SPECULATION, ARBITRAGE, AND THE TERM STRUCTURE OF FOREIGN EXCHANGE RATES

by

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§1. Introduction

This paper presents theoretical explanations for discrepancies between current forward Foreign Exchange (FX) rates and current market expectations of spot FX rates for corresponding future dates. Such discrepancies are generally referred to as "risk premiums," the assumption being that they must arise from risk characteristics somehow inherent in FX rates. The basic argument seems unassailable: these discrepancies represent opportunities for speculative profits, which in equilibrium should be commensurate with the speculative risks.

Nevertheless, a consensus on the origin and determinants of these discrepancies has not yet emerged in either the theoretical or empirical literature. Most recent theoretical studies have employed some sort of portfolio approach, defining risk in terms of certain covariances. For example, Stulz (1980) relates risk premiums to covariances of exchange rates with world real consumption; Solnik (1974) to covariance of (real) exchange rates with the world portfolio of assets, and thence to net foreign investment; and Grauer, Litzenberger, and Stehle (1976) to covariances of exchange rates with gross world product, and thence to nominal "outside" assets. Older theoretical studies following Hicks (1946) and Keynes (1930), tended to explain risk premiums as insurance which hedgers pay speculators.

In order to test the various theories empirically, one must somehow identify the market expectation of future spot rate. The usual approach has been to regard the forward rate, adjusted perhaps by some appropriate risk premium, as the expectation or "forecast" held by all market participants. With this notion of market expectations, one can test theories of the risk premium by comparing the ex-post forecast errors -- that is, differences between adjusted forward rates and subsequent spot rates -- under various specifications of the risk premium adjustment. Unfortunately, the variance of the unadjusted forecast errors generally swamps the variance of reasonable specifications of the risk premium, so tests of this sort are usually inconclusive. See Dooley and Isard (1979) and Bilson (1980) for recent discussions of this problem, and Kohlhagen (1978) for a broader survey. Indeed, the variance of spot rates is so large that it is not even clear that the unadjusted forward rate is a better predictor of subsequent spot rates than is the current spot rate (Hsieh (1981) provides evidence that it may be worse!). On the other hand, Levich (1978) among others, finds that certain forecasting models can out-perform the forward rate, and presumably even a "risk-adjusted" forward rate.

Given this unsatisfactory state of affairs, it seems appropriate to take a closer look at the market processes which determine spot and forward FX rates. In the sections which follow, I introduce a rather explicit micro model of FX rate determination, emphasizing the arbitrage possibilities arising from the fact that currencies are storable goods and the role of possibly heterogeneous expectations. I classify the roles of FX market participants in into three types: traders (i.e., exporters and importers), speculators and arbitrageurs. I consider the supply and demand for FX contracts for a range of maturities and market dates. I restrict attention to a single foreign currency (denoted £) traded against home currency (parochically denoted \$). The approach is in some ways reminiscent of Tsiang (1959) and McCormick (1977).

The framework allows one to define market expectations in terms of speculators' probability distributions, and to address issues involving the "thickness" of markets. The principal conclusion of the analysis is that there may well be discrepancies between forward rates and corresponding expected future spot rates which cannot properly be described as "risk premiums." Consequently, there is additional reason to believe that forward rates, though perhaps not consistently biased, may not be optimal predictors of future spot rates.

Despite the fact that only a single bilateral FX market is examined, the consideration of many agents, dates and contract maturities creates a substantial burden of notation and exposition. The organization of the paper is intended mainly to ease this burden as much as possible. In Section 2, I briefly introduce traders and discuss the determination of equilibrium spot FX rates in the absence of other types of agents. The following section first discusses some general considerations regarding speculation, and introduces a simple parametric specification of speculator behavior. Then equilibrium spot and forward FX rates are computed (Proposition 1) for the parametric example, under the assumption that hedging and arbitrage are absent; it turns out that "discrepancies" are also absent in this case. Section 4 enriches the role of the speculator by allowing for the possibility of hedging. The equilibrium forward rates now may contain risk premiums of the Keynes-Hicks sort, as shown in Proposition 2.

The role of arbitrage is introduced in the following section. After a brief discussion of covered interest arbitrage and interest rate parity, I argue that the entire term structure (spot and all forward) FX rates are jointly determined in the presence of arbitrage, and compute the equilibrium values for the parametric specifications of trade and speculation in Proposition 3. The equations incorporate a new type of discrepancy between forward and expected future spot rates which I refer to as the "blurring effect" of arbitrage. (Roughly speaking, this discrepancy arises if the market expression of speculators' expectations is more "yielding" than the term structure of interest rates enforced by covered interest arbitrage.) I close the section with a supply and demand diagram of the determination of the FX rate term structure. The following section illustrates the compatibility of the model with "Rational Expectations" requirements by means of two numerical examples.

The final section briefly discusses some limitations of the analysis and its implications for empirical work. The basic definitions and assumptions are collected and presented more formally in the first part of the Appendix, principally as an aid to the reader who loses track of the notation. Some technical notes and computations are collected at the end of the Appendix.

§2. Trading

Let us first consider agents who participate in foreign exchange markets only to facilitate international trade. We will not consider their behavior in detail; for the purposes of this paper it suffices to specify how their aggregate net demand for foreign exchange (Ⱡ) depends on the spot exchange rate p(t) (measured in \$/Ⱡ), and other factors e(t) external to the FX market. A rather general specification is

$$\text{(T1)} \qquad \sum_{k\in T} G_k^*(t) = z(p(t),e(t)),$$

where $G_k^*(t)$ represents agent k's current desired spot contract to buy Ⱡ (negative if k contracts to sell Ⱡ for \$), and T is the set of all agents classified as traders. For the next two sections, we assume that these agents do not make use of forward markets.

