independent, then $\underset{e}{E}(v_2) = E(v_2)$ regardless of e, so once again the covariance is zero and hence L = 0 (Proposition #3). Equation (30) similarly verifies propositions #2a and #3a for the solidity premium.

Finally, both equations (29) and (30) help explain the somewhat unexpected property that L and S cannot both be positive. Since the denominator in the ratio that is the first variable in each covariance expression appears also as the other covariable, there is a certain tendency for the covariance to be negative unless there is a strong positive influence from the denominator of the first covariable.[8] Thus, L and S tend to be both negative unless there are forces to induce (for example) positive serial correlation in income.

We have shown that L and S are each a function of interest-rate risk (a kind of price risk) with income risk, each at a different date. With this we can now provide a more complete intuitive interpretation of the meaning of L and S, and in particular of the various possible sign combinations of L and S.

Consider two "routes" for obtaining date-2 consumption via market transactions, as viewed from the standpoint of the initial date-0. The "long-term route" LT yields the investor $(1 + {}_0r_1)(1 + {}_0r_2)$ at date-2. The "short-term route" ST yields $(1 + {}_0r_1)(1 + {}_1r_2)$, the latter factor being a random variable. Thus, the difference between the attractiveness of the two routes is the difference between the known ${}_0r_2$ and the unknown ${}_1r_2$.

If $L > 0$, then ${}_0r_2 > E({}_1r_2)$. Thus the yield on the LT route to date-2 consumption is on average higher. The implication is that the underlying circumstances would have made ST a "superior route" to date-2 income, had $L > 0$ not obtained. Thus, a positive liquidity premium must be paid whenever individuals desirous of date-2 income must be induced to hold long-term bonds.

Conversely, now consider two "routes" to date-1 consumption. Here the ST route simply yields $1 + {}_0r_1$. The LT route yields $(1 + {}_0r_1)\,\dfrac{1 + {}_0r_2}{1 + {}_1r_2} =$

$(1 + {}_0r_1)\,\dfrac{{}_0P_1/{}_0P_2}{{}_1P_1/{}_1P_2}$. Here the difference between the two routes is represented by the expectation of the ratio $\frac{{}_0P_1/{}_0P_2}{{}_1P_1/{}_1P_2}$. If the LT route yields on average more, the ratio on average exceeds unity, and $\frac{{}_0P_1}{{}_0P_2} < E(\frac{{}_1P_1}{{}_1P_2})$. It then follows (applying Jensen's Inequality as before) that $\frac{{}_0P_2}{{}_0P_1} < \frac{1}{E(\frac{{}_1P_1}{{}_1P_2})} < E(\frac{{}_1P_2}{{}_1P_1})$, and $S < 0$.

In conclusion, the size combination $L > 0$, $S < 0$ indicates a case where (were it not for these liquidity/solidity premia) the short-term instruments would be unambiguously preferred by all investors -- i.e., by those seeking date-2 income and by those seeking date-1 income.

If $L < 0$ and $S > 0$, exactly the opposite holds. The short-term bonds are on the average less expensive for providing for consumption at both dates. The premium is paid on the short-term bonds, since the long-term bonds are unambiguously "superior".

If $L < 0$ and $S < 0$, the short-term bonds are on the average less expensive a means of acquiring date-2 consumption, but more expensive for date-1 consumption. There is no unambiguous preference. Each bond is the more expensive instrument for acquiring consumption at it's own date."

Finally, Jensen's Inequality rules out the case where $L > 0$ and $S > 0$. The economic interpretation of this case would be that long-term bonds are the cheaper instrument for providing for date-2 consumption while short-term bonds are the cheaper for providing date-1 consumption. We are spared this conundrum.

IV. STORAGE AND THE LIQUIDITY PREMIUM: THE NON-INFORMATIVE CASE

So far we have been considering an economy with immutable social endowments. Though individuals could trade current for future income claims among themselves, there were no productive investment opportunities, thus no way of physically transforming society's real income of an earlier date into real income of a later date. We now introduce productive transformation opportunities in the very simple form of costless storage over time. (Costless storage is an investment that always pays off 1:1, so that its net rate of return is of course exactly zero.) It is the storage option that makes earlier-maturity assets more "liquid" (flexible) than later-maturity assets. Claims c_0 to current income can be consumed at dates 0, 1, or 2; c_1-claims at dates 1 or 2; but c_2-claims can be consumed only at date-2. There are thus two storage decisions to be made: at date-0 and at date-1.

