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Multi-unit Auctions, Price Discrimination and Bundling*

by

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Over the last few years the optimal selling strategy of a monopolist who has imperfect information about buyers has been the focus of a number of papers. Specific applications range from the design of an optimal auction for the sale of a unique object to the monopoly pricing of product quality. While these problems appear at first to be very different, they have several basic elements in common. First, the seller brings to market some predetermined quantity of goods, possibly of varying quality. Second, buyers are risk neutral and their preferences are parameterized by a scalar v. Third, the underlying distribution of v, F(v), is assumed known to the seller and all the buyers. Fourth, the seller is also assumed to be risk neutral so that his objective is to choose a pricing scheme which maximizes expected profit.

Despite these common elements the published and unpublished literature divides naturally into three distinct parts with essentially no cross-referencing. In the earliest relatively complete analysis of the optimal pricing problem, Mussa and Rosen [1978] consider a commodity class from which each consumer, if he purchases at all, will purchase a single unit. The parameter v reflects differences in tastes for quality q. The monopolist announces what price R(q) must be paid for each quality level, q, and each consumer responds by selecting his optimal quality level $q_R(v)$. To focus on the demand side of the market the cost of producing a unit of quality level q, c(q) is assumed to be independent of the level of aggregate demand for this or any other quality level.

The constant-marginal-cost assumption is also central to recent reexamination of the Pigouvian question of optimal second degree price discrimination. In this analysis v is analysis v is a parameter which reflects differences in consumers' demand functions for a product, q becomes the number of units purchased and R(q) the total tariff function.

While there is a considerable literature on the special case in which R(q) is of the form $\alpha + \beta q$ (see, for example Oi [1971] and Feldstein [1972]), it is only very recently that general non-linear revenue functions have been analyzed. Spence [1978] provides the first characterization of an optimal tariff function. However, as Goldman, Leland and Sibley [1980] have since shown, Spence's conclusions must be modified unless the distribution function F(v) and the family of demand curves p(q;v) together satisfy a monotonicity restriction. Goldman, Leland and Sibley obtain an explicit solution when the restriction is satisfied and illustrate the problems that arise when it is violated. Their central point is that it is not generally desirable to price discriminate in such a way as to separate out each different type of demander. An analogous point has been made in the optimal income tax literature (see, for example, the survey by Mirrlees [1981]). Only in special cases is it optimal to separate out all skill classes with a smoothly increasing tax schedule.

The third strand of the literature focuses upon monopoly selling strategies making the extreme opposite assumption about the cost of production. Instead of constant marginal costs, aggregate supply is treated as fixed, at least in the short run. The paradigm case is that of the sale of an object d'art or estate. Then there is a single indivisible unit for sale and the parameter value, v_i, is to be interpreted as buyer i's dollar valuation of this unit. Allowing for the possibility that the outcome of the selling scheme may be random, q is now to be reinterpreted as the expected number of objects purchased. Then each buyer can be thought of as having a demand curve of the special form

$$(1) \quad p_i(q) = \begin{cases} v_i, & q \leq 1 \\ 0, & q > 1 \end{cases}$$

In the original theoretical examination of auctions, Vickrey [1961] compared two common auctions; the sealed or 'high bid' auction and the open or 'second bid' auction. He showed that both are efficient in the ex-post sense that the agent with the highest valuation always ends up with the item for sale. Moreover the two auctions yield exactly the same expected revenue to the seller.

More recent work by Harris and Raviv [1979], Myerson [1981], Riley and Samuelson [1981] and Maskin and Riley [1980] considers the choice of auction from the seller's viewpoint.

A central result of this work is that for a wide class of continuously differentiable distribution functions, the optimal selling strategy involves the use of either of the auction schemes considered by Vickrey. However each is modified by the introduction of a minimum price which is strictly greater than the seller's personal valuation.

For the special case of a uniform distribution, Harris and Raviv [1981] have shown that this result continues to hold if there are multiple units for sale and each buyer has the simple demand curve given by (1). In the open auction version the seller simply announces the optimal minimum price. Then, if the number of bidders exceeds supply, the asking price is raised continuously until just enough bidders drop out.

In providing this brief overview of the literature we have emphasized the close links between the different models. Indeed a major conclusion of this paper is that, from a mathematical viewpoint, the models are essentially equivalent. Each can be solved as a special case of an optimal control problem. In the following section we show how the problem of selecting an optimal selling strategy can be reduced to a relatively straightforward class of optimization problems. Then, in section II, we present a general method for solving such problems.

The approach is related to one recently used in the public decision literature (see, for example, d'Aspremont and Gerard-Varet [1979] and Laffont and Maskin [1979]). That there should be a strong connection between these literatures is not surprising. In both cases the goal is to maximize some objective function that depends on the action of other agents (buyers in the monopoly problem, consumers in the public goods framework) subject to the constraint that these other agents in turn pursue their own private ends. The only substantive difference between the two classes of problems is that in the one the objective function is an index of social welfare while in the other it is simply profit.

In addition to clarifying the relationship between the different strands of literature our approach makes several generalizations possible. In section III we are able to provide a complete solution to the Pigouvian price discrimination problem for a wide class of demand functions, even when the simplifying "monotonity restriction" is violated. In general the optimal total tariff function, $R(q)$, has an initial slope exceeding the marginal cost of production so the proportion of buyers purchasing is less than with marginal cost pricing (or perfectly discriminating monopoly). A second general property is that pricing is efficient for the biggest demanders, that is, the slope of the tariff function declines towards the marginal cost of production. Third, the tariff function may have one or more kinks so that marginal price $\frac{dR}{dq}$ declines discontinuously.

Section IV reconsiders the problem when there are only a small number of buyers and the seller knows the distribution, $F(v)$, from which each buyer's parameter v_i is drawn, but not the population histogram. For the special case in which each buyer wishes to purchase only one unit, expected profit is maximized in an open auction. With reservation values distributed uniformly, Harris and Raviv [1981] have shown that in the optimal auction the initial asking price is positive even if the seller places no personal value on the objects for sale. The asking price is then raised continuously until the number of buyers remaining equals the number of objects available.

In general, however, it may also be necessary to prohibit bids, not just below some minimum, but also over other predetermined intervals. If after jumping the asking price the auctioneer finds that demand is less than supply he returns to the previous price and asks who would like to buy a unit at that price. Successful bidders are then determined randomly.

The case of downward sloping demands is somewhat more complicated. An optimal strategy of the seller is to begin by asking buyers to pick a point on a tariff function R(q). If aggregate demand is less than supply the monopolist fills each order. If, however, preliminary orders exceed supply the monopolist scales down each buyer's demand, in a predetermined way, until the capacity constraint is met. At the same time the total tariff is reduced below R(q), and any buyer squeezed out of the market completely has his tariff reduced to zero.

Finally, in section V we extend the Mussa and Rosen analysis by examining a one parameter model in which differences in preferences result in differences in demand for both quality and quantity. In this case the optimal strategy of the monopolist is to sell multiple units in bundles. Higher quality units are sold in packages of different sizes. For a simple parametrization of preferences we again provide a complete characterization of the optimal bundling strategy.

I. Formulation of the Seller's Optimization Problem

We begin by considering the alternative selling strategies of a monopolist with $\bar{q}$ units of a commodity for sale. Later we shall see how to incorporate variable product quality and consider the monopolist's choice of $\bar{q}$.

Each buyer is characterized by a parameter value v_i drawn from some continuously differentiable distribution function $F(v)$. For mere convenience we also assume that $F'(v)$ is positive everywhere in the unit interval and zero elsewhere. Buyer i's demand price for q units of the product is assumed to be a non-increasing function of the form

$$(2) \quad p_i(q;v_i) = v_i a(q) - b(q).$$

It is assumed that there is some q such that, for all v_i and all $q > \underline{q}$, $p_i(q;v_i) = 0$. It is also assumed that $a(q)$ and $b(q)$ satisfy

$$(3) \quad a(q) \geq 0, \text{ and } b(q) \geq 0, \text{ for all } q \geq 0$$

Condition (3) implies that the demand price, for any level of q, is nondecreasing in v_i. Therefore a buyer with a high v is a "high demander."[1]

The functional form (2) incorporates a wide range of interesting special cases. First of all, with $b(q)$ equal to zero and $a(q)$ equal to unity for $q \leq 1$ and zero for $q > 1$, (2) reduces to

$$(2') \quad p_i(q;v_i) = \begin{cases} v_i \,, & q \leq 1 \\ 0 \,, & q > 1 \end{cases} .$$

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[1] This assumption can be significantly weakened with only minor changes in the analysis. All we require is that

$$p(q;v_i) = \alpha(v_i)a(q) - \beta(v_i)b(q)$$

and that $\alpha(x)/\beta(x)$ should be a monotonic function. With $\alpha(x)$ and $\beta(x)$ both positive and increasing the demand curves for any two levels of v_i necessarily intersect.

With aggregate supply, $\bar{q}$, equal to unity this is the optimal auction problem. with $\bar{q}$ greater than unity we have the generalization to multi-object auctions considered by Harris and Raviv [1981].

Second, with $a(q) = 1$, the family of demand curves are vertically parallel. Without loss of generality we may assume $b(0) = 0$ so that the parameter v_i becomes the intercept with the vertical axis.

A third example is provided by the assumptions that $b(q) \equiv 0$ and $a(q) > 0$ if and only if $q < \underline{q}$. Then all the demand curves intersect the horizontal axis at $q = \underline{q}$ and a higher value of the parameter v_i indicates a steeper demand curve.

Buyers are assumed to respond independently to the monopolist's selling strategy. Any such strategy is then a set of rules for a non-cooperative game among the buyers. Associated with a particular set of rules is some strategy space S_i for each buyer. On the basis of the strategies chosen $(s_1, \ldots, s_n)$, the rules assign some quantity $\hat{q}_i(s_1,\ldots,s_n)$ to buyer i and require him to make an expected payment $\hat{T}_i(s_1,\ldots,s_n)$.

The seller must choose rules which satisfy the feasibility constraint

$$(4) \quad \sum_{i=1}^{n} \hat{q}_i(s) \leq \bar{q} \ .$$

It is also natural to assume that the seller cannot force a buyer to participate. Formally this option can be expressed by including in each strategy space a null strategy, which ensures the buyer a zero quantity of the good and a zero payment, independent of what other buyers do.