A simple parametric example of an aggregate trader excess demand function is

$$\text{(T2)} \qquad z(p(t),e(t)) = (\bar{p}-p(t))d + e(t),$$

i.e., linear with additive "shocks." The parameter $\bar{p}$ can be thought of as a "normal" spot rate, which would equate supply and demand for foreign exchange by traders in the absence of "shocks." We will assume the parameter d is positive, since ceteris paribus (i.e., e held constant), traders presumably would demand fewer (more) Ⱡ to buy foreign goods and more (fewer) \$ to buy U.S. goods if p(t) is high (low).[1] We make no special assumptions about the distribution of e.

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[1] See McKinnon [1980, p. 151] for the contrary position that d may be negative at least in the very short run.

Our main focus in the next several sections will be the determination of equilibrium exchange rates as new types of agents are introduced into the FX market. Equilibrium will always mean that desired contracts sum to zero. With only traders present, the equilibrium price at time t will be the solution to $z(p,e(t)) = 0$; in particular, we have

(T3) $$p(t) = \bar{p} + e(t)/d,$$

as the equilibrium spot price under (T2). This simplest case will be a handy benchmark for later computations.

§3. Speculation

We will be quite interested in the behavior of risk-averse agents who take uncovered positions in the forward FX markets with the intention of profitably reversing these positions at a later date. We assume that each such speculator i is able to contract at each date t to accept delivery at any maturity date s, $s \geq t$, of any quantity $G_i(t,s)$ of £. Of course, if $G_i(t,s) < 0$, i is contracting to make delivery of £ (and accept delivery of \$), and spot contracts $G_i(t,t) \equiv G_i(t)$ are also permitted. The \$ price of £ in a forward contract (*i.e.*, the forward FX rate) is denoted p(t,s). Thus, if speculator i holds his contract to maturity and settles at the then prevailing spot rate p(s), his \$ profit or loss will be $G_i(t,s)[p(s)-p(t,s)]$.

It will often be convenient to refer to a speculator's *net position* as of date t for a particular maturity s, $g_i(t,s) = \sum_{u \leq t} G_i(u,s)$. That is, i's net position $g_i(t,s)$ is the sum of all contracts made at past dates $u \leq t$ which mature on the given date s.

Presumably speculators chose their contracts $G_i(t,s)$ in order to maximize profits in some sense. They all face the same forward rates p(t,s), but may have different risk preferences, information, financial resources, and previously established forward positions. Let $A_i(t) = (U_i, I_i(t), M_i(t), g_i(t-1,\cdot))$ represent these agent-specific features for the ith speculator. The basic assumption will be that speculators have some decision rule for selecting their desired forward contracts $G_i^*(t,\cdot) = \{G_i^*(t,s),\ s \geq t\}$ based on $A_i(t)$ and current FX rates $\{p(t,s),\ s \geq t\}$, *viz*:

(S1) $G_i^*(t,\cdot) = H(p(t,\cdot), A_i(t))$, for each $i \in \mathcal{S}$.

S represents the set of all speculators; (S1) asserts that each speculator at each date t decides jointly on desired contracts for all maturities, given the current term structure of FX rates p(t,•) and his personal situation $A_i(t)$.

Economists generally regard such decisions as arising from a two step process: Sepculators first use their information to form (personal, subjective) probability distributions of FX rates which will obtain in future periods. Then they chose their desired contracts so as to maximize expected utility of profits, expectations taken with respect to these "forecast" probability distributions.

The first step of the process is a bit tricky: all speculators have the current forward FX rates p(t,•) as part of their information set, and they will, under some circumstances, be able to use this public information to make important inferences regarding others' private information. For the most part we will not delve into these matters; we will take the forecasting process as given, and our results will not depend on whether the forecasts are "rational" (in the sense of Tirole (1980), for example), or relatively naive.

It will again be helpful to have a parametric example, at least of the second step of the speculator's decision process. Let $E_{it}p(s)$ be the mean of speculator i's forecast of the date s spot rate, the forecast using in some fashion the information available to him at date t. Suppose he enters into contracts so as to make his new <u>net position</u> proportional to the expected unit profitability $E_{it}p(s) - p(t,s)$, <u>viz</u>:

(S2) $g_i^*(t,s) = a_{its}(E_{it}p(s) - p(t,s)),$

$a_{its} = h(p(t,s),A_i(t) > 0,$ for each $i \in S$, $s > t$.

In most versions of (S2), the positive coefficient a_{its} depends mainly on risk aversion and the forecast variance of p(s) (cf. Muth (1961, p. 324)). In the Appendix we derive such a specification from a simple expected-utility-of-\$-profits maximization problem, for which

$$\text{(S2a)} \quad a_{its} = \frac{R_i(t,s)}{\mathrm{Var}_{it}\ p(s) + (E_{it}p(s) - p(t,s))^2} \simeq R_i(t,s)/\mathrm{Var}_{it}\ p(s),$$

where $1/R_i$ is i's absolute risk aversion evaluated at the \$ market value of his inherited net position, and Var_{it} is the conditional variance operator, i.e., the imprecision of i's forecast. The Appendix also shows that it matters little whether speculators of this sort use \$ or £ to denominate their profits.

In any case, (S2) implies that, given their inherited positions $g_i(t-1,s)$, speculators chose their forward contracts $G_i^*(t,s)$ so as to attain their desired net positions:

$$\text{(S3)} \quad G_i^*(t,s) = g_i^*(t,s) - g_i(t-1,s), \quad i \in S, \quad s > t.$$

As part of the definition of speculation (and as also suggested by (S2)), we will assume that speculators desire zero net spot positions, hence

$$\text{(S4)} \quad G_i^*(t) = -g_i(t-1,t), \quad i \in S.$$

Thus (S4) tell us that aggregate net spot speculative activity is $-g(t-1,t) \equiv -\sum_{i\in S} g_i(t-1,t) = \sum_{i \in S} G_i^*(t)$.