The crucial proposition to be developed in this section is that, given time-separable utility functions and statistical independence of date-1 and date-2 endowments, if storage takes place at date-1 the liquidity premium can never be negative. Statistical independence means that the occurrence of any state-e at date-1 tells us nothing about the probabilities at date-2. Thus, while the advent of any state at date-1 is potentially an information-event as well as an income-event, in this "non-informative" case the former property is lacking. We saw in Section III above that before the storage option was introduced, statistical independence of endowments dictated $L = 0$. Thus, storage opportunities introduce a tendency toward a positive liquidity premium, though $L = 0$ remains possible as a limiting case. Moreover, when $L > 0$, then $S \leq 0$ (by Jensen's Inequality), so the storage option generates an unambiguous preference for the short-term bonds.

Let $\bar{v}_{1e}$ and $\bar{v}_{2s}$ represent marginal utilities of the endowed incomes at the subscripted date and state. Let $\hat{v}_{1e}$ be the marginal utility of income at date-1 in state-3 after the first storage decision at date-0. (Note that storage at date-0 will have incremented income equally in all states e = 1,...,E at date-1, but does nothing to change endowments at date-2.) Let $\overset{*}{v}_{1e}$ and $\overset{*}{v}_{2s}$ be the marginal utilities after the storage decisions at date-1; thus, these are the marginal utilities at the final consumptive solution.

Consider for simplicity a representative individual, after some state of the world e has occurred at date-1. The representative individual cannot trade, since everyone is in the same situation he is in, and can only contemplate the storage option. In doing so he will compare $\hat{v}_{1e}$, the marginal utility of corn available at date-1, with $\underset{e}{E}(\bar{v}_2)$, the conditional expectation of marginal utility of endowed corn at date-2. If corn at date-1 is plentiful so that $\hat{v}_{1e}$ is relatively small, storage will take place until $\overset{*}{v}_{1e} = \underset{e}{E}(\overset{*}{v}_2)$. And since we saw in (33) that $1 + {}_{1e}r_2 = v_{1e}/\underset{e}{E}(v_2)$, if storage occurs:

$$\frac{\overset{*}{v}_{1e}}{\underset{e}{E}(\overset{*}{v}_2)} = 1 \qquad \text{and} \qquad {}_{1e}r_2 = 0 \tag{34}$$

If resources at date-1 are relatively scarce, however, nothing will be stored and an inequality will hold instead:

$$\frac{\overset{*}{v}_{1e}}{\underset{e}{E}(\overset{*}{v}_2)} > 1 \qquad \text{and} \qquad {}_{1e}r_2 > 0 \tag{35}$$

It will be convenient notationally to let <e'> be the set of states over which storage occurs and <e"> the set of states where storage does not occur. The date-0 expectation of the marginal utility of endowed date-2 income is:

$$E(\bar{v}_2) = E(\underset{e}{E}(\bar{v}_2)) = \pi\langle e'\rangle \underset{\langle e'\rangle}{E}(\bar{v}_2) + \pi\langle e''\rangle \underset{\langle e''\rangle}{E}(\bar{v}_2) \tag{36}$$

Furthermore, given statistical independence of the date-1 and date-2 endowments:

$$E(\bar{v}_2) = \underset{\langle e'\rangle}{E}(\bar{v}_2) = \underset{\langle e''\rangle}{E}(\bar{v}_2) \tag{37}$$

Similarly, the date-0 expectation of the marginal utilities of the consumed quantities is:

$$E(\overset{*}{v}_2) = E(\underset{e}{E}(\overset{*}{v}_2)) = \pi\langle e'\rangle \underset{\langle e'\rangle}{E}(\overset{*}{v}_2) + \pi\langle e''\rangle \underset{\langle e''\rangle}{E}(\overset{*}{v}_2) \tag{38}$$

However:

$$\underset{\langle e'\rangle}{E}(\overset{*}{v}_2) < \underset{\langle e'\rangle}{E}(\bar{v}_2) = E(\bar{v}_2) \text{ whereas } \underset{\langle e''\rangle}{E}(\overset{*}{v}_2) = \underset{\langle e''\rangle}{E}(\bar{v}_2) = E(\bar{v}_2) \tag{39}$$

It follows, of course, that:

$$E(\overset{*}{v}_2) < E(\bar{v}_2) \tag{40}$$

We will partition the covariance of equation (29) over the set of storage states $\langle e'\rangle$ and non-storage states $\langle e''\rangle$.

Over the storage states, the covariance can be written:

$$\sum_{e'} \pi_{e'}[1 - E(1 + {}_1r_2)][\underset{\langle e'\rangle}{E}(\overset{*}{v}_2) - E(\overset{*}{v}_2)] \tag{41}$$

The first bracket is based upon equation (33), which indicated the equivalence of $1 + {}_{1e}r_2$ to the first variable in the covariance of (29), together with the fact that ${}_{1e}r_2 = 0$ for states in which storage occurs. The first bracket is thus a constant; furthermore, it is negative in sign if there is at least one non-storage state (i.e., a state where ${}_{1e}r_2 > 0$). That the second bracket is also negative follows immediately from (38), since $E(\overset{*}{v}_2)$ is an average of

$\underset{<e'>}{E}(\overset{*}{v}_2)$ and the larger magnitude $\underset{<e''>}{E}(\overset{*}{v}_2)$ -- provided again that there is at least one non-storage state. (If storage occurs in every state, both brackets will equal zero and so the covariance must be zero.)