We now consider the response of the buyers to some particular set of rules, $\langle \hat{q}_i(s), \hat{T}_i(s) \rangle$ $i = 1, \ldots, n$. Each buyer knows his own parameter value v_i but knows only that the other valuations are drawn independently from the distribution $F(\cdot)$. (F is common knowledge among the sellers and buyers.) Thus buyers play a game of incomplete information. A natural extension of Nash equilibrium to non-cooperative games is the Bayesian equilibrium of Harsanyi [1967-68].

To define a Bayesian equilibrium, we introduce the notion of a strategy rule for buyer i: a function $s_i^*(\cdot)$ that, for each possible parameter value v_i, assigns a strategy $s_i^*(v_i)$. With buyers behaving non-cooperatively the vector of strategy rules $(s_1^*(\cdot), \ldots, s_n^*(\cdot))$ is an equilibrium if, when adopted by all buyers but one, the latter's best response is to adopt it also. Consider then the return to buyer 1 when he adopts some alternative strategy. In particular suppose that when his parameter value is v_1 he adopts the strategy $s_1^*(x)$. If assigned $\hat{q}_1$ units for a total tariff of $\hat{T}_1$ his net gain is just the area under the demand curve less the tariff, that is,

$$(5) \quad v_1\int_0^{\hat{q}_1} a(z)dz - \int_0^{\hat{q}_1} b(z)dz - \hat{T}_1 \equiv v_1A(\hat{q}_1) - B(\hat{q}_1) - \hat{T}_1 \ .$$

Then, with all other buyers using their equilibrium strategy rules, the first buyer's expected gain can be expressed as[2]

$$(6) \quad \Pi_1(x,v_1) = \underset{v_{-1}}{E}\ \{ v_1A(\hat{q}_1(s_1^*(x),\ s_{-1}^*(v_{-1}))) - B(\hat{q}_1(s_1^*(x),\ s_{-1}^*(v_{-1})))$$
$$- \hat{T}_1(s_1^*(x), s_{-1}^*(v_{-1}))\}$$

that is

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[2] $s_{-i}^*(v_{-i}) \equiv (s_1^*(v_1), \ldots, s_{i-1}^*(v_{i-1}), s_{i+1}^*(v_{i+1}), \ldots, s_n^*(v_n))$ and $\underset{v_{-i}}{E}$ is the expectation taken over $(v_i, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$.

$$\left\{\begin{matrix}\text{expected}\\ \text{buyer}\\ \text{gain}\end{matrix}\right\} = \left\{\begin{matrix}\text{expected area}\\ \text{under}\\ \text{demand curve}\end{matrix}\right\} - \left\{\begin{matrix}\text{expected}\\ \text{tariff}\end{matrix}\right\}$$

Below, we obtain simple expressions for both the expected area under the demand curve and for the expected buyer gain. Then, from (6) we are able to derive the expected tariff paid by buyer 1.

Since $s_1^*(\cdot)$ is defined as this buyer's equilibrium strategy rule it must be the case that buyer 1 can do no better than adopt $s_1^*(v_1)$, that is, choose $x = v_1$. Then a necessary condition for a Bayesian equilibrium is

$$\Pi_1(x,v_1) \leq \Pi_1(v_1,v_1), \quad \text{for all } x \text{ and } v_1.$$

In particular we have

$$\Pi_1(v_1',v_1) \leq \Pi_1(v_1,v_1) \text{ and } \Pi_1(v_1,v_1') \leq \Pi_1(v_1',v_1').$$

Adding these two inequalities, rearranging and substituting from (5) and (6) we then have the following necessary condition

$$(v_1'-v_1)(G_1(v_1') - G_1(v_1)) \geq 0,$$

where

$$(7) \quad G_i(x) = \underset{v_{-i}}{E} A(\hat{q}_i(s_1^*(v_1), \ldots, s_i^*(x), \ldots, s_n^*(v_n))$$

Of course we can make exactly the same kind of argument for each buyer. Thus in choosing his optimal selling strategy the monopolist must also satisfy the constraint that each buyer's associated G_i function must be non-decreasing. Since a non-decreasing function is differentiable, almost everywhere, we may write this constraint as

$$(8) \quad \frac{dG_i(x)}{dx} \geq 0, \text{ almost everywhere.}$$

Moreover, from (3), $A(\cdot)$ is non-negative. Therefore the maximized net gain to buyer 1, $\Pi_1(v_1,v_1)$, is non-decreasing on [0,1]. It is therefore differentiable almost everywhere.

Then almost everywhere, we may write

$$\frac{d\Pi_1(v_1,v_1)}{dv_1} = \left.\frac{\partial\Pi_1(x,v_1)}{\partial x}\right|_{x=v_1} + \left.\frac{\partial\Pi_1(x,v_1)}{\partial v_1}\right|_{x=v_1}$$

Since $\Pi_1(x,v_1)$ must take on its maximum at $x = v_1$ the first term is zero, almost everywhere. From (6) and (8) we therefore have

$$(9) \quad \frac{d\Pi_1(v_1,v_1)}{dv_1} = G_1(v_1), \text{ almost everywhere.}$$

In the appendix this result is formally derived. It is also shown that $\Pi_1(v_1,v_1)$ is absolutely continuous on [0,1]. Then $\Pi_1(v_1,v_1)$ is an indefinite integral and we have

$$(10) \quad \Pi_1(v_1,v_1) - \Pi_1(0,0) = \int_0^{v_1} \frac{d\Pi_1(x)}{dx}\, dx = \int_0^{v_1} G_1(x)dx$$

We are now ready to consider the game from the seller's viewpoint. From (6) the expected tariff paid by buyer 1 can be expressed as

$$(11) \quad \bar{T}_1(v_1) = v_1G_1(v_1) - \underset{v_{-1}}{E}\, B(\hat{q}_1(s^*(v))) - \Pi_1(v_1,v_1)$$

As far as the seller is concerned, v_1 and hence the expected revenue, $\bar{T}(v_1)$, is a random variable. The seller's expected revenue from buyer 1 is therefore the expectation of $\bar{T}_1(v_1)$. Since the seller knows that v_1 has distribution $F(v_1)$ his expected revenue is

$$\bar{R}_1 = \int_0^1 \bar{T}_1 dF$$

Substituting from (9) and (10) and integrating by parts, the seller's expected revenue can be rewritten as follows

$$(12) \quad \bar{R}_1 = \int_0^1 J(x)G_1(x)dF(x) - \underset{v}{E}B(\hat{q}_1(\overset{*}{s}(v)) - \Pi_1(0,0)$$

where

$$(13) \quad J(x) \equiv x + (F(x) - 1)/F'(x).$$

Finally summing over all n buyers the expected revenue of the seller is

$$(14) \quad \bar{R}_0 = \int_0^1 J(x_i) \sum_{i=1}^{n} G_i(x_i)dF(x_i) - \underset{v}{E} \sum_{i=1}^{n} B(\hat{q}_i(\overset{*}{s}(v))) - \sum_{i=1}^{n} \Pi_i(0,0)$$

At this point it is convenient to subsume the vector of buyers' strategy rules, $\overset{*}{s}(v)$, and write the monopolist's selling strategy as a set of rules $\langle q_i(v), T_i(v)\rangle \equiv \langle\hat{q}_i(s^*(v)), \hat{T}_i(s^*(v))\rangle$ $i = 1, \ldots, n$. We may then summarize the results of this section by describing the monopolist's problem as follows.[3]

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[3]Constraint (β) is a necessary condition for buyer i's reward function $\Pi_i(x,v_i)$ to have its maximum at $x = v_i$. In lemma 2 of the appendix it is demonstrated that this condition is also sufficient.

Choose $q_i(v)$, $i = 1, \ldots, n$ to maximize

$$(14) \quad \sum_{i=1}^{n} \{ \int_{v_i} J(v_i)G_i(v_i)dF(v_i) - \int_{v_1}\cdots\int_{v_n} B(q_i(v))dF(v_1)\ldots dF(v_n) - \Pi_i(0,0)\}$$

where $J(x) \equiv x + (F(x) - 1)/F'(x)$

subject to the constraints

$$(\alpha) \quad G_i = E_{v_{-i}} A(q_i(v_1, \ldots, x, v_{i+1}, \ldots v_n))$$

$$(\beta) \quad \sum_{i=1}^{n} q_i(v) \leq \bar{q}$$

$$(\gamma) \quad G_i(x) \text{ is non-decreasing}$$

$$(\delta) \quad \Pi_i(0,0) \geq 0$$

Constraint (α) defines G_i. Constraints (β) and (γ) simply repeat conditions (4) and (8). Constraint (δ) is the requirement that no buyer can be forced to make a purchase.

A formal solution to this problem is provided in the following section. Since there is a large family of distributions, F, for which the derivation is relatively straightforward we consider this case first. To obtain the general result we make use of the standard techniques of control theory.

II. The Optimal Selling Strategy

In the previous section we showed how the monopolist's optimal choice reduces to solving a constrained optimization problem. Here we provide the solution. Constraint (δ) may be taken into account immediately. Since $\Pi_i(0,0) \geq 0$ and this enters negatively in the objective function, it is optimal to set $\Pi_i(0,0)$ equal to zero. This is simply the requirement that a buyer who is unwilling to pay a positive price, even for a small amount of the good, should not have an expected gain.

Substituting (α) into the objective function we may rewrite the maximum as

$$(15) \quad \int_{v_1} \cdots \int_{v_n} \sum_i [J(v_i)A(q_i(v)) - B(q_i(v))]dF(v_1) \ldots dF(v_n).$$

Moreover, since

$$G_i(y) - G_i(x) = \underset{v_{-i}}{E} \int_0^{q_i(y,v_{-i})-q_i(x,v_{-i})} a(z)dz \quad ,$$

a sufficient condition for constraint (γ) to be satisfied is

(16) $\quad q_i(v)$ is non-decreasing in v_i, $i = 1, \ldots, n$

Before treating the general case we consider the solution for distribution functions, F, in the following class.

Definition: J-monotone Distribution Functions

The distribution function $F(x)$, continuously differentiable on $[0,1]$ is J-monotone if, for any $x \in (0,1)$ such that $J(x) \equiv x + (F(x)-1)/F'(x)$ is non-negative, $J(x)$ is increasing.