Proposition 1: If speculators chose their contracts according to (S2), traders according to (T2), and no other types of agents are present, then for each date t and maturity $s > t$, equilibrium spot and forward rates are:

(1S) $p(t) = \bar{p} + (e(t)-g(t-1,t))/d$, and

(1F) $p(t,s) = \sum_{i\varepsilon S} w_{its}E_{it}p(s)$, where the weights w_{its} satisfy $0 \leq w_{its} \leq 1$ and $\sum_{i\varepsilon S} w_{its} = 1$.

The proof is straightforward. The equilibrium spot price will satisfy the equation

$$0 = \sum_{i\varepsilon T} G_i^*(t) + \sum_{i\varepsilon S} G_i^*(t) = (\bar{p}-p(t))d + e(t) - g(t-1,t);$$

the solution is clearly (1S). The equilibrium forward price likewise will satisfy, for $s > t$:

$$0 = \sum_{i\varepsilon T\cup S} G_i^*(t,s) = \sum_{i\varepsilon S} G_i^*(t,s) = \sum_{i\varepsilon S} g_i^*(t,s) - \sum_{i\varepsilon S} g_i(t-1,s).$$

The last term is zero (because $\sum_{i\varepsilon S} G_i(u,s) = 0$ for all $u < t$, and $g_i(t-1,s) = \sum_{u<t} G_i(u,s)$), so the equilibrium condition by (S2) reduces to:

$$0 = \sum_{i\varepsilon S} g_i^*(t,s) = \sum_{i\varepsilon S} a_{its}(E_{it}p(s) - p(t,s)).$$

Define $w_{ts} = \sum_{i\varepsilon S} a_{its}$ and $w_{its} = a_{its}/w_{ts}$, and solve the last equation to obtain (1F).

The implications of Proposition 1 are very much in accord with common understanding of the effects of speculation. To the extent that shock e(t) has been anticipated by speculators and reflected in their contracts $G_i(u,t)$,

$u < t$, its effect on the spot rate will be offset; thus (1S) is the same as (T3) with $e(t)$ replaced by $e(t) - g(t-1,t)$. Forward rates are determined entirely be speculators; in fact (1F) asserts that each forward rate is a certain weighted average of speculators' current expectations of the future spot rates, and therefore may be regarded as the "market expectation."

The weight of each speculator reflects his willingness to speculate (a_{its}) relative to the aggregate willingness (w_{ts}). Under assumption (S2a) we can be even more specific: each agent's willingness to speculate is the product of his risk acceptance R_i and the precision of his forecast $1/Var_{it}$. In general, a speculator's weight will vary according to the maturity s, principally because his forecast precision may drop off with increasing s at a different rate than his fellow speculators. For instance, a speculator with a relatively large R and rapid decline with s in $Var_{it}^{-1}p(s)$ can be regarded as a "near-term specialist": his influence in terms of market share and price impact will be felt primarily in near term (smaller s) forward markets.

The proof of (1F) also contains an explanation of the well-known empirical phenomenon that forward markets are thinner for more distant maturities. If one measures the thickness of a market by either $\frac{1}{2}\sum_i |g_i|$ = contracts outstanding, or $\frac{1}{2}\sum_i |G_i|$ = new contracts, and if (2Sa) applies with $Var_{it}p(s)$ an increasing function of s, while $R_i(t,s)$ is relatively constant, we obtain markets which thin. The rate of thinning will be exponential ("radioactive decay") if $Var_{it}p(s)$ increases exponentially, as would be the case if speculators have lognormal diffusion models of $p(s)$. The Appendix contains further remarks on this subject.

The explicit formulas of Proposition 1 arise, of course, from the simple parametric form of (S2) and (T2). Under the more general assumptions (S1) and (T1) one cannot be so explicit, but it should be clear that qualitatively the same forces are at work.

§4. Hedging

The definition of speculation advanced in the previous section allows for the possibility of hedging. Suppose that due perhaps to activities in international trade, at date t an agent i schedules for some quantity $X_i(t,s)$ of Ⱡ (possibly negative) to arrive at date $s > t$. In this case, $g_i(t,s)$ as defined previously in terms only of forward contracts does not accurately represent his net Ⱡ position. If $x_i(t,s) = \sum_{u \le t} X_i(u,s)$ represents i's net scheduled Ⱡ inflow for date s as of date t due to activities outside the FX markets, then his Ⱡ position is actually $\hat{g}_i(t,s) = g_i(t,s) + x_i(t,s)$. Specification (S2) now should read

$$\text{(SH2)} \quad \hat{g}_i^*(t,s) = \hat{a}_{its}(E_{it}p(s) - p(t,s)),$$

where $\hat{a}$ differs from a only in that the speculator's "financial resources" now include $x_i(t,s)$. In (S2a), for instance, the only change is that R_i must be evaluated at the market value of $\hat{g}$ rather than g. See the Appendix for details.

Although the speculator's desired net position may not be much affected, his desired forward contracts $\hat{G}^*$ will be. Assuming that differences between a and $\hat{a}$ are small, we have

$$\hat{G}_i^*(t,s) \equiv \hat{g}_i^*(t,s) - \hat{g}_i(t-1,s) \simeq g_i^*(t,s) + x_i(t,s) - g_i(t-1,s) - x_i(t-1,s)$$

$$= G_i^*(t,s) - X_i(t,s).$$

In words, a speculator will first offset the newly scheduled Ⱡ payment (hedge 100%) and then choose his speculative contracts essentially as before. See Danthine (1978, pp. 82-3) for an analogous result.