Over the non-storage states the covariance can be written:

$$\sum_{e''} \pi_e[1 + {}_{1e}r_2 - E(1 + {}_1r_2)][\underset{<e''>}{E}(\bar{v}_2) - E(\overset{*}{v}_2)] \qquad (42)$$

That the second bracket of (42) is positive, so long as there is some storage state, follows immediately from (40) above, since in the non-informative case $\underset{<e''>}{E}(\bar{v}_2) = E(\bar{v}_2)$ as shown in (39). As it is a constant, it can be passed outside the summation. The first bracket might not be positive in every non-storage state, but <u>on average</u> it must be positive: that is, $\underset{<e''>}{E}({}_1r_2) > E({}_1r_2)$, as the latter expectation includes storage states for which ${}_{1e}r_2 = 0$. So, if there is some storage state as well, the covariance over the non-storage states is positive. (By analogy with the preceding, if <u>every</u> state is a non-storage state, the covariance must be zero.)

We can, finally, assert the desired theorem:

PROPOSITION #4: If date-1 and date-2 endowed incomes are statistically independent (the "non-informative" case), if tastes regarding date-1 and date-2 incomes exhibit neither positive nor negative complementarity (utility functions are separable over time), <u>and if endowed incomes can be stored</u> from t=1 to t=2, the liquidity premium cannot be negative. Furthermore, it will be positive except in two extreme cases: (1) if every state at date-1 is so poor that individuals never store; or, (2) if every state at date-1 is so rich that individuals always store.

Note that in the first extreme case L = 0 because there is no randomness in conditional expected marginal utility at date-2 (the second variable in the covariance expression of equation (29) is in fact a constant). In the second extreme case, L = 0 because there is no randomness in contingent interest rates, which must equal zero because of the zero rate of return on storage -- corresponding to absence of variation in the first variable of the covariance expression of (29).

Apart from the restrictive assumptions stated therein, plus of course the standard assumption of risk-aversion, this theorem is very robust. In particular, it is more powerful than the analogous result of Hirshleifer [1972]. Hirshleifer was basically concerned to show that the storage option makes for an ascending term structure (other things equal), which translates into a positive liquidity premium only if there is also a level trend of short-term interest rates.[9] In contrast, Proposition #4 above shows that, regardless of the term structure, the presence of physically liquid (storable) assets in the economy makes the liquidity premium positive.

What if storage is costly? Then the rate of return on stored assets will be negative. Alternatively, if the asset improves with storage as in the case of vintage wine, the rate of return could be positive. Proposition #4 will hold for any constant rate of return, positive or negative (more generally, for any linear intertemporal production process). In the set of states <e'> where the process is used, the contingent interest rates will all equal the constant rate of return. And in the set of states <e"> where no storage occurs, the conditional expected marginal utility of income at date-2 will be constant. If neither set is empty, the previous argument continues to apply and L > 0.

Our analysis of the storage option yields further insights into the essential nature of liquidity. One point of importance (already emphasized by Hirshleifer) is that, for L to be positive, the decision to store at date-1 must be deferrable until it is know which state has occurred at that date. If the amount to be stored at date-1 must be decided at time-0, the amount so stored to be independent of which state occurs at date-1, then $\underset{e}{E}(\overset{*}{v}_2)$ would be the same for every state-e. Then once again there is no variation in conditional expected marginal utility at date-2; the second variable of the covariance in (29) becomes a constant so that the liquidity premium L disappears. Note that while we have been calling the case here "non-informative," that adjective refers only to the effect of occurrence of state-e at date-1 upon beliefs as to the probabilities of states <u>at date-2</u>. We have been assuning throughout a situation which is nevertheless informative in that the storage decision at date-1 can be made after the state at the date is known.