It is readily confirmed that any convex distribution function is J-monotone, as is the family $F(v) = v^c$, $c > 0$.

Proposition 1: If $F(v)$ is J-monotone there exists an optimal selling strategy $\langle q_i^*, T_i^* \rangle_{i=1}^n$ such that

$$(\#)\quad \begin{cases} J(v_i)a(q_i^*) - b(q_i^*) \leq \mu \\ \text{(strict inequality implying that } q_i = 0) \\ \mu(\bar{q} - \Sigma q_i^*) = 0 \end{cases}$$

$$(\#\#)\qquad T_i^* = v_i A(q_i^*) - B(q_i) - \int_0^{v_i} A(q_i^*(x, v_{-i}))dx$$

Proof:

We consider first the maximization of the integrand of (15)

$$I(v,q) \equiv \sum_i J(v_i)A(q_i) - B(q_i)$$

subject to the aggregate supply constraint, and then show that the solution, $q^*(v)$, satisfies (16).

From the definition of J, $J(0) < 0$ and $J(1) > 0$. Then, since F is J-monotonic there is a unique x^*, greater than zero, such that $J(x)$ changes sign at $x = x^*$. Since $A(q_i)$ and $B(q_i)$ are non-negative, it follows by examination of $I(v,q)$ that $q_i^*(v) = 0$ for $v_i \leq x^*$. By assumption $p(v,q_i) = v_i a(q_i) - b(q_i)$ is a non-increasing function of q_i for all $v_i > 0$. Then the integrand is a concave function of q_i for all $v_i > x^*$. It follows that the first-order conditions define the global optimum. To obtain these conditions we form the Lagrangian

$$L = \Sigma J(v_i)A(q_i) - b(q_i) + \mu(\bar{q} - \sum_i q_i)$$

We have

(17) $\frac{\partial L}{\partial q_i} = J(v_i)a(q_i) - b(q_i) - \mu \leq 0$

with the strict inequality implying that $q_i^* = 0$.

(18) $\frac{\partial L}{\partial \mu} = \bar{q} - \sum_i q_i \geq 0$,

with the strict inequality implying that $\mu = 0$.

Together these conditions imply (#). Making use of equations (10) and (11) of the previous section, we know that the expected tariff paid by buyers must be of the form

$$(19)\quad \bar{T}_i(v_i) = v_i G_i(v_i) - \underset{v_{-i}}{E}\, B(q_i(v)) - \int_0^{v_i} G_i(x)dx$$

$$= \underset{v_{-i}}{E}\, \{v_i A(q_i(v)) - B(q_i(v)) - \int_0^{v_i} A(q_i(x,v_{-i})dx\}$$

Comparing (19) and (##) it follows immediately that the latter indeed defines the optimal tariff function.

Finally we wish to show that $q_j(v)$ is a non-decreasing function of v_j. Consider $q_j^*(v) > 0$ and

$$v' = (v_1, \ldots, v_j', \ldots v_n) \text{ where } v_j' > v_j.$$

The necessary condition (17) becomes

(20) $J(v_i')a(q_i) - b(q_i) - \mu' = 0$, for $q_i > 0 \quad i = 1, \ldots, n$

Suppose first that $\mu' \leq \mu$. Since $J(v_i') \geq J(v_i)$, for all i, and the left-hand side of (20) is non-increasing in q, it follows that $q_i^*(v') \geq q_i^*(v)$, for all i. Moreover $J(v_j') > J(v_j) > 0$ so that $q_j^*(v') > q_j^*(v)$. But if μ is strictly positive this is impossible for then $\sum_i q_i^*(v) > \bar{q}$, and the new allocation violates the aggregate supply constraint. Therefore, if $\mu > 0$, $\mu' > \mu$ and from (20), $q_i^*(v') < q_i^*(v)$, $i \neq j$. But $\mu' > \mu > 0$ implies that $\sum_i q_i^*(v') = \sum_i q_i^*(v) = \bar{q}$. Then $q_j^*(v') > q_j^*(v)$.

Q.E.D.

In addition to providing a proof of Proposition 1, the above analysis provides a clear indication of the problems that arise in the absence of J-monotonicity. The first-order conditions (17) and (18) then yield a solution $\hat{q}_i(v)$, $i = 1, \ldots, n$, which is no longer monotonically increasing and the sufficient condition, (16), is violated. We shall see below that the optimum in such cases no longer has the property that

$$q_i^*(v) > q_j^*(v) > 0 \text{ if and only if } v_i > v_j.$$

In economic terms, it is no longer optimal for the monopolist to separate out all the different types of buyers. The intuition behind this result is spelled out in section 3. We conclude this section by solving for the optimum when F is not J-monotonic.[4]

Since the constraint (γ) is binding, at least over some subinterval, we incorporate it as follows

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[4]Some readers may wish to skip the derivation and move immediately to Section 3.

$$(21) \quad \frac{dG_i}{dv_i} = u_i, \quad u_i \geq 0, \text{ almost everywhere.}$$

Because the distribution function $F(\cdot)$ is differentiable, as are the functions $J(\cdot)$, $A(\cdot)$, and $B(\cdot)$, there is, for any $G_i(v)$ differentiable almost everywhere, some piecewise differentiable function $G_i^*(v_i)$ for which the value of the integral in (14) is the same. Thus there is no restriction implied in seeking a solution, $G_i^*(v)$, from among the class of piecewise differentiable functions. We may therefore apply the standard techniques of control theory to prove the following result.

Proposition 2: Characterization Theorem

Define the increasing function

$$K(x) = \begin{cases} J(x) \equiv v + F(v)-1)/F'(v), & x \in [x^{2j}, x^{2j+1}],\ j=0,\dots,m \\ J(x^{2j-1}), & x \in [x^{2j-1}, x^{2j}],\ j=1,\dots,m \end{cases}$$

where $x^0, \dots, x^{2m+1}$ with $0 = x^0 < x^1 < \dots < x^{2m+1} = 1$ satisfy

(A) $J(x)$ is non-decreasing on $[x^{2j}, x^{2j+1}]$, $j = 0, \dots, m$ and $J(x^{2j-1}) = J(x^{2j})$, $j = 1, \dots, m$

(B) $\displaystyle\int_{x^{2j-1}}^{x} [J(z) - J(x^{2j-1})]dF(z) \geq 0$

and the constraint is binding at $x = x^{2j}$.

Also define x^* to be the largest value of x such that $K(x^*) = 0$.

Then there exists an optimal selling strategy $\langle q_i^*(v),\ T_i^*(v)\rangle_{i=1}^{n}$ such that

(#) $q^*(v)$, $i = 1, \ldots, n$ is the solution of

$$\max_{q_1,\ldots,q_n} \left\{ \sum_{i=1}^{n} K(v_i)A(q_i) - B(q_i) \,\middle|\, \sum_{i=1}^{n} q_i \le \bar{q} \right\}$$

(##) $$T_i^* = v_i A(q_i^*) - B(q_i^*) - \int_0^{v_i} A(q_i^*(x, v_{-i}))dx$$

Proof: See appendix.

Comparing our two propositions it should be clear that (#) of Proposition 1 may be rewritten as

(#') $q_i^*(v)$, $i = 1, \ldots, n$ is the solution of

$$\max_{q_1,\ldots,q_n} \left\{ \sum_{i=1}^{n} J(v_i)A(q_i) - B(q_i) \,\middle|\, \sum_{i=1}^{n} q_i \le \bar{q} \right\}$$

Then the method of solution is exactly the same for a general distribution function, $F(v)$, as for a J-monotonic distribution function. The only additional step is the derivation of the monotonic function $K(x)$. This is illustrated for an example in Figure 1. The segment $[x^1, x^2]$ is chosen so that the average value of J over this interval is equal to the value of J at each endpoint.

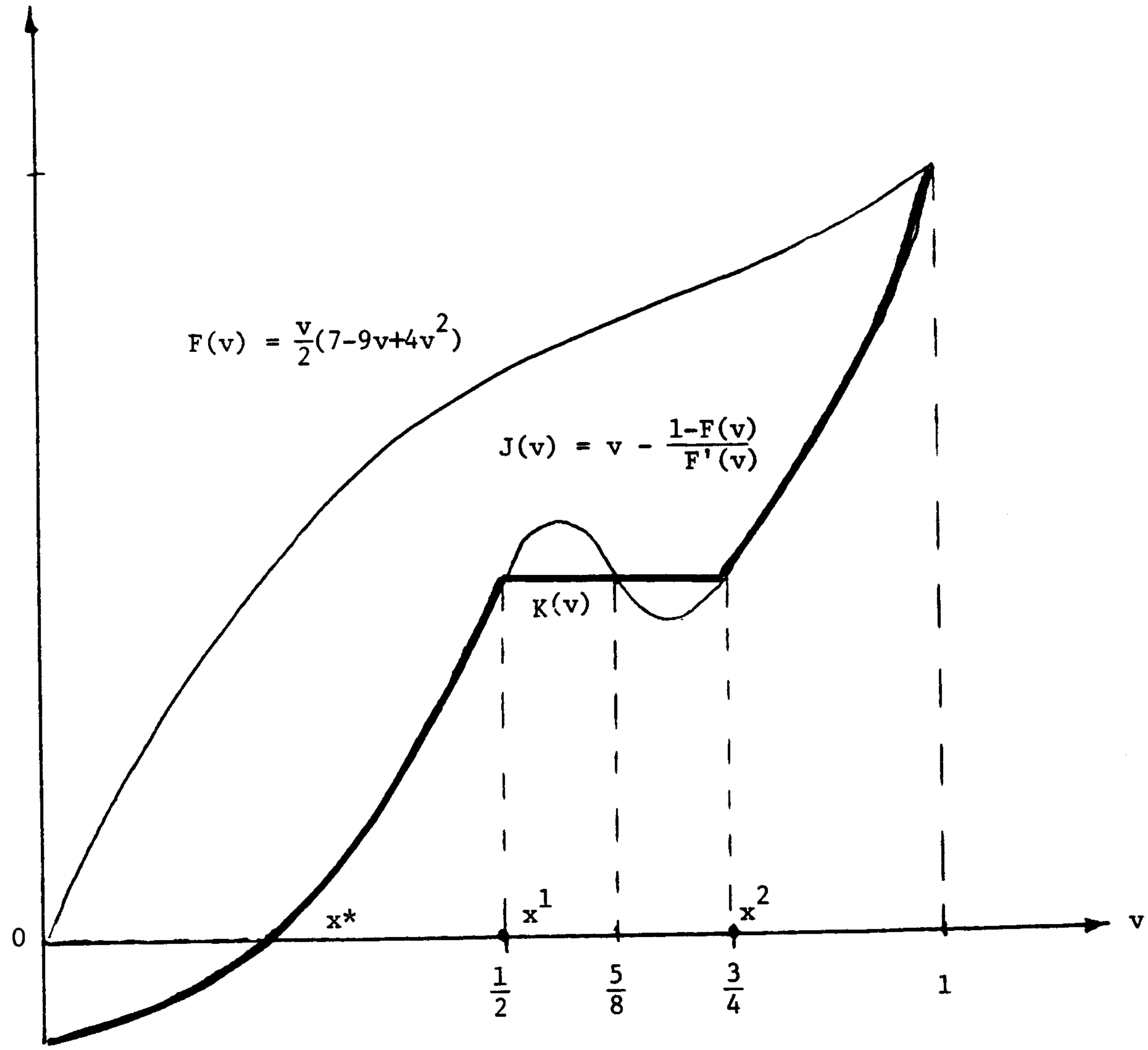

Figure 1: Determination of K(v)