What happens when these scheduled payments finally arrive? By date t, agent i's scheduled b current inflows are $x_i(t) \equiv x_i(t-1,t) = \sum_{u<t} X_i(u,t)$, and the aggregate current inflow is $x(t) \equiv \sum_{i\varepsilon S} x_i(t)$. Since $x(t)$ doesn't respond to $p(t)$, it may be lumped together with the "shock" $e(t)$. Therefore it makes sense to define $\hat{e}(t) = e(t) - x(t)$ as the "unhedgable shock." The spot rate will then respond to $\hat{e}(t)$ rather than $e(t)$ because the hedgable inflows $x(t)$ will be offset by the corresponding maturing forward contracts.

How about forward rates? We have argued that the "pure speculative demands" $G_i^*(t,s)$ will be offset by "hedge demands" $X_i(t,s)$, the aggregate of which is $\sum_{i\varepsilon S} X_i(t,s) \equiv X(t,s)$. Incorporation of these modifications into the argument for Proposition 1 yields

<u>Proposition 2</u>: If speculator/hedgers chose their contracts according to (SH2), traders according to (T2), and no other agents are present, then for each date t and maturity $s > t$, equilibrium exchange rates are:

(2S) $\quad p(t) = \bar{p} + (\hat{e}(t) - g(t-1,t))/d$, and

(2F) $\quad p(t,s) = \sum_{i\varepsilon S} \hat{w}_{its} E_{it} p(s) - X(t,s)/\hat{w}_{ts}$.

The weights $\hat{w}$ are defined from the $\hat{a}$'s exactly as the w's of Proposition 1 were defined from the a's.

Equation (2S) suggests that the effect of the "shock" $e(t)$ on the spot price may be offset in either of two ways. To the extent that it is actually scheduled in advance ($x(t)$), it can be completely offset by hedging. To the extent that it has been merely anticipated by speculators given their information $I_i(u)$, they will, according to (S2), estimate its likely impact in forming $E_{iu}p(t)$ and take positions based on their willingness to assume risk and on anticipated profits, probably resulting in only a partial offset $g(t-1,t)$ of the unhedgable portion $\hat{e}(t)$. Of course, in practice it will be difficult or impossible to distinguish aggregate hedging from aggregate speculation.

Equation (2F) suggests a first reason why a forward rate may be a biased predictor of the corresponding future spot rate: the "market expectation" $\Sigma\hat{w}_i E_i$ differs from the forward rate by a "risk premium" $-X/\hat{w}$, as in the Keynes-Hicks theory of speculation. In (2F), the magnitude of the bias depends directly on the magnitude of aggregate hedging, and inversely on the aggregate willingness to speculate. The direction of the bias is of opposite sign to aggregate hedging; e.g., if aggregate £ net inflows scheduled on date t for date s are positive (so hedgers are net forward sellers of £), then the forward rate will exceed the "market expectation" (so speculators can expect to profit by buying £ forward).

The approach to hedging adopted here basically regards it as a subspecies of speculation associated with agents who are also traders. If one wanted to distinguish sharply between hedging and speculation, one could define a "pure" hedger as an agent who has the role of trader as well as that of a speculator, but who is unwilling to speculate ($a_{its} = 0$, all t,s). Likewise, a "pure" speculator would be one for whom $X_i(t,s) = 0$, all t,s. Such distinctions would have no effect on our analysis, so we will not pursue them any further. Henceforth, when we mention speculation, we generally mean to include hedging activities.

§5. Arbitrage

Although we have not yet explicitly acknowledged their existence, a third set of agents has already been at work. Arbitrageurs try to take advantage of price discrepancies to make riskless profits; their activities tend to unify prices. Our basic assumption that all speculators can buy and sell £ forward at the same price $p(t,s)$, for instance, is tenable to the extent that arbitrage is effective. We will take this sort of arbitrage for granted here (i.e., ignore bid-ask spreads) and concentrate instead on an intertemporal form of arbitrage made possible by the fact that currencies are storable goods. Henceforth, "arbitrage" will refer specifically to this latter type (properly known as "covered interest arbitrage"), which unifies the term structure of forward FX rates with the term structure of \$ and £ interest rates.

In order to spell this out, let $r_{\$}(t,s)$ be the nominal Eurodollar interest rate ("LIBOR") prevailing at date t for deposits or loans maturing at date s (again ignoring bid-ask spreads), expressed as a continuously compound per period rate. Similarly, let $r_{£}(t,s)$ represent the corresponding Eurosterling rate, and let $r(t,s) = r_{\$}(t,s) - r_{£}(t,s)$ represent the interest rate differential.[1] Then, the term structures are unified by:

$$\text{(A1)} \qquad p(t,s) = p(t)e^{(s-t)r(t,s)}, \qquad \text{for all } t \text{ and } s \geq t.$$

The interest rate parity relationship (A1) will <u>always</u> hold under the assumptions: (1) bid-ask spreads and default risks are negligible, (2) arbitrageurs can borrow and lend unlimited quantities of \$ and £ at the (possibly endogenous) rates $r_{\$}$ and $r_{£}$, and (3) arbitrageurs can buy and sell £ for \$ spot and forward in unlimited quantities at the (possibly endogenous) rates $p(t)$ and $p(t,s)$.

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[1] This may be thought of as the "storage cost" of £ (relative to \$).