A numerical example at this point may be helpful in seeing the price-risk/quantity-risk interaction induced by the storage option. Suppose we have the simple preference-scaling function $v = \ln c_0c_1c_2$, which has the separable properties necessary for our theorems. Suppose the representative individual's endowments are (for two equally probable states at date-1):

Date:	0	1	2
State "a"		150	
	100		100
State "b"		75	

The only variables that enter into our analysis are incomes at dates-1 and -2, so assume no storage takes place at date-0. If state "a" occurs at date-1, individuals will all store to equalize marginal utility at the two dates. (Since the assumed utility function has zero time-preference, and since the natural rate of interest in production -- our costless storage -- is also zero. If state "b" occurs, no storage takes place because date-1 is relatively poor. The after-storage contingent consumption plans are therefore:

Date:	0	1	2
State "a"		125	125
	100		
State "b"		75	100

The marginal utilities are:

Date	0	1	2
State "a"		.008	.008
	.01		
State "b"		.0133	.01
Expected Marginal Utility	.01	.0108	.009

The current short-term rate of interest is found by $1 + {}_0r_1 = v_0/E(v_1) = .01/.0108 = .926$, so ${}_0r_1 = -7.4\%$. The forward rate is found by $1 + {}_0r_2 = E(v_2)/E(v_2) = .0108/.009 = 1.20$, so ${}_0r_2 = 20\%$. The contingent rates are 0% in state "a", where the individual stores, and in state "b", where she doesn't, $1 + {}_1r_{2b} = v_{1b}/v_{2b} = .0133/.01 = 1.33$, so the interest rate is 33.3%. Since the states are equally probable, the expected future rate of interest is 6.7%, clearly lower than the forward rate of 20%. Thus $L > 0$.

By Jensen's Inequality, if $L > 0$, then $S \leq 0$, and we can confirm this with the numbers here. The forward discount is $-16.6\% = (1/1.20 - 1)$. The two contingent future discounts are $-24.8\% = (1/1.33 - 1)$ and $0\% = (1/1 - 1)$. The expected future discount is then -12.4%, so the forward discount is lower than the expected future discount, and $S < 0$.

The crucial factor here is the interaction of contingent interest rates with contingent endowments. Consider first the problem of providing for date-2 consumption. If one buys long-term bonds, one simply gets $(.926)(1.20) = (1.11)$ units of date-2 consumption for each unit of date-0 consumption. If one buys short-term bonds and rolls over, she gets .926 = (.926)(1.0) units of consumption if "a" occurs, and if "b" occurs, she gets (.926)(1.33) = 1.235. Note however, that 'a', the state in which she rolls over at a low rate, is the richer state of date-2. When she rolls over at a high rate, it is to the poorer state at date-2, when she values the option to roll over more. Thus, the short-term bond is an attractive instrument for providing for date-2 income because interest rates covary <u>against</u> the date-2 income risk.

Consider the same interactions for date-1. If one bought long-term bonds, planning on liquidating and consuming at date-1, in state-"a" he would liquidate at a low discount and in "b" at a high one. The capital-value risk of the long-term bond therefore covaries <u>with</u> the date-1 income, and hence it is not an attractive instrument. The short-term bonds are, in terms of interest-rate or price-risk interaction with income or quantity risk, the better instrument for <u>both</u> dates' consumption. A premium therefore must be paid on the long bonds to clear the market.

Note how this analysis contrasts with that of Modigliani and Sutch. Modigliani and Sutch had in mind that individuals would want to hold bonds that matched the maturity of the net negative cash flows from their permanent incomes. A premium had to be paid to individuals to get them to hold bonds of any other maturity. My analysis here shows that this sort of force could affect the term structure, but not necessarily the liquidity premium. Here the premium is paid to the long-term bonds because individuals would want short-term bonds not only to provide for date-1 but also for date-2. We have to persuade them to hold an asset that does in fact <u>match</u> the maturity of their habitat!

V. THE INFORMATIVE CASE

In this section I concentrate upon the implications for the liquidity premium of events at date-1 that lead to revisions of individuals' beliefs as to the probabilities of states at date-2 -- so that $\pi_{s.e}$ is no longer assumed equal to π_s. This informational effect has seemingly been overlooked in the entire previous literature on the liquidity premium,[10] yet its implications are (we shall see) quite drastic.

To bring out the principle involved, we can first consider the special case where there is no endowment risk at t=1. Then the states e at date-1 differ from one another only as information-events, not as income-events. (This is the reverse of the situation in Section IV, where each state-e was associated with a distinct income endowment at date-1, but thanks to the assumption of "statistical independence" this occurrence provided no information about states at date-2.)

To begin with, let us take up the pure-exchange or no-storage case. Dusting off equation (22) once again, since $v_1 = \hat{v}_1$ is now constant we can write:

$$L = \hat{v}_1 \left[\frac{1}{E(\underset{e}{E}(\bar{v}_2))} - E\left(\frac{1}{\underset{e}{E}(\bar{v}_2)}\right)\right] \tag{43}$$

Applying Jensen's Inequality (27) here to $\underset{e}{E}(\bar{v}_2)$ as random variable, since $\hat{v}_1$ is a positive constant, we see immediately from (32) that $L < 0$. And looking again at equation (26), our expression for S, we have

$$S = \frac{E(\bar{v}_2)}{E(\hat{v}_1)} - E\left(\frac{\underset{e}{E}(\bar{v}_2)}{\hat{v}_1}\right) \tag{44}$$

If $v_1 = \hat{v}_1$ is a constant, S is identically zero. So we have:

PROPOSITION #5: With separable tastes, no endowment risk at date-1 but information about date-2 endowments arriving at date-1, in the absence of storage opportunities L will be negative and S will be zero.