III. Second Degree Price Discrimination

In this section we consider the special case in which monopolist, with constant marginal cost of production c, faces a family of strictly downward sloping continuously differentiable demand curves

$$p(q;v_i) = v_i a(q) - b(q)$$

Since we shall establish that the optimal selling strategy involves charging different marginal prices for different total purchases, it will be helpful to review the Pigouvian price discrimination problem. If the total amount paid for q units is R the net gain to the consumer is just the area under his demand curve less R, that is,

$$(22) \quad u_i(q,R) \equiv U(q,R;v_i) = \int_0^q p(z;v_i)dz - R$$

Several indifference curves for this family of indirect utility functions are depicted in Figure 2. Through any point <q,R> the steepness of buyer i's indifference curve is

$$\left.\frac{dR}{dq}\right|_{u_i} = -\frac{\frac{\partial U}{\partial q}}{\frac{\partial U}{\partial R}} = p(q;v_i)$$

Since the demand price function $p(q;v_i)$ is increasing in v_i, it follows that "high demanders" have indifference maps that are everywhere steeper than low demanders.

The non-discriminating monopolist charges a fixed unit price $\bar{p}$ so that each buyer faces the linear schedule $R(q) = \bar{p}q$. Any discriminating pricing scheme is then a non-linear tariff function R(q), such as the one

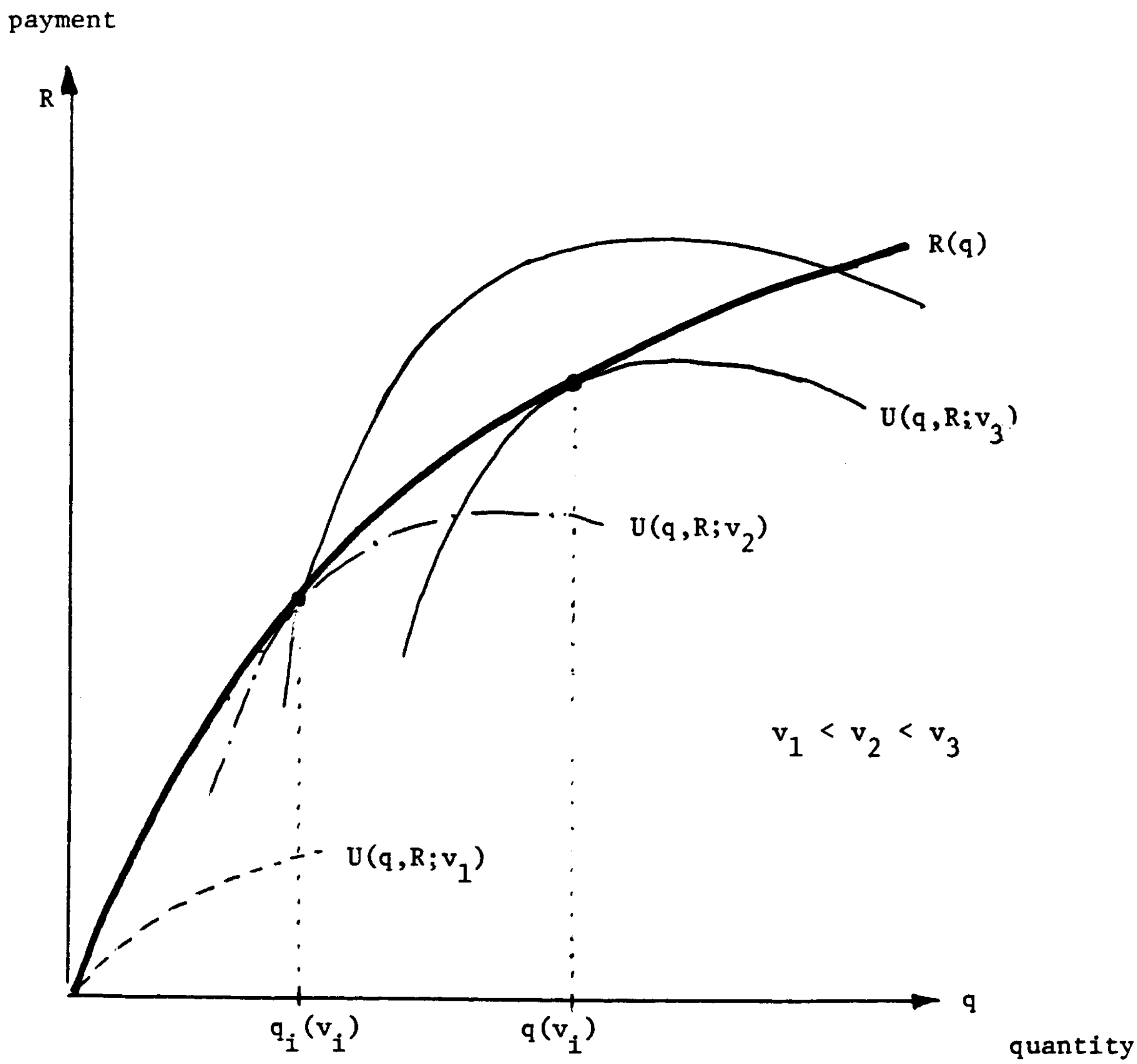

Figure 2: Second Degree Price Discrimination

depicted. We shall see that, for a broad class of distribution functions, the optimal pricing scheme separates out all those buyers actually making purchases so that $q^*(v)$ is strictly increasing in v whenever it is strictly positive. However, complete separation is not always optimal. To understand this consider the simple situation in which there are just two types, as illustrated in Figure 3.

A buyer of type v_1, with $v_1 < v_2$, will accept any offer on or below his indifference curve through the origin. Suppose then that the seller offers each buyer a choice between $\langle q_1^*, R_1^*\rangle$ and $\langle q_2^*, R_2^*\rangle$. It should be clear that revenue from type 2 is maximized by selecting $\langle q_2^*, R_2^*\rangle$ at B the maximum of type 2's indifference curve through $\langle q_1^*, R_1^*\rangle$. It should also be clear that $\langle q_1^*, R_1^*\rangle$ must lie on the upward sloping section of type 1's indifference curve through the origin.[5]

Now suppose that a very small fraction of the population has a parameter value v' where $v_1 < v' < v_2$. Given our assumptions a typical indifference curve for this third type (dashed in the figure) is of intermediate steepness. Reinterpret $\langle q_1^*, R_1^*\rangle$ as being optimal for type 1 with all three types. Then a typical schedule of separating offers is the triple {A,C,D}. Alternatively the seller can continue to offer the pair {A,B} in which case type 3 choose A. With the separating schedule the seller extracts more revenue from type 3 but at the cost of extracting less from type 2. With only a small proportion of type 3 buyers the cost of separation exceeds the benefit so that only incomplete separation is optimal.

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[5]To see this note that for any pair {A',B'}, where A' is on the downward sloping section of type 1's indifference curve, there exists an alternative pair {A,B} generating greater revenue .