The well-known argument runs as follows. Suppose $p(t,s) - p(t)e^{(s-t)r} = \varepsilon > 0$. Then an arbitrageur j can simultaneously make the following transactions:

(1) Sell forward a quantity $|G_j(t,s)|$ of £ (causing downward pressure on $p(t,s)$);

(2) Lend the quantity $D_£(t,s) = -G_j(t,s)e^{-(s-t)r_£}$ of £ to mature at s, yielding $G_j(t,s)$ at maturity (causing downward pressure on $r_£(t,s)$;

(3) Buy spot the quantity $D_£(t,s)$ of £ (causing upward pressure on $p(t)$; and

(4) Borrow the quantity $D_\$(t,s) = -p(t)\, D_£(t,s)$ of dollars to mature at date s (causing upward pressure of $r_\$(t,s)$). One can readily verify that the arbitrageur has "covered" his forward contract (1) by the borrow/spot/lend contracts (2) - (4). Specifically, the net flow of £ for these transactions is zero at both dates, while the net flow of \$ is zero at date t and is $p(t,s)|G_j(t,s)| - D_\$(t,s)e^{(s-t)r_\$} = \varepsilon|G_j(t,s)|$ at date s. Thus, our arbitrageur will acquire a riskless \$ profit which can be made arbitrarily large as long as $\varepsilon > 0$. However, the larger G_j he chooses, the stronger the indicated pressures, all of which tend to drive ε to 0, <u>i.e.</u>, cause (A1) to hold. The case $\varepsilon < 0$ is exactly similar with "buy" and "sell," and "borrow" and "lend" interchanged. A more careful version of this argument would reveal that $|\varepsilon|$ can't exceed the sum of the unit transactions costs (bid-ask spreads) on spot FX, Eurodollar and Eurosterling deposits.

As an empirical matter, (A1) is an excellent approximation; see Aliber (1973). There is a simple institutional reason: commercial banks are the primary market makers in both forward FX and Eurocurrency deposits, and they price forward FX off LIBOR differentials (McKinnon (1980), pp. 216-7).

The extent that (A1) holds with respect to domestic interest rates is another matter which we will discuss briefly in Section 7.

We characterize arbitrage by the assumption that forward transactions are always covered and hence riskless. The paradigmatic scheme is covering a given forward contract by the borrow/spot/lend contracts specified above. Alternatively, the covering could be a bit more complicated, involving forward contracts for different maturities together with appropriate \$ and Ł borrowing and lending. In any case, one can readily verify that arbitrageurs' net forward transactions $G_j(t,\cdot)$ will be riskless if and only if they are supported by appropriate borrowing and lending and satisfy:

$$\text{(A2)} \qquad \sum_{s \geq t} e^{-(s-t)r_Ł} G_j(t,s) = 0, \qquad \text{for all } j \in \mathcal{A},$$

where $\mathcal{A}$ is the set of arbitrageurs. In words, the (Ł) discounted present value of each arbitrageur's FX contracts (j's "open position") must be zero. The careful reader may observe that (A1) and (A2) together imply that arbitrage, while riskless, is also profitless (see Appendix). This is, of course, due to our neglect of bid-ask spreads; presumably arbitrageurs do earn a normal return in the real world.

For present purposes, the key observation is that, in the presence of arbitrage, spot and forward prices can't be determined separately. Specifically, we no longer may assume that net speculative demand is zero for any particular maturity; it may be offset by net arbitrage activity, which is constrained by (A2). The upshot is contained in the next proposition, which employs the following special notation:

$$\mathfrak{g}(t) = \sum_{i \in \mathcal{S}} \sum_{s > t} e^{-(s-t)r_Ł} \mathfrak{g}_i(t-1,s) = \text{(Ł)present value of aggregate old speculative positions;}$$

$$W_t = \sum_{i\varepsilon S} \sum_{s>t} e^{-(s-t)(r-r_b)} a_{its} = \text{a discounted value of aggregate willingness to speculate}$$

Proposition 3: If speculators chose their contracts according to (SH2), traders according to (T2), and arbitrageurs obey (A2) while enforcing (A1), then for each date t and maturity s, equilibrium exchange rates are:

(3S) $p(t) = \bar{p}(\frac{d}{d+W_t}) + (\hat{e}(t)-\hat{g}(t))/(d+W_t) + \sum_{i\varepsilon S} \hat{W}_{it}\widehat{E_{it}p}$, and

(3F) $p(t,s) = p(t)e^{(s-t)r(s,t)}$.

Note: The "weights" $\hat{W}_{it}$, defined below, are non-negative, but do not in general sum to unity. The "discounted average expected FX rates" $\widehat{E_{it}p}$ are also defined below.

Proof: The forward rate equation (3F) is just assumption (A1), so only (3S) must be established. The equilibrium condition is that desired spot contracts sum to zero, i.e.,

$$0 = \sum_{k\varepsilon T} G_k^*(t) + \sum_{j\varepsilon A} G_j^*(t) + \sum_{i\varepsilon S} G_i^*(t).$$

The first term is, of course, $(\bar{p}-p(t))d + e(t)$ by (T2). (A2) implies that $G_j^*(t) = -\sum_{s>t} e^{-(s-t)r_b} G_j^*(t,s)$ for each $j \varepsilon A$. But all forward contracts must have counterparties, so for each t and s > t, $\sum_{j\varepsilon A} G_j(t,s) = -\sum_{i\varepsilon S} G_i(t,s)$. These last two equations tell us that the second term above is

$$\sum_{i\varepsilon S} \sum_{s>t} e^{-(s-t)r_b} G_i^*(t,s) = \sum_{i\varepsilon S} \sum_{s>t} e^{-(s-t)r_b} \{a_{its}(e_{it}p(s) - p(t)e^{(s-t)r})$$

$$- g_i(t-1,s) - X_i(t,s)\},$$

where we have used (SH2) and (A1) for the last equation.

Finally, the third term is $-\sum_{i\varepsilon S} \hat{g}_i(t-1,t)$, as noted in the proof of Proposition 1. Putting these expressions together and rearranging slightly, we obtain from the equilibrium condition

$$(3^*) \quad 0 = d(\bar{p}-p(t)) + (\hat{e}(t) - \hat{g}(t)) + \sum_{i\varepsilon S} \sum_{s>t} \hat{a}_{its}(E_{it}p(s)e^{-(s-t)r} - p(t)),$$

where $\hat{a}_{its} = a_{its}e^{(s-t)(r-r_b)}$. Set $W_{it} = \sum_{s>t} a_{its}e^{-(s-t)r_b}$, and define

$$\widehat{E_{it}p} = \sum_{s>t} a_{its}e^{-(s-t)r_b} E_{it}p(s)/W_{it}.$$ Then the equilibrium condition yields

$$dp(t) + W_t p(t) = d\bar{p} + (\hat{e}(t) - \hat{g}(t)) + \sum_{i\varepsilon S} W_{it}\widehat{E_{it}p}.$$ Defining $\hat{W}_{it} = W_{it}/(d+W_t)$ and solving yields (3S).