(This result was foreshadowed at the end of Section I above, where we noticed that fluctuations in $\underset{e}{E}(v_2)$ tend, other things equal, to induce a negative liquidity premium.)

The economic interpretation is as follows. When the only income risk is at date-2, in the absence of earlier information arrival (as to what will happen at date-2) everyone would know at date-0 what the future short-term rate ${}_1r_2$ would be. But the prospect of early information arrival makes the rate ${}_1r_2$ uncertain, and thus requires individuals to adjust to price risk at date-1. They will attempt to insure against this price risk by "locking in" (purchasing certain claims to) date-2 consumption at date-0 which in the absence of physical storage opportunities will tend to raise ${}_0P_2$ and thus reduce ${}_0R_2$ and ${}_0r_2$ relative to $E({}_1r_2)$. What we have here is the reverse of the situation envisaged by Keynes and Hicks. In their original argument the price risk at date-1 led to a positive liquidity premium because, as we saw in Part I, it was implicitly assumed that investors were concerned only to avoid income risk at date-1. While our investors are generally concerned with income risk at all dates, I have constructed a special situation without income risk at date-1. Thus, the date-1 price risk here is relevant only in that it increases uncertainty of consumption at date-2, hence, individuals are willing to undertake long-term commitments to assure date-2 income even where ${}_0r_2$ is low relative to $E({}_1r_2)$ -- i.e., even where the liquidity premium is negative.

Note here that the date-2 income risk covaries with the interest-rate risk. At date-1 when date-2 has been discovered to be richer than previously thought, those who bought short-term bonds to provide for date-2 consumption find themselves able to roll over at high rates, but this just adds to their already augmented wealth. When news arrives that interest rates are to be low, it is because date-2 has been found to be poorer than previously thought. Those who bought short-term bonds for date-2 consumption must now roll over at unfavorable rates, adding to their misery. The short-term bonds are not an attractive instrument for providing for date-2 consumption. Remember, however, that $S = 0$, implying that long-term bonds and short-term bonds are equally costly on average in providing for date-1 consumption. There is no income risk at date-1, so interest rates can neither covary with nor against income at date-1.

Still within the special case of no income risk at date-1, we now can consider in combination the effects of information arrival and of storage opportunities. The former, by Proposition #5, tends to make the liquidity premium negative; the latter, by Proposition #4, tends to make it positive. Can we say which will be the more powerful?

With storage opportunities the amount to be consumed at date-1, and therefore the magnitude of v_1, are now uncertain. Some of the risk at date-2 is thus shifted back to date-1, not only in prices but in amounts consumed. We will use the technique once again of segregating the covariance of equation (29) between the set of storage states <e'> and the set of non-storage states <e"> at t=1.

Since the conditional interest rate ${}_{1e}r_2$ is always zero in all the storage states, the expression (41) continues to apply for the covariance over these states. As before the first bracket is always negative (if there is at least one non-storage state). But now the second bracket is positive (again, if there is some non-storage state as well). The reason is that the storage states are precisely those for which the new information has revealed $E(\bar{v}_2)$ to be high, and in particular higher than $\hat{v}_1$ (the marginal utility of the time-1 endowment plus any storage from t=0). More explicitly, equation (38) continues to apply. So, to show that $\underset{<e'>}{E}(\overset{*}{v}_2) > E(\overset{*}{v}_2)$, we need only demonstrate that $\underset{<e'>}{E}(\overset{*}{v}_2) > \underset{<e''>}{E}(\overset{*}{v}_2)$. In every storage state e', once storage occurs, $\overset{*}{v}_{2e''} = \bar{v}_{2e''} > \hat{v}_1$. But in every non-storage state e", $\overset{*}{v}_{2e''} = \bar{v}_{2e''} < \hat{v}_1$. Hence the second bracket in (41) is now surely positive and so the covariance is negative over the storage states, if there is some non-storage state.

Over the non-storage states, rather than use (42) it will be simpler to notice directly that the covariance in (29) becomes (since $v_{1e''} = v_1$,) a constant:

$$\mathrm{Cov}[\frac{v_{1e''}}{\underset{<e''>}{E}(v_2)}, \underset{<e''>}{E}(v_2)] = \hat{v}_1 \ [\mathrm{Cov} \frac{1}{\underset{<e''>}{E}(\bar{v}_2)}, \underset{<e''>}{E}(\bar{v}_2)] \qquad (45)$$

Since the bracketed expression on the R.H.S. is of the form $[\frac{1}{x}, x]$, the covariance is negative over the non-storage states. Since it is also negative for the storage states, we have a negative liquidity premium.