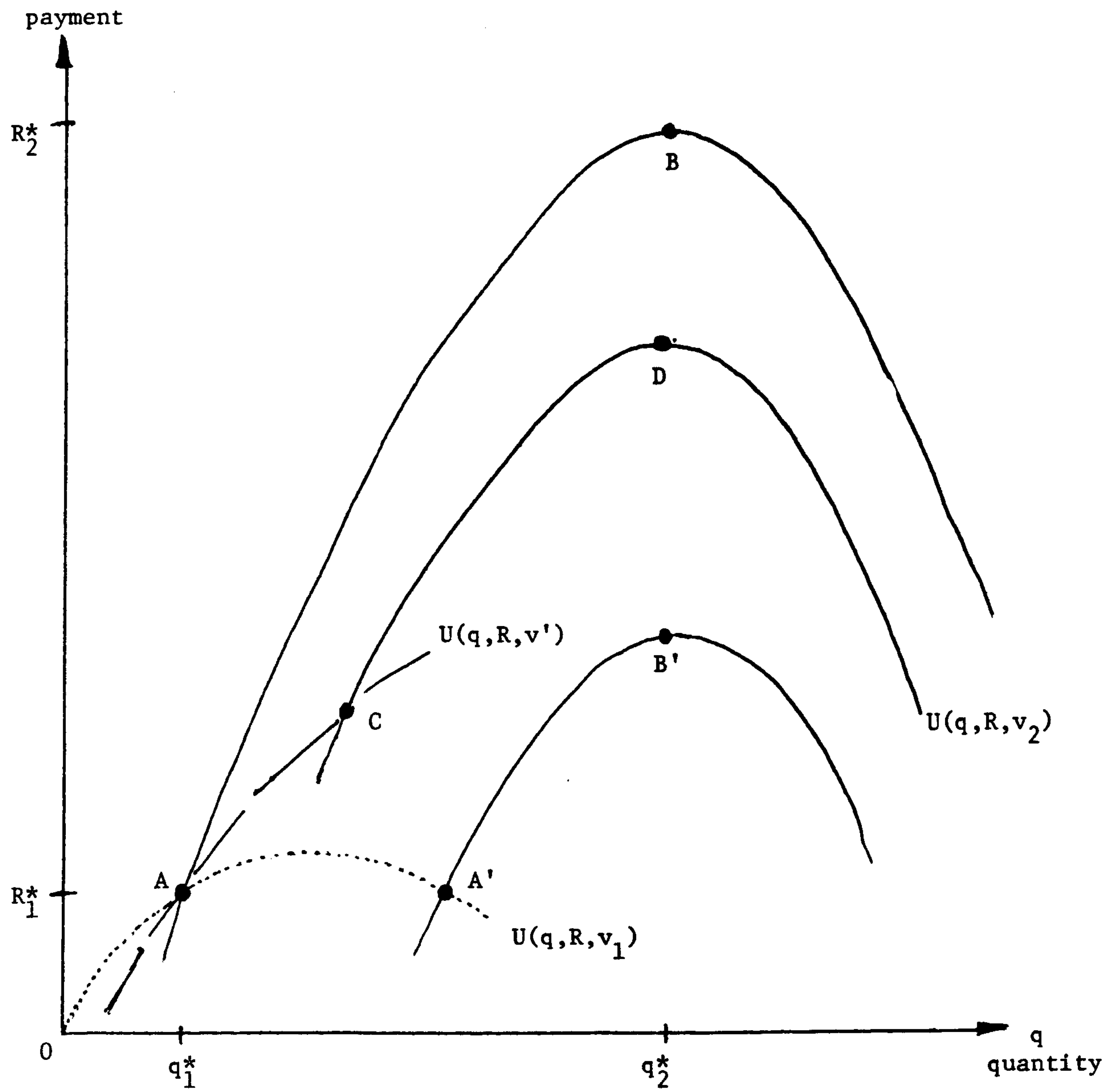

Figure 3: Incomplete Separation of Three Types of Buyers ($v_1 < v' < v_2$)

We now turn to the general analysis. First we shall prove

Proposition 3: Consider the family of demand curves

$$p(q;v_i) \equiv v_i a(q) - b(q)$$

such that p is a strictly decreasing continuous function of q. Then if the marginal cost of production is c, the optimal selling strategy of the monopolist is to annouce a tariff function

$$R^*(q) = \int_0^q [a(z)K^{-1}(\frac{b(z)+c}{a(z)}) - b(z)]dz$$

where K(x) is defined in Proposition 2.

Proof:

In the previous section we were able to characterize the optimal selling strategy which maximizes expected revenue $R(\bar{q})$ given some fixed supply $\bar{q}$. From Proposition 2 there is, for each realization of parameter values $v = (v_1, \ldots, v_n)$, a shadow price $\mu(v)$ reflecting the opportunity cost of the fixed supply constraint. But with a constant marginal cost equal to c the optimal selling strategy is to choose total output so that the shadow price is always equal to c. Then from Proposition 2 there exists an optimal selling strategy $\langle q_i^*(v), T_i^*(v)\rangle_{i=1}^n$ such that

(23) $K(v_i)a(q_i^*) - b(q_i^*) \leq c$

(strict inequality implying that $q^*(v_i) = 0$)

(24) $T_i^* = v_i A(q_i^*) - B(q_i^*) - \int_0^{v_i} A(q_i^*(x))dx$.

The importance of the simplifying assumption of a constant marginal cost of production should now be apparent. From (23) we can solve for q_i^* as a function of v_i alone. Then, from (##) T_i^* is a function of v_i alone and can be expressed as (24). Given the symmetry of the solution we shall henceforth write the optimal selling strategy as $< q^*(v_i),\ T^*(v_i)>_{i=1}^{n}$. From Proposition 2 K(v) is zero for $v = x^*$ and monotonically increasing for $v > x^*$. Then, from (23), there is a unique $\underline{x} > x^*$ such that $q^*(v_i) > 0$ if and only if $x > \underline{x}$.

Then we may rewrite (24) as

$$T^*(v_i) = v_i A(q^*) - B(q^*) - \int_{\underline{x}}^{v_i} A(q^*(x)dx$$

Since $q^*(x)$ is a continuous piecwise differentiable function we may integrate the final term by parts to obtain

$$(25) \quad T^*(v_i) = v_i A(q^*) - B(q) - A(q^*)v_i + \int_{\underline{x}}^{v_i} a(q^*)\frac{dq^*}{dx} \cdot xdx$$

For $x > \underline{x}$ we may invert (23) and rewrite (25) as

$$(26) \quad T^*(v_i) = \int_0^{q^*(v_i)} a(a)K^{-1}(\frac{b(q)+c}{a(q)})dq - B(q^*(v_i)).$$

Finally we may define

$$R^*(q(v_i)) = T^*(v_i)$$

Then, from (26) the expected payment by a buyer purchasing q units is

$$R^*(q) = \int_0^{q} a(z)K^{-1}(\frac{b(z)+c}{a(z)})dz - B(q)$$

Q.E.D.

For any particular family of demand curves and distribution of preferences, Proposition 3 yields immediately the optimal tariff function.

Example 1:

$$p(q;v_i) = v_i - q$$

$$F(v_i) = v_i$$

For this simple case $a(q) = 1$, $b(q) = q$ and $J(x) = 2x - 1$. Then $J(x)$ is a strictly increasing function so, from Proposition 2, $K(x) = J(x)$. Substituting into (23) we have

$$(27) \qquad q^*(v_i) = \begin{cases} 0 & , \quad v_i < \underline{x} = \frac{1}{2}(1+c) \\ 2v_i - 1 - c & , \quad v_i \leq \underline{x} \end{cases}$$

From the statement of Proposition 3 we have

$$(28) \qquad R^*(q) = \int_0^q \left(\frac{1+c+z}{z}\right) dz - \frac{1}{2}q^2$$

$$= q\left(\frac{1+c}{2} - \frac{1}{4}q\right)$$

Thus, for this example, the average unit price, $p^*(q) = \frac{1}{2}(1+c) - \frac{1}{4}q$ declines linearly with q: the optimal selling strategy is a quantity discount strategy.

From (27) the highest demanders (v_i=1) purchase $q^*(1) = 1 - c$ units. Then, from (28) the marginal price paid,

$$\frac{dR^*(q)}{dq} = \frac{1}{2}(1+c) - \frac{1}{2}q$$

$$\geq \frac{dR^*(1)}{dq} = c.$$

Thus the marginal price paid by all buyers except the very highest demanders exceeds marginal cost.

Applying exactly the same argument for the general case we have the following result

Proposition 4: For the optimal selling strategy the marginal price, $\frac{dR^*(q^*(v_i))}{dq}$ paid by each buyer exceeds marginal cost except for the very highest demanders ($v_i = 1$).

To obtain a simple partial ordering of the tariff functions for different distributions we utilize a ranking of distributions which implies (but is not implied by) first-order stochastic dominance. Let $\rho_F(x)$ be the "decay rate" of the distribution F(x), that is

$$(29) \quad \rho_F(x) = \frac{F'(x)}{1 - F(x)}$$

Since higher values of x are associated with higher demanders, a distribution with a low decay rate is obviously preferable from the seller's viewpoint. By maintaining the same tariff function the seller provides buyers the same incentives and the greater density of the higher demanders results in greater revenue. What is less evident, however, is that if one distribution function has a decay rate which is everywhere lower, the optimal tariff is strictly larger for all positive q.

Proposition 5: If the distribution $F_\alpha(x)$ has an everywhere lower decay rate than $F_\beta(x)$, then for all $q > 0$

$$R^*_{F_\beta}(q) > R^*_{F_\alpha}(q)$$

Proof:

Consider $J(x) = x - \frac{(1-F(x))}{F'(x)}$. If F_α has an everywhere lower decay rate it follows immediately that for all $x < 1$

$$J_\alpha(x) < J_\beta(x)$$

Then, from the definition of K(x) in Proposition 2

$$K_\beta(x) > 0 \quad \rightarrow \quad K_\alpha(x) > K_\beta(x),$$

and hence

$$z > 0 \quad \rightarrow \quad K_\beta^{-1}(z) < K_\alpha^{-1}(z).$$

It follows immediately that the integrand in the statement of Proposition 3 is larger for the distribution F_α than for the distribution F_β.

Q.E.D.

Finally, it should be noted that, while Proposition 3 assumes a non-stochastic allocation $(q_1(s), \ldots, q_n(s))$, there are no gains to making $q_i(s)$ stochastic. To see this suppose the contrary so that $\tilde{q}_i(s)$ is a random variable with expectation $\bar{q}_i(s)$. The expected gain to buyer i is

$$EU(\tilde{q}_i(s), R_i(s); v_i) = \int_0^{\tilde{q}_i} p(z,v_i)dz - R_i(s)$$

Since the demand price $p(z,v_i)$ is decreasing in z, U is a concave function of q_i and $-R_i$. Then, for some $\delta \geq 0$

$$EU(\tilde{q}_i(s), R_i(s); v_i) = U(\bar{q}_i(s), R(s)+\delta, v_i)$$

Thus for any stochastic allocation rule $\tilde{q}_i(s)$, there is an alternative rule which provides the consumer with the same expected quantity and which generates greater expected revenue to the seller.

IV. Auction mechanisms

In section III we studied the monopolist's optimal selling strategy under the assumption of constant marginal cost. This greatly simplified the analysis since it was possible to consider each buyer in isolation. However, with rising marginal cost, the opportunity cost of selling an additional unit to one buyer depends on the demands of all other buyers. In this section we turn to the polar case of a rising marginal cost schedule: the case of fixed aggregate supply.