The thrust of this proposition is that, because of arbitrage, the spot rate is affected by <u>all</u> current speculation, and therefore any new information which prompts revisions in speculators' expectations of future spot prices $E_{it}p(s)$ will cause corresponding changes in the current spot rate. In fact, by (3F), the entire term structure of exchange rates will shift in response to new information. The strength of the response depends on the "willingness to speculate" parameters a_{its}, attenuated by the time horizon (s-t) for the information via the discount factors. Equation (3S) expresses these responses in terms of the individual speculators' influences: each will tend to move the spot rate towards his "discounted average" expectation $\widehat{E_{it}p}$ (so his expectations concerning a particular p(s) matter only insofar as they affect this average); i's "opinion" $\widehat{E_{it}p}$ will be incorporated by the market to the extent his discounted willingness to speculate W_{it} is significant relative to the aggregate willingness W_t, augmented by traders' aggregate responsiveness d to spot prices.

To the extent that aggregate current willingness to speculate W_t is small (and past speculation g(t) has also been small), (3S) approximates (2S). To the extent these factors are large relative to d, the spot rate (indeed, the entire term structure) is a creature of market expectations. In either case, though, we have no analogue of equation (1F); specific forward rates according to (3F) apparently are not tied any more closely to expectations concerning the corresponding future spot rates than they are to expectations concerning spot rates which will prevail at other dates. Thus we have a second, and new, explanation of why a forward rate might not be a particularly good predictor of its corresponding future spot rate: arbitrage "blurs" the market impact of expectations across maturities.

Given the relative thickness of near-term forward markets, it follows from arguments of this section that the beliefs of speculators regarding near-term developments are more important than "long-run" beliefs in determining the current level of the FX rate term structure. Thus, for example, the one-month forward rate should be a relatively good indicator of the "market expectation" of the spot rate one month hence, as compared to the six-month forward rate as an indicator of the "expected" spot rate six months hence.

A technical point should be mentioned. In defining W_t and elsewhere, it was implicitly assumed that certain sums over all forward dates s converged. This assumption is justified if one either assumes that forward markets will not exist beyond some arbitrary horizon h (so $(s-t) \leq h$), or else makes assumptions which ensure asymptotic convergence. The latter type of assumption is discussed further in the Appendix.

We have not included in Proposition 3 the usual proviso concerning the absence of other types of agents, because our classification system is now complete. Any agent, I claim, can be decomposed into the three roles. For instance, an "investor" who buys £ assets spot with the intention of reselling can be regarded as a speculator-cum-arbitrageur, who, qua speculator, buys the £ forward from himself (qua arbitrageur), and qua arbitrageur, borrows \$ (the opportunity cost of the £ assets), buys £ spot and £ assets and sells £ forward (to himself, qua speculator). The decomposition is discussed further in the Appendix. It should be admitted that it is a bit Procrustean when applied to Government agencies.

Much of the argument on the determination of the term structure of FX rates is summarized diagrammatically in Figure 1. On the vertical axis is the spot rate $p(t)$; given the interest rates $r_{\$}$ and $r_{£}$, all forward rates can be inferred from $p(t)$ by (A1), so they need not be depicted. The horizontal axis measures net quantities of £ "supplied" and "demanded." The z curves are related to excess trading demand for £; $z(\cdot,0)$ is the graph of (T2) for the "unshocked" case $e = 0$, $z(\cdot,\bar{e})$ is the same for a given shock $\bar{e}$, and $z(\cdot,\hat{e})$ is the graph of $z(p,\bar{e}) - x(t) - g(t-1,t)$, the current excess trading demand net of hedging and previous speculation. The S curves represent the spot supply of £ due to current "pure" speculation in forward FX, transmitted to the spot market by arbitrage. Specifically, S and $\bar{E}$ are defined by $S(p,\bar{E}) = -W_t(\bar{E}_t - p(t)) = -\sum_{i\in S}\sum_{s>t}\hat{a}_{its}(e^{-(s-t)r_£}E_{it}p(s) - p(t))$. Thus z incorporates the first two terms of equation (3S) and S the last term so that their intersection (p,Q) represents the equilibrium spot rate p and net amount of arbitrage-mediated speculation Q.

For a given shock $\bar{e}$, the equilibrium spot rate would be p_0 if there were no hedging or speculation, but (by Proposition 2) would be p_2 had there been speculation (including hedging) at previous dates. If one also takes into account current speculation (and hedging) oriented towards future dates, the outcome depends on the "market average expectation" $\bar{E}$ (as well as aggregate willingness to speculate, which determines the slope of S). If $\bar{E} = \bar{p}$, then in the case illustrated, the equilibrium spot rate is reduced to p_1, but if speculators expect some higher average $\bar{E}$, then the spot rate is raised to p_3. The corresponding speculative supplies are $Q_1 > 0$ (*i.e.*, speculators are in some average sense, "short" Ł) and $Q_3 < 0$ ("long" Ł, implying negative supply of Ł).