Does this produce a situation with an unambiguous premium on the short-term bonds? To answer, we must know the sign of S. By the same partitioned covariance methods we can show that indeed $S > 0$, so we have:

PROPOSITION #6: With separable tastes, no endowment risk at date-1 but information about date-2 arriving at date-1, the liquidity premium will be negative and the solidity premium positive when there are storage opportunities.

Note that we need make an exception here only for one extreme case, where storage occurs in every state (in which case $L = 0$ and $S = 0$). In the other extreme case, where storage <u>never</u> occurs, we are back in the world of Proposition #5 where $L < 0$ and $S = 0$ still holds.

Once again, a numerical example may be helpful in displaying the interaction of price or interest-rate risk with income risk. Again consider $v = \ln c_0c_1c_2$ with the following endowment, and equally probable states:

Date:	0	1	2
State "a"			150
	100	100	
State "b"			75

At date-1 the great economist in the sky will reveal conclusively the state to occur at date-2. If it is to be "a", rejoicing ensues, no storage occurs, and the rate of interest is 50%. If it is to be state "b", storage occurs and the interest rate is zero. The contingent consumption plans are then:

Date	0	1	2
State "a"		100	150
	100		
State "b"		87.5	87.5

The marginal utilities for the state/date combinations are:

Date	0	1	2
State "a"		.01	.0066
	.01		
State "b"		.0143	.0143
Expected Marginal Utility	.01	.0121	.0104

With equally probable states the forward rate of interest is 16.3%, which is lower than the expected future rate, 25%, so $L > 0$.

Once again, let us examine the interaction of interest rates with income. If one buys short-term bonds planning to roll them over for date-2, news of a high rate at which to roll over comes when its marginal value is relatively small -- in the richer state at date-2. News of the low rate at which to roll over comes with news of the poorer state at date-2, just when the marginal value of a high rate would have been high.

On the other hand, suppose one bought long-term bonds planning to liquidate them and consume at date-1. When news of a high rate arrives, with it is news that the richer state at date-1 obtains, in which the marginal value of liquidating at a favorable rate is lower. But if the poorer state at date-1 occurs, it is accompanied by the low interest rate. One gets to liquidate on more favorable terms in the state when its value is higher. Thus, the long-term bond's capital value covaries <u>against</u> the date-1 income, making it an attractive instrument for date-1 consumption, as well as for date-2 consumption. A premium is therefore paid on the short bond.

In the previous section in which we showed a clear preference in short-term bonds (and consequent $L < 0$, $S > 0$) we examined only incoming information about the near future, interacting with storage option. Just above, in which we showed a clear preference for long-term bonds (and consequent $L < 0$, $S > 0$) we examined only incoming information about the far future, suppressing information about the near future except to the degree it was affected by the storage option. What if we have information about <u>both</u> dates? The result can go either way, or we can end up with $L < 0$, $S < 0$. We obtain as our last proposition:

PROPOSITION #7: With separable tastes, endowment risk at both date-1 and date-2, information arriving about date-2 at date-1, and storage opportunities, the signs of the liquidity premium and the solidity premium are not in general determinate.

To see the ambiguity resulting when both near-future and far-future information interact with storage, consider one more numerical example, again with $v = \ln c_0 c_1 c_2$, conclusive information at date-1 regarding date-2, and a storage option. Endowments and contingent consumption plans are:

	Endowments			Consumption Plan		
Date	0	1	2	0	1	2
State "a"		75	150		75	150
	100			100		
State "b"		150	75		112 1/2	112 1/2

Here in the storage state "b" the interest rate is zero as before, while in the nonstorage state it is 100%. The forward rate is 42%, the expected future rate if 50%, so $L < 0$. But the forward discount is -.3 and the expected future discount is -.25, so $S < 0$ also. There is no clear premium on either bond. To understand why this is so merely note that the capital value of the long bond covaries with the date-1 income and the future value of a short bond rolled over covaries with the date-2 income. Thus, the risk interaction of prices and income tends to make each bond the more expensive investment instrument for obtaining consumption at its own maturity date.

In the two previous examples where we suppressed information about one date or the other, storage tended to induce positive serial correlation in planned consumption streams. There were two contingent streams -- one richer, one poorer. What then determined the sign of L and S was whether the high interest rates came in the richer stream or the poorer one. In the example here, the high interest rate is associated with poorer date-1 and richer date-2, hence the ambiguity.

VI. CONCLUSION

This paper indicates, I believe for the first time, the correct interpretation of the liquidity premium and the solidity premium, and analyzes the factors that determine the sign and magnitude of these liquidity/solidity measures.