We begin by considering the special case in which each buyer wishes to purchase a single unit. The unknown parameter, v_i, is then his reservation value. As before we assume that each v_i is an independent random draw from $F(v)$. When the seller has a single unit for sale the problem reduces to the choice of a set of rules for the auctioning of a unique object. More generally, suppose that there are $\bar{q}$ units for sale where $\bar{q}$ is some positive integer. This is precisely the problem considered by Harris and Raviv [1981] although these authors make the important simplifying assumption that reservation values are distributed uniformly.[6]

As in the previous section we appeal directly to Proposition 2 in order to characterize the optimal selling strategy. With

$$p(q;v_i) = \begin{cases} v_i, & q < 1 \\ 0, & q \geq 1, \end{cases}$$

condition (#) of Proposition 2 reveals that $q_i^*(v)$ is the solution of

$$(30)\quad \max_{q_1,\ldots,q_n} \{ \sum_{i=1}^{n} K(v_i)q_i \mid q_i \leq 1, \ \sum_{i=1}^{n} q_i \leq \bar{q}\} \qquad .$$

[6] This appears to be an important assumption in their very different approach to the problem.

By inspection it is clear that the objective function is maximized by setting q_i equal to unity for those individuals having high values of $K(v_i)$ and hence v_i. (Remember that $K(x)$ is a non-decreasing function.) Formally, for any vector of reservation values $(v_1, \ldots, v_n)$ let $\bar{v}$ be the $\bar{q}^{th}$ highest reservation value. Also let $\tilde{q}$ be the number of buyers for whom $K(v_i) > K(\bar{v})$ and let r be the number of buyers for whom $K(v_i) = K(\bar{v})$.

Then the $q_i^*(v)$ which maximizes (30) is given by

$$(31) \qquad q_i^*(v) = \begin{cases} 1, & K(v_i) > K(\bar{v}) \text{ and } \bar{v} \geq x^* \\ \dfrac{\bar{q} - \tilde{q}}{r}, & K(v_i) = k(\bar{v}) \text{ and } \bar{v} \geq x^* \\ 0 \;, & \text{otherwise} \end{cases}$$

If the distribution function, F, is J-monotonic, $K(x)$ is equal to $J(x)$ for $x > x^*$ and is therefore increasing for all $x > x^*$. Then (31) indicates that expected revenue is maximized by any scheme which assigns one unit to a reservation value which is both greater than x^* and is among the $\bar{q}^{th}$ highest.

One simple way of achieving this is an open auction. The auctioneer opens the bidding at x^* and continues raising the asking price until only $\bar{q}$ buyers remain. Since this takes place when the asking price is equal to the $\bar{q} + 1^{th}$ buyer's reservation value, each of the bidders then pays this reservation value. Of course if there are no more than $\bar{q}$ bidders at the opening price all sales are made at that price.

If $F(x)$ is not J-monotonic then with finite probability several buyers have the same value of K. If these buyers are pivotal in the sense that one has the $\bar{q}+1^{th}$ reservation value, (31) indicates that at least one unit is allocated randomly. All this is summarized in the following proposition.

Proposition 6: Optimal multi-unit auctions with unit demand

Let each buyer have a reservation value for a single unit which is an independent random draw from $F(x)$. Let $x^*, x^1, \ldots, x^{2m}$ be defined as in Proposition 2. Then

(A) If $F(x)$ is J-monotonic the seller can do no better than announce a minimum price x^*, and, if there are more than $\bar{q}$ buyers at this price, raise the asking price continuously until only $\bar{q}$ buyers remain.

(B) If $F(x)$ is not J-monotonic and $J(x) \equiv x + (F(x)-1)/F'(x)$ is decreasing for some $x > x^*$, there is at least one interval $[x^{2i-1}, x^{2i}]$ over which the asking price is raised discontinuously. Let the number of buyers remaining, after the asking price is raised to x^{2i} be $\tilde{q}$. Then if $\tilde{q}$ is less than $\bar{q}$ there are two selling prices. Those remaining pay x^{2i} while those dropping out are given an equal chance of obtaining the $\bar{q} - \tilde{q}$ units at a price of x^{2i-1}.

The first point to be made about Proposition 6 is that it is only one of a host of possible auction mechanisms which would satisfy (31) and hence yield the greatest expected revenue. The obvious alternative is a sealed bid auction. If $F(x)$ is J-monotonic the seller simply announces the minimum price x^* and assigns a unit to each buyer submitting one of the $\bar{q}^{th}$ highest bids. Each successful bidder pays his actual bid $b_i(v_i)$. When $F(x)$ is not J-monotonic the seller also prohibits bids over certain intervals corresponding to intervals $[x^{2i-1}, x^{2i}]$, $i = 1, \ldots, m$.

For a discussion of some unusual optimal auctions when there is just a single unit for sale see Riley and Samuelson [1981]. Their results are readily extended to cover the multi-unit case.

However the auction mechanism described in Proposition 6 has the advantage that the equilibrium strategies of the buyers are dominant strategies of a very simple form. The computational requirements of determining the equilibrium bidding strategy are therefore smaller and doubts about whether other bidders are actually adopting their equilibrium strategies do not affect behavior.

A third point is that if F is not J-monotonic so that the outcome may be stochastic for some buyers, there is a finite probability that the buyer with the $\bar{q}+1^{th}$ highest reservation value is assigned one unit. The proof of Proposition 6 therefore assumes that the seller can enforce a no-resale provision. In the absence of such a provision the prospect of resale changes buyers' behavior. While the seller's expected revenue declines, Maskin and Riley [1980] have shown that the conditions under which it is optimal for the seller to utilize a stochastic auction are the same with and without a no-resale provision.

We now consider the general problem in which demand curves slope downward. Perhaps the most common auction of this type is the U.S. Treasury bill auction. Buyers may submit orders at one or more prices. Thus, in principle a buyer can approximate any demand curve arbitrarily closely. Current practice is for the Treasury to fill orders at the prices submitted until total orders equal the size of the offering. However the treasury has also experimented with a sealed bid auction in which all buyers pay the bid price of the nearest unsuccessful bidder.

As we shall see below neither of these auctions are optimal, even with an announced minimum price. Moreover, there is no obvious reason why the expected revenue from the two auctions should be the same.[7]

To simplify the exposition a little, we assume that the family of demand curves

$$p(q;v_i) = v_i a(q) - b(q)$$

has a negative slope and that no one buyer would wish to purchase all $\bar{q}$ units, even at a zero price. Formally, the optimal selling strategy is already characterized in Proposition 2. What remains is to interpret this characterization and then to translate it into something which is economically more familiar. From (#) of Proposition 2 the seller can solve for $(q_1^*(v), \ldots, q_n^*(v))$. Substituting into (##) the seller can then announce the tariff function and allocation rule

$$(32) \qquad \begin{cases} T_i^*(v_1,\ldots,v_n) = v_i A(q_i^*(v)) - B(q_i^*(v)) - \int_0^{v_i} A(q_i^*(x,v_{-i})dx \\ q_i^*(v_1,\ldots,v_n) \end{cases}$$

Each buyer is then asked to submit his parameter value v_i. What we have established is that if every buyer but the ith submits his true parameter value, truthtelling is optimal for buyer i as well.

Since buyers are risk neutral an alternative scheme is for the seller to announce the tariff function

$$(33) \qquad \bar{T}_i^*(v_i) = \underset{v_{-i}}{E}\; T_i^*(v)$$

Also note that when the aggregate supply constraint is not binding, the allocation rule (#) becomes

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[7] The treasury has not yet announced the results of its experiment with the one-price auction. In further work we plan to use the results of this paper to make comparisons of the two forms of Treasury bill auction with the theoretical optimum. For a discussion of the use of the one-price auction when buyers bid for a share of an indivisible object see Wilson (1979).

$$(33)\quad \begin{cases} q_i^* = 0 & v_i \leq x^* \\ K(v_i)a(q_i^*)-b(q_i^*)=0, & v_i > x^* \end{cases}$$

Given our assumptions there exists a unique $q_i < \bar{q}$ satisfying (33). Then, instead of asking for each buyer to submit his parameter value the seller can ask each to choose a point on the schedule

$$(34)\qquad R^*(q_i) = \bar{T}_i^*(K^{-1}(\frac{b(q_i)}{a(q_i)}))$$

If total orders are less than supply, orders are filled and buyer i, with order q_i, pays $R^*(q_i)$. If total orders exceed supply the final allocations are chosen so that, for some $\mu > 0$,

$$(35)\quad \begin{cases} \sum_{i=1}^{n} q_i^* = \bar{q} \\ K(v_i)a(q_i^*)-b(q_i^*) \leq \mu \\ \text{with the strict inequality implying } q_i^* = 0 \end{cases}$$

Comparing (34) and (35) it is clear that each buyer's final allocation is reduced. Thus we may ignore those buyers who do not submit an initial order ($v_i < x^*$). Substituting from (34) the final allocation rule is

$$\begin{cases} q_i^*[\frac{b(q_i)}{a(q_i)}\, a(q_i^*) - b(q_i^*) - \mu] = 0 \\ \sum_{i=1}^{n} q_i^* = \bar{q} \end{cases}$$

To summarize we have proved the following result.

Proposition 7: An optimal selling strategy of the monopolist is to announce the tariff schedule

$$R^*(q_i) = \underset{v_{-i}}{E}\; T_i^*(K^{-1}(\frac{b(q_i)}{a(q_i)}))$$

where $K(\cdot)$ and $T_i^*(\cdot)$ are defined in Proposal 2.

Each buyer submits an initial order, q_i. If total orders exceed supply, final allocations are reduced according to the following rationing scheme

$$(\dagger)\quad \begin{cases} q_i^*[\frac{b(q_i)}{a(q_i)}\, a(q_i^*) - b(q_i^*) - \mu] = 0 \\[2ex] q_i^* = \bar{q} \end{cases}$$

For the special case in which the demand curves have the simple form

$$p(q;v_i) = v_i - qb$$

the allocation rule is especially straightforward. From (†) we have

$$q_i^* > 0 \rightarrow b(q_i - q_i^*) - \mu = 0 \rightarrow q_i - q_i^* = \mu/b$$

Thus the monopolist simply reduces orders by an equal absolute amount. Once any particular order reaches zero the remainder are reduced by an equal absolute amount until aggregate demand equals supply.

V. The Pricing of Product Quality and Optimal Bundling

Our general model can easily be utilized to determine the optimal selling strategy of a monopolist selling products of differing quality levels. We assume that each consumer wishes to purchase only one quality level. Furthermore, following Mussa and Rosen [1978], we begin by assuming that consumers either do not buy or purchase just one unit.