Figure 1: Determination of FX Rates

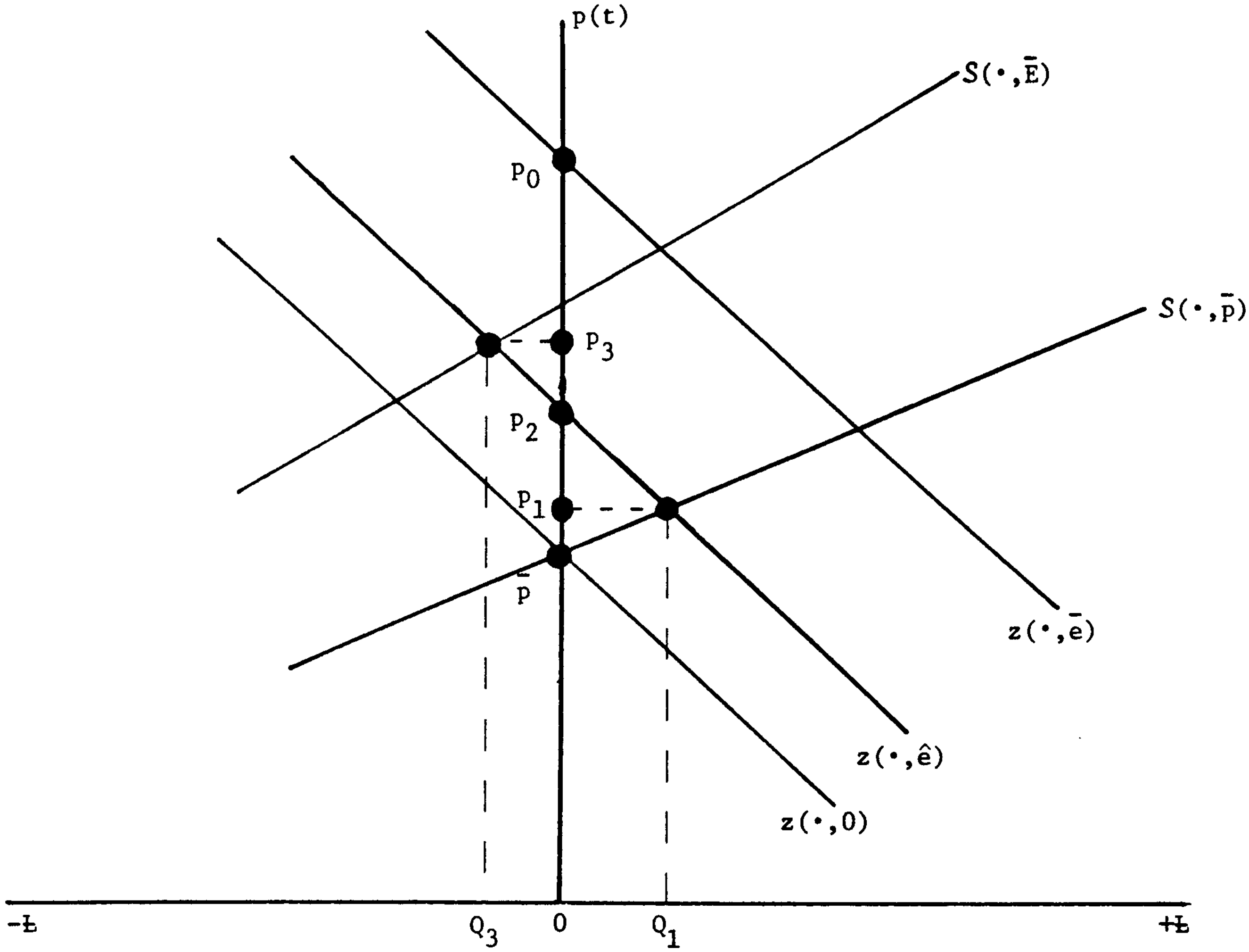

Figure 1

§6. Numerical Examples

Consider the following simple numerical example. There are only two market dates, today ($t = 0$) and tomorrow ($s = 1$). Traders obey (T2), with $e(0) = 0$ for certain, and $e(1) = 0$ or 1 with equal probability. There is only one speculator, who forms his expectations "rationally," knows the distribution of the shock e, and follows (S2) with $a_{ts} = R/\mathrm{Var}_t p(s)$, with constant risk acceptance parameter R. An arbitrageur stands ready to accommodate the speculator, and interest rate differentials are zero, so $p(0) = p(0,1) \equiv p$. There are no inherited positions or scheduled (hedgable) cash flows.

Under these assumptions, (3*) implies $p_0 = \bar{p} + G/d$, $p_{10} = \bar{p} - G/d$, and $p_{11} = \bar{p} + (1-G)/d$, where $G = G(0, 1)$ is the speculator's contract, and p_{10} and p_{11} denote tomorrow's spot prices for the realizations $e(1) = 0$ and $e(1) = 1$, respectively. Clearly $E_0 p_1 = \bar{p} + (\frac{1}{2} - G)/d$ and $\mathrm{Var}_0 p_1 = (\frac{1}{2}/d)^2 = \frac{1}{4}d^2$. Hence $G \equiv (E_0 p_1 - p_0)\cdot R/\mathrm{Var}_0 p_1 = ((\frac{1}{2} - 2G)/d)\cdot R\cdot 4d^2$, so $G = 2dR/(1+8dR)$. Consequently,

$$p_0 = \bar{p} + 2R/(1+8dR),$$

and $E_0 p_1 = \bar{p} + (\frac{1}{2}d + 2R)/(1+8dR)$.

Note that as "risk acceptance" $R \to \infty$, we have $G \to \frac{1}{4}$, and tomorrow's expected spot rate $E_0 p_1 \to p_0 = p(0,1)$, today's forward rate. For finite values of R, however, speculation only partially compensates for the expected "shock," and the forward rate is a biased predictor of tomorrow's spot rate.

It is more difficult to compute such "rational-expectations" equilibrium prices when there are several forward dates. In principle, one can solve for speculators positions and spot rates for further term dates contingent upon nearer-term realizations of e, p and G, and recursively work backward towards an explicit solution of the current spot rate. One finds that the computational burden increases fairly rapidly (e.g. the solution to a simple 3 date example requires finding the roots of a 6th degree polynomial).