First, as to meaning. The liquidity premium, L, is the difference between the forward short-term rate of interest (that is implicit in the term structure of current rates on short-term and long-term bonds) and the expected future short-term rate of interest. The solidity premium, S, is the difference between the forward discount and the expected future discount, where the discount is defined in terms of the corresponding interest rate r by $1 + d \equiv 1/(1 + r)$. When L is positive and S is negative, long-term bonds are on average (on average because the future interest rate and corresponding discount are uncertain) the less expensive instrument for providing consumption at both the near-maturity date (the date the shorter bond matures) and at the far-maturity date (the date the long bond matures.) That is, a long-term bond yields more at the later date than does the purchase of a short-term bond subsequently rolled over, on average, and a long bond liquidated at the earlier date yields more, on average, than does the shorter bond that matures at that date. When L is negative and S is positive, it is the shorter bonds that are on average the less expensive instrument for providing consumption at both the earlier and the later future dates. That is, a short-term bond rolled over at maturity yields more on average at the later date than does a bond that matures at the later date, and a bond that matures at the earlier date yields more than does a longer bond liquidated at the earlier date, on average.

When, as in these cases, S and L take on opposite signs, either the shorter-term or the longer-term instrument is on average definitely less expensive for providing consumption at both near-future and far-future dates. How can this be the case? Clearly, the "more expensive" instrument must be providing some other desirable characteristic, to wit, protection against a kind of risk.

Yet each bond is clearly a riskless instrument for providing consumption at its own maturity date. How can anything be less risky than riskless? Easily, if it is a form of insurance. Suppose the investor is concerned with consumption at the near-future date. If interest rates fluctuate randomly in such a way that the capital-value of the long-term bond will covary <u>against</u> one's own endowed-income risk at the earlier date, such a bond provides insurance against this endowment risk. Similarly, short-term bonds (rolled over at maturity) can provide insurance against far-future endowment risk. The bond providing this insurance feature may therefore be held in equilibrium despite being more expensive on average in terms of consumption yield at either date.

This interpretation enables us to explain certain "paradoxes" in the relations between L and S: e.g., that a negative liquidity premium does not <u>necessarily</u> imply a positive solidity premium. If short-term instruments are the more expensive route to consumption at the later date, this by no means implies that long-term instruments constitute the cheaper route to consumption at the earlier date. In fact, limiting and degenerate cases aside, the following sign combinations are possible:

(i) $L < 0$, $S < 0$: Short-term bonds are more expensive for providing the earlier date's consumption, and long-term bonds are the more expensive for providing for the later date's consumption. Interest rate/discounts vary in such a way that the capital value of the bond that does not mature at each date covaries <u>with</u> endowed income at that date. Hence the bond maturing at that date is less risky (riskless, in fact) and is on average more expensive.

(ii) $L > 0$, $S < 0$: The shorter bond's capital value (matured and rolled over) covaries against endowed income risk at the later date, and the longer bond's capital value covaries with endowed income at the earlier date. The short bond is therefore riskless with regard to the earlier date and constitutes a form of insurance for the later date. Hence it is on average more expensive.

(iii) $L < 0$, $S > 0$ The opposite of case (ii) above. The short bond's capital value covaries with the later date's income risk, and the longer bond's capital value covaries against the earlier date's income risk. Here the long bonds have the insurance aspect, and thus are on average more expensive.

What about the case $L > 0$, $S > 0$? By Jensen's Inequality, L and S cannot both be positive. This means that the long bond cannot provide insurance for the earlier date while at the same time the short bond provides insurance for the later date.

Three main elements determining the signs and magnitudes of L and S were analyzed here: (1) the initial distribution of beliefs about time-and-state income endowments, (2) the prospective arrival of information regarding the near-future and far-future endowments, and (3) the possibility of physically transferring income endowments from earlier to later dates.

With respect to the first item, the mean and variance of the income distributions at earlier and later dates can affect the size of L and S, but not the sign. What does affect the sign is the covariance of the contingent future interest rate/discount with each of the two dates' incomes. The arrival of information about earlier and later dates does not, in itself, tend to command a premium. Rather, this information tends to make interest rates/discounts vary in such a way that the capital value of the longer bonds covaries with income risk at the earlier date, and likewise the short-term bonds (plus roll-over) with the later date, and hence to make each bond the best instrument for providing consumption at its own maturity date. Thus, such information tends to generate the pattern $L < 0$, $S < 0$.