Consider the Marshallian utility function[8]

$$\tilde{u}_i(x,q,z) = x + z(v_i A(q) - B(q))$$

where x is spending on other goods, q is the quality level of the single unit purchased and z is a dichotomous variable equal to unity with a purchase and zero otherwise. If a consumer with income level I_i pays T for a unit of quality level q we can rewrite his indirect utility as

$$(36) \quad u_i(q,T,z) = z(v_i A(q) - B(q) - T) + I_i$$

As long as A(q) is a positive increasing function the marginal utility of higher quality is, ceteris paribus, higher for consumers with higher levels of v.

With little loss of generality we define units of quality in such a way that the marginal cost of a unit of quality level q is cq. Then the monopolist's problem is identical to the problem considered in section III except that q is now interpreted as quality rather than quantity.

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[8]If $A(q) = q$ and $B(q) = 0$ this reduces to the problem considered by Mussa and Rosen.

The natural generalization of this problem is to incorporate the choice of both quality, q and the number of units purchased, z. Replacing z in (36) by the concave increasing function h(z) we have the indirect utility function.

(37) $u_i(q,T,z) = h(z)(v_i A(q) - B(q)) - T + I_i$

where T is the total cost to the consumer of the z units of quality level q. Adapting only slightly the proof of Proposition 2 we have the following result.

Proposition 8: If each buyer parameter value, v_i is an independent random draw from $F(\cdot)$, expected revenue of the monopolist is maximized by the sale of $z^*(x)$ units of quality $q^*(x)$ to any buyer with parameter value x, where z^* and q^* are the solution of

$$\max_{z,q} \{h(z)[K(x)A(q) - B(q)] - czq\},$$

and K(x) is defined in Proposition 2. The total tariff for this bundle of goods is

$$T^*(x) = h(z^*(x))(xA(q^*(x)) - B(q^*(x))) - \int_0^x h(q^*(y))A(q^*(y))dy$$

Example 2:

Suppose $h(z) = z^{1/2}$, $A(q) = q$, $B(q) = 1/3q^2$, $F(x) = x$.

Using Proposition 8 we can solve for $q^*(x)$ and $z^*(x)$ to obtain

(38) $$q^*(x) = \begin{cases} 0 & , x < 1/2 \\ J(x) = 2x - 1, & x \geq 1/2 \end{cases}$$

$$(39) \quad z^*(x) = \begin{cases} 0 & , x < 1/2 \\ \frac{k^2}{9c^2} = (\frac{q^*(x)}{3c})^2 & , x \geq 1/2 \end{cases}$$

Substitution these into the expression for $T^*(x)$ we have

$$(40) \quad T^*(x) = \frac{q^*(x)^4}{8c}$$

Note that, from (39) and (40), we may write

$$z^*(x) = \tilde{z}(q^*(x)) \text{ and } T^*(x) = \tilde{R}(q^*(x))$$

Thus the monopolist can do no better than announce that he will sell $\tilde{z}(q) = (\frac{q}{3c})^2$ units of quality level q for a total tariff of $\tilde{R}(q) = \frac{q^4}{8c}$; an optimal bundling strategy.

VI. Concluding Remarks

In this paper we have presented a general method for solving a broad class of problems in which one agent announces the rules of the game while the others, acting non-cooperatively, choose their optimal responses. We conclude with a few remarks about the crucial assumptions.

First of all agents are assumed to be risk neutral. With risk averse buyers, the analysis is much more complicated. For the relatively simple case in which a single item is up for auction several papers have compared specific auction rules (See, for example, Matthews [1979], Holt [1980], and Riley and Samuelson [1981]). Under the assumptions of Section IV it has been shown that, for any minimum price, the high bid auction yields a higher expected revenue than the second bid auction. Furthermore, Maskin and Riley [1980] have established that the seller can raise expected revenue still higher by utilizing both a minimum price and an entry fee. However there are no results on the nature of the expected revenue maximizing auction.

The second crucial assumption is that of parameter value independence. Any pair of buyers, with possibly very different parameter values, have the same beliefs about the parameter value of a third buyer. While this is the natural first approximation there are situations in which it is clearly deficient. Again it is helpful to consider the auction application. Suppose, as in the auctioning of mineral rights, the true value of the item is unknown. Each buyer has an estimate based on research. In this case it is natural to assume that a buyer with a low estimate will have more conservative beliefs about the estimates of other buyers than a buyer with a high estimate. Recently, in an elegant paper, Milgrom and Weber [1980] have developed the concept of positive association to formalize this idea, and have used it to compare sealed auctions (both high and second bid) and the open ascending bid auction. A central result is that information revealed as the open auction progresses, raises the expected selling price. With risk neutral buyers there is no equivalent effect in either of the sealed bid auctions so the open auction dominates in terms of expected revenue.[9]

This conclusion suggests that the seller might be able to exploit the positive associatedness of buyer's reservation values with an auction very different from either of the usual auctions. Some examples of Myerson [1981] tend to strengthen this inference. Indeed, he provides a dramatic illustration of an auction in which the seller is able to extract essentially all of the consumer surplus.

The final crucial assumption is that the underlying family of demand curves can be approximated by a one parameter family. Preliminary work suggests that it will not be easy to extend the method utilized here to incorporate additional unknown parameters.

Despite these limitations, we believe that the methods presented will prove useful in examining broad class of principal-agent problems. Moreover, while our paper focusses on the case in which the agent chooses the rules of the game to maximize his own expected gain, our methods are very easily modified to incorporate a social objective. For example, instead of a monopolist price discriminating to maximize expected profit, the rules of the game might be chosen to maximize consumer surplus subject to a constraint on the expected profit of the regulated firm.

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[9] Since risk aversion has the effect of raising expected revenue from the high bid auction, there is no simple ranking of the three auctions except under the assumption of risk neutral buyers.

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Appendix.

Lemma 1: The maximized payoff to buyer i, $\Pi_i(v_i,v_i)$ is absolutely continuous on [0,1]. Moreover

$$\Pi_i(v_i,v_i) - \Pi_i(0,0) = \int_0^{v_i} G_i(x)\,dx$$

Proof:

Suppose $\tilde{v}_i \geq v_i$. Since

$\Pi_i(v_i,\tilde{v}_i) \equiv \tilde{v}_i G_i(v_i) - \underset{v_{-i}}{E}\,(B(q_i(v)) + T_i(v))$ and

$G_i(v_i) \geq 0$ it follows that

$$\Pi_i(v_i,\tilde{v}_i) \geq \Pi_i(v_i,v_i).$$

Moreover, since $\Pi_i(v_i,v_i)$ is the maximized payoff to buyer i we also have

$$\Pi_i(\tilde{v}_i,\tilde{v}_i) \geq \Pi_i(v_i,\tilde{v}_i) \text{ and } \Pi_i(v_i,v_i) \geq \pi_i(\tilde{v}_i,v_i).$$

Combining these inequalities we have

$$\text{(A1)}\quad 0 \leq \Pi_i(\tilde{v}_i,\tilde{v}_i) - \Pi_i(v_i,v_i) \leq \Pi_i(\tilde{v}_i,\tilde{v}_i) - \Pi_i(\tilde{v}_i,v_i)$$
$$= G_i(\tilde{v}_i)(\tilde{v}_i - v_i)$$
$$\leq A(\bar{q})(\tilde{v}_i - v_i),$$

since $q_i \leq \bar{q}$ and $G_i(\tilde{v}_i) \equiv \underset{\tilde{v}_{-i}}{E}\,A(q_i(\tilde{v}))$.

Therefore for any $\varepsilon > 0$ and every m disjoint open subintervals $(v_j,\tilde{v}_j)$ of [0,1], n = 1, 2, ..., the sum of whose lengths is less than $\varepsilon/A(\bar{q})$,

$$\sum_{j=1}^{m} \left|\Pi_i(\tilde{v}_j,\tilde{v}_j) - \Pi_i(v_j,v_j)\right| < \varepsilon.$$

Thus $\Pi_i(v_i,v_i)$ is absolutely continuous on [0,1]. It follows that $\Pi_i(v_i,v_i)$ is an indefinite integral and we may write

$$\text{(A2)} \quad \Pi_i(v_i,v_i) - \Pi_i(0,0) = \int_0^{v_i} \frac{d\Pi_i(x,x)}{dx}\,dx$$

It remains to confirm that where the derivative of $\Pi_i(x,x)$ exists it is equal to $G_i(x)$. We have

$$\frac{\Pi_i(\tilde{v}_i,\tilde{v}_i) - \Pi_i(v_i,v_i)}{\tilde{v}_i - v_i} = \frac{\Pi_i(\tilde{v}_i,\tilde{v}_i) - \Pi_i(\tilde{v}_i,v_i)}{\tilde{v} - v_i} + \frac{\Pi_i(\tilde{v}_i,v_i) - \Pi_i(v_i,v_i)}{\tilde{v}_i - v_i}$$

$$= G_i(\tilde{v}_i) + \frac{\Pi_i(\tilde{v}_i,v_i) - \Pi_i(v_i,v_i)}{\tilde{v}_i - v_i}$$

Since $G_i(x)$ is a non-decreasing function it is continuous almost everywhere. Moreover, since $\Pi_i(v_i,v_i)$ is differentiable almost everywhere we have

$$\frac{d\Pi_i}{dv_i} = \lim_{\tilde{v}_i \to v_i} \frac{\Pi_i(\tilde{v}_i,\tilde{v}_i) - \Pi_i(v_i,v_i)}{\tilde{v}_i - v_i} = G_i(v_i) + \lim_{\tilde{v}_i \to v_i} \frac{\Pi_i(\tilde{v}_i,v_i) - \Pi_i(v_i,v_i)}{\tilde{v}_i - v_i}$$

But $\Pi_i(\tilde{v}_i,v_i)$ achieves its maximum for all $\tilde{v}_i \in [0,1]$ at $\tilde{v}_i = v_i$. Therefore the final term is zero and we have

$$\frac{d\Pi_i(v_i,v_i)}{dv_i} = G_i(v_i), \quad \text{almost everywhere.}$$

Q.E.D.