Computation is much simpler when one takes the marginal probability distributions for future spot prices as given. For example, suppose that there are now two forward dates (s = 1, 2) and one speculator, who again has no hedgables or inherited positions and chooses his G's as above. Assume now that current his information leads him to believe that the possible realizations of p(1) are distributed normally (μ_1,σ_1) and those of p(2) are distributed $N(\mu_2,\sigma_2)$. If all interest rates are zero and there is no current period shock, (3S) and (3F) yield

$$p(0) = p(0,1) = p(0,2) = \left(\frac{d}{d+w}\right)\bar{p} + \left(\frac{w}{d+w}\right)E_p \text{, where}$$

$$w = a_{01} + a_{02} = R(\sigma_1^2 + \sigma_2^2)/\sigma_1^2\sigma_2^2 \text{ , and } Ep = \left(\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)\mu_1 + \left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)\mu_2 .$$

Thus, today's spot rate and forward rates are all weighted averages of expected future spot rates (μ_1 and μ_2) and today's "non-speculative" spot rate ($\bar{p}$), the relative weights being σ_1^{-2}, σ_2^{-2} and $\frac{d}{R}$, respectively. Of course, μ_1 and μ_2 depend on our speculator's plans, but as long as $\frac{d}{R}$ is not insignificant, it seems clear that there must be distributions of e(s) which "rationalize" given (μ,σ)'s. The main lesson to be drawn from this example is that, in the presence of term structure relations in forward rates induced by arbitrage, it doesn't make sense to regard any particular forward rate as an indicator of the corresponding expected spot rate. Rather, all expectations "blur" together and jointly influence the entire term structure.

§7. Discussion

The analysis presented above, in its context of a single floating FX rate, is limited to several respects. It does not examine the process by which FX market participants form their forecasts, thus sidestepping important issues raised by the "rational expectations" literature. For instance, one might wish to know under which circumstances the prices (p(t,•)) are "revealing," *i.e.*, they convey sufficient information for agents to achieve the same set of contracts they would form if they all had access to all private information (see Grossman (1979), for example). In the numerical examples just presented, prices are trivially revealing, but preliminary work suggests that in models of the sort presented here they generally will not be; in particular, the net arbitrageur positions (or, equivalently, arbitrageurs' net deposits and loans in $ and Ł) seem to constitute a crucial independent source of information.

In order to resolve such matters in a satisfactory manner in our framework, one would have to specify carefully what information is publicly available concerning future realizations of the "shock" e (*e.g.*, data affecting price levels, terms of trade, tariffs, . . .), and privately available (*e.g.*, corporate plans, inside information on domestic stabilization policies, . . .), as well as speculators' knowledge of how the markets work. Then one would have to define an appropriate notion of equilibrium between forecasts and information, taking into account the "feedback" of market generated data (p(t,•) and perhaps G(t,•)'s) into the information set. This task seems too ambitious at present.

Another important issue we have sidestepped is the formation of interest rate term structures in the two currencies. Clearly, some information relevant to FX markets (e.g., that affecting expected inflation) is also relevant to money markets. Given interest rate parity (A1), it is clear that FX rates $p(t,\cdot)$ and Eurocurrency rates $r_{\$}(t,\cdot)$ and $r_{£}(t,\cdot)$ are formed simultaneously. For completeness, one must also consider the linkages between domestic and Eurocurrency interest rates. In the absence of regulatory barriers (in particular, actual or anticipated capital controls), arbitrage between domestic and external money markets should lead to relatively small and predictable interest rate discrepancies between the two markets. (Incidentally, it would seem that the most natural way to study violations in interest rate parity with respect to domestic interest rates would be to focus on such regulatory barriers.) Of course, a serious account of domestic and Eurocurrency interest rates would again excessively complicate the model.

Although the analysis is limited, it still allows one in some respects to take a broader view of discrepancies between forward and expected future spot FX rates than allowed by previous models. In particular, it shows that covered interest arbitrage may produce "blurring effects." For the reasons just discussed, the analysis does not provide a definitive account of exactly when these effects may occur, nor of their exact magnitude.

Indeed, a skeptical reader could assert that market expectations might actually all line up according to interest rate differentials, i.e., for given t and all s, that $E_t p(s) = p(t)e^{(s-t)r}$, where E_t is some weighted average of the E_{it} operators. Two arguments could be advanced to support this assertion. First, one or both of the interest rate term structures could be extremely malleable (i.e., $r(t,\cdot)$ could be easily reshaped by arbitrage borrowing and lending, induced by net speculation). Such would be the case if Ł is the currency of a small open country, for instance (see Beenstock and Longbottom (1981) for an extended discussion of this case).

The other argument is that speculation could be extremely elastic (e.g., in specification (S2), $\sum_{i \in S} a_{its} = w_{ts} \to \infty$ for given t and all s). This entails essential unanimity of beliefs among speculators (i.e., $E_{it}p(s) \simeq E_t p(s)$ given t, "almost all" $i \in S$, and all $s > t$). Speculation of this sort could overwhelm domestic influences on interest rate term structures. However, even if interest rates were completely rigid, such speculation could still eliminate discrepancies between the forward and expected future spot rates by means of a "feedback" effect on the latter rates. To illustrate this effect, suppose speculative purchases of date s_0 contracts mount in anticipation of a large excess "trader" demand for Ł at that date. Then arbitrageurs would acquire an equivalent number of Ł deposits maturing at the same date. Thus the date s_0 supply of Ł is increased by speculation and arbitrage, and expectations should adjust accordingly.

These arguments lead to the conclusion that "blurring effects" will be important only to the extent that interest rates are not very malleable and that speculators are heterogeneous in their beliefs or unwilling to assume very large positions. It is my impression that these conditions have generally been met in recent times, but clearly the question can only be resolved empirically.

The analysis presented in the body of this paper does suggest some directions for empirical