However, when the arrival of information is combined with the option to transfer real resources to a later date, then a clear "insurance" advantage can emerge -- but whether this advantage favors the short-term or the long-term bonds depends upon whether the information relates to the near-future or the far-future endowment. The option to store brings about low real interest rates (in fact, zero if storage is costless) in states of the world in which storage occurs. If the information concerns the near-future date, storage will take place <u>from the richer states</u> at that date; if it concerns the far-future date, storage will take place <u>to the poorer states</u> at that date. The option to store, combined with information about the near future, therefore causes the low interest rates to occur in the richer branches of the tree of possible income streams, thus making the "insurance" feature attach to the short-term bonds. The consequence is a positive liquidity premium and negative solidity premium. On the other hand, information about the far-future combined with the option to store means that storage and therefore low interest rates will occur in the poorer branches of the tree -- making for a negative liquidity premium and positive solidity premium.

Whether the one case is more probable than the other must be left an open question, depending as it does upon the relative weight of anticipated information arrival in resolving uncertainty as to near-future or far-future endowments. If there were no systematic difference in this regard, we would tend to observe a negative liquidity premium <u>and</u> a negative solidity premium, a result that does not seem to have been noted by anyone who has thought publicly on these issues.

FOOTNOTES

[1]The price ratio $\frac{{}_0P_1}{{}_0P_2} = 1 + {}_0r_2 = e^{{}_0r^*_2}$, where the star indicates continuously-compounded interest rate. And of course, $\frac{{}_0P_2}{{}_0P_1} = \frac{1}{1+{}_0r_2} = e^{-{}_0r^*_0}$.

The continuously compounded rate ${}_1r^*_2$ is similarly defined in terms of the price ratio $\frac{{}_1P_2}{{}_1P_1}$. Thus, with continuous-compounding interest:

$$L = {}_0r^*_2 - E({}_1r^*_2) = \ln \frac{{}_0P_2}{{}_0P_1} - E \ln \left(\frac{{}_1P_2}{{}_1P_1}\right).$$

A corresponding continuous version of S can be written

$$S = \left(\frac{1}{1+{}_0r^*_2} - 1\right) - E\left(\frac{1}{1+{}_1r^*_2} - 1\right)$$

Thus, S = -L under continuous compounding.

[2]Thus, we are assuming E markets (for date-1 claims) plus S markets (for date-2 claims), each tradable against current corn at date-0. These "Arrow-complete" markets seemingly represent a more restricted trading regime than "Debreu-complete" markets that would incorporate also ability at date-0 to purchaseclaims to consumption c_{2s} contingent upon the advant of state-e at date-1. But our model Arrow-complete markets suffice for achieving preferred consumption vectors, that is, no Pareto-preferred improvements are made available by opening more markets.

[3]For a discussion of this point in the context of the theory of speculation see Salant [1976] and Hirshleifer [1976].

[4]Condition (20) follows directly from maximization of U = E(v) subject to the following budget constraint at date-C (where the overbars indicate endowed quantities):

$$ {}_0P_0\, c_0 + \sum_e {}_0P_{1e}c_{1e} + \sum_s {}_0P_{2s}c_{2s} = {}_0P_0\bar{c}_0 + \sum_e {}_0P_{1e}\bar{c}_{1e} + \sum_s {}_0P_{2s}\bar{c}_{2s}$$

[5]This follows immediately from equation (31) below and the definition (16).

[6]If $L \geq 0$, then $\frac{{}_0P_1}{{}_0P_2} \geq E(\frac{{}_1P_1}{{}_1P_2})$ from equation (11). Taking reciprocals:

$$\frac{{}_0P_2}{{}_0P_1} \leq \frac{1}{E(\frac{{}_1P_1}{{}_1P_2})}$$

Assuming the price ratio is not a degenerate random variable, we can use the strong form of Jensen's Inequality so that:

$$\frac{1}{E(\frac{{}_1P_1}{{}_1P_2})} < E(\frac{{}_1P_2}{{}_1P_1})$$

Thus: $\frac{{}_0P_2}{{}_0P_1} < E(\frac{{}_1P_2}{{}_1P_1})$, or $S < 0$ from equation (10).

A similar development shows that $S \geq 0$ implies $L < 0$.

[7]The development that follows is due to McCulloch [1973].

[8]The correlation coefficient r between two variables $\frac{x}{y}$ and y is approximately (neglecting cubic terms) given by:

$$r = \frac{r_{xy} - \omega_y^2}{\omega_y\sqrt{\omega_x^2\omega_y^2 - 2r_{xy}\omega_x\omega_y}}$$

where ω here represents the coefficient of variation and r_{xy} the ordinary correlation coefficient between x and y (see Pearson [1897]). In particular, $r_{xy} = 0$ implies $\rho < 0$ (unless $\omega_y = 0$).

[9]An ascending term structure implies ${}_0r_2 > {}_0r_1$. A positive liquidity premium is defined by ${}_0r_2 > E({}_1r_2)$. With a level trend of short-term interest rates, ${}_0r_1 = E({}_1r_2)$, so $L > 0$ if and only if the term-structure is ascending.

[10]But I would like to acknowledge a classroom exercise of John Riley which brought this point to my attention.

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