Lemma 2: If $G_i(x)$ is a non-decreasing function $\Pi_i(y,x)$ is pseudo concave, that is, for all x and y ε [0,1]

$\Pi_i(y,x) \leq \Pi_i(x,x)$

Proof:

Utilizing equations (6) and (7) we may write buyer i's reward function as

$$\Pi_i(y,v_i) = \Pi_i(y,y) + (v_i-y)G_i(y)$$

From Lemma 1 we can rewrite this as

$$\text{(A3)}\quad \Pi_i(y,v_i) = \int_0^y G_i(x)\,dx + (v_i-y)G_i(y) + \Pi_i(0,0)$$

Consider y, z such that $y < z < v_i$. Making use of (A3) we have

$$\text{(A4)}\quad \Pi_i(z,v_i) - \Pi_i(y,v_i) = \int_y^z (G_i(x)-G_i(y))\,dx + (v_i-z)(G_i(z)-G_i(y))$$

Since $G_i(x)$ is non-decreasing the right-hand side is non-negative. Thus $\Pi_i(x,v_i)$ is a non-decreasing function of x on $[0,v_i]$.

An almost identical argument establishes that $\Pi_i(x,v_i)$ is a non-increasing function of x on $[v_i,1]$.

Q.E.D.

Proposition 2: Characterization Theorem

Define the increasing function

$$K(x) = \begin{cases} J(x) \equiv v + (F(v)-1)/F'(v), & x \in [x^{2j},x^{2j+1}],\ j=0,\dots,m \\ J(x^{2j-1}), & x \in [x^{2j-1},x^{2j}],\ j=1,\dots,m \end{cases}$$

where $x^0, \dots, x^{2m+1}$ with $0 = x^0 < x^1 < \dots < x^{2m+1} = 1$ satisfy

(A) $J(x)$ is non-decreasing on $[x^{2j},x^{2j+1}]$, $j = 0, \dots, m$ and $J(x^{2j-1}) = J(x^{2j})$, $j = 1, \dots, m$

(B) $$\int_{x^{2j-1}}^{x} [J(z) - J(x^{2j-1})]\,dF(z) \geq 0$$

and the constraint is binding at $x = x^{2j}$.

Also define x^* to be the largest value of x such that $K(x^*) = 0$.

Then there exists an optimal selling strategy

$<q_i^*(v),\ T_i^*(v)>_{i=1}^{n}$ such that

(#) $q^*(v)$ is the solution of

$$\operatorname*{Max}_{q} \{ \sum_{i=1}^{n} K(v_i)A(q_i) - B(q_i) \mid \sum_{i=1}^{n} q_i \leq \bar{q} \}$$

$$(\#\#) \quad T_i^* = v_i A(q_i^*) - B(q_i^*) - \int_0^{v_i} A(q^*(x, v_{-i}))dx$$

Proof:

Consider the general statement of the problem at the end of section 1. It is useful to incorporate the first two constraints by forming the Lagrangian

$$L = \sum_i \int_{v_i} \dots \int_{v_n} [J(v_i)G_i - B(q_i(v))]dF(v_1) \dots dF(v_n)$$

$$+ \sum_i \int_{v_1} \dots \int_{v_n} \lambda_i A(q_i(v))dF(v_1) \dots dF(v_n) - \int_{v_i} \lambda_i G_i dF(v_i)$$

$$+ \int_{v_1} \dots \int_{v_n} [\bar{q} - \sum_i q_i(v)]\mu dF(v_1) \dots dF(v_n)$$

This is more conveniently rewritten as

$$L = \bar{q}\mu + \int_{v_1} \dots \int_{v_n} \sum_i [J(v_i)G_i - \lambda_i G_i - \mu q_i + \lambda_i A(q_i) - B(q_i)]dF(v_1) \dots dF(v_n)$$

To incorporate the constraint that $G_i(v_i)$ is a non-decreasing function we then form the following Hamiltonian

$$H = \sum_i [J(v_i)G_i - \lambda_i G_i - \mu q_i + \lambda_i A(q_i) - B(q_i)] F'(v_1) \dots F'(v_n) + \phi_i u_i$$

Except at points of discontinuity we have the necessary conditions,

(a2) $\frac{\partial G_i}{\partial v_i} = \frac{\partial H}{\partial x} = u_i \geq 0,$

(a3) $\frac{\partial \phi_i}{\partial v_i} = -\frac{\partial H}{\partial q} = (\lambda_i - J(v_i))F'(v_1)\ldots F'(v_n)$

(a4) $\frac{\partial H}{\partial q_i} = \lambda_i a(q_i) - b(q_i) - \mu \leq 0$, strict inequality $q_i^* = 0$,

(a5) $\mu \geq 0$, strict inequality $\rightarrow \sum_i q_i^* < \bar{q}$,

(a6) $\frac{\partial H}{\partial u_i} = \phi_i \leq 0$, strict inequality $\rightarrow u_i^* = 0$.

Finally the corner conditions are satisfied if, for all $v_i \varepsilon [0,1]$ and all $i = 1, \ldots, n$,

(a7) $\hat{H}(v) = \sum_{i=1}^{n} J(v_i)A(q_i(v)) - B(q_i(v))$ is continuous.

It is convenient here to introduce some additional notation. Let $I^j = [x^{j-1}, x^j]$, $j = 0, \ldots, 2m + 1$ where $x^0 = 0$ and $x^{2m+1} = 1$. Also let

$f(v_{-i}) = F(v_1) \ldots F(v_{i-1}) \;\; F(v_{i+1}) \ldots F(v_n)$

Take

$$\lambda_i^* = K(v_i)$$

From (a3)

$$\frac{\partial \phi_i^*}{\partial v_i} = 0 \text{ on } I^1 \cup I^3 \cup \ldots, \cup I^{2m+1}$$

Also, for $v_i \varepsilon I^{2j} = [x^{2j-1}, x^{2j}]$

$$\phi_i^*(v_i, v_{-i}) - \phi_i^*(x^{2i-1}, v_{-i}) = f(v_{-i}) \int_{x^{2j-1}}^{v_i} (J(x^{2j-1}) - J(z))dF(z)$$

From condition (B) the right-hand side is less than or equal to zero for $v_i \varepsilon I^{2j} = [x^{2j-1}, x^{2j}]$ and equal to zero for $v_i = x^{2j}$. Therefore if we define

$$\phi_i^*(x^*, v_{-i}) = 0$$

it follows that $\phi_i^*(v_i, v_{-i})$ is everywhere non-positive so condition (a6) is satisfied. Substituting for λ_i^* in (a6) we have

$$\text{(a9)} \quad K(v_i)a(q_i) - b(q_i) - \mu \leq 0$$

with the strict inequality implying $q_i = 0$.

Conditions (a9) and (a5) are the necessary conditions for the maximization of $\sum_{i=1}^{n} K(v_i)A(q_i) - B(q_i)$ subject to the aggregate supply constraint. Arguing exactly as in the proof of Proposition 1 these conditions, which define a vector of allocation functions $<q_1^*(v), \ldots, q_n^*(v)>$, are also sufficient. Thus statement (#) of the Proposition is satisfied. Moreover, since $K(v_i)$ is non-decreasing, $q_i^*(v)$ is non-decreasing in v_i. From (16) we know that this is a sufficient condition for inequality (a2). Finally, since $\sum_{i=1}^{n} K(v_i)A(q_i^*(v) - B(q_i^*(v))$ is the solution to a maximization problem and $K(v_i)$ is continuous it follows that $\hat{H}(v)$, defined by (a7) is everywhere continuous. We have therefore obtained a vector of allocation functions $q^*(v)$ which satisfy all the necessary conditions.

Just as in the proof of Proposition 1 we can make use of equations (10) and (11) to establish that the expected tariff paid by buyers must be of the form

$$T_i^*(v_i) = \sum_{v_{-i}} \{v_i A(q_i(v)) - B(q_i(v)) - \int_0^{v_i} A(q_i(x, v_{-i})dx\}$$

Comparing this with (##) it follows immediately that the latter defines the optimal tariff function.

It remains to establish that $q^*(v)$ yields the global optimum. From (14), the expected revenue from any feasible allocation rule is

$$\text{(a10)}\quad \sum_{i=1}^{n}\{\int_0^1 J(v_i)G_i(v_i)dF(v_i) - \underset{v}{E}\, B(q_i(v)) - \Pi_i(0,0)\}$$

where $G_i(v_i)$ is a non-decreasing function.

Below we shall show that this expression is no greater than

$$\text{(a11)}\quad \sum_{i=1}^{n}\{\int_0^1 K(v_i)G_i(v_i)dF(v_i) - \underset{v}{E}\, B(q_i(v)) - \Pi_i(0,0)\}$$

It is evident that $q_i^*(v)$, the solution to (#), satisfies the necessary conditions for maximizing (a11) as well as (a10). Furthermore, for $v_i < x^*$ so that $K(v_i) \leq 0$, $q_i^*(v) = 0$ maximizes the integrand in (a11). Finally the integrand is a concave function of q_i for $v_i > x^*$. Then the necessary conditions indeed define the global optimum for (a11) and hence for (a10).

It remains to confirm that (a10) is less than or equal to (a11). From the definition of $K(x)$ the second integral minus the first is

$$\text{(a12)}\quad \sum_{j=1}^{n}\sum_{i=1}^{n}\int_{x^{2j-1}}^{x^{2j}} [J(x^{2j}) - J(x)]G_i(x)dF(x)$$

Let $y^1, \ldots, y^c$ be the points of non-differentiability of $G_i(x) = \underset{v_{-i}}{E}\; A(q_i(x))$ and define $y^0 = x^{2j-1}$, $y^{c+1} = x^{2j}$. Then, integrating by parts, the integral in (a12) can be rewritten as

$$\text{(a13)} \quad \sum_{k=1}^{c} \{ \int_{y^k}^{y^{k+1}} [J(x^{2j-1}) - J(x)] G_i(x) dF(x) =$$

$$\sum_{k=1}^{c} [G_i(y_-^k) - G_i(y_+^k)] \int_{x^{2j-1}}^{y^k} [J(x^{2j-1}) - J(x)] dF(x)$$

$$- \sum_{k=1}^{c} \int_{y^k}^{y^{k+1}} G_i'(x) \int_{x^{2j-1}}^{x} [J(x^{2j-1}) - J(z)] dF(z)$$

Since $G_i(x)$ is a non-decreasing function it follows from condition (B) that the right-hand side of (a13) is non-negative.

Q.E.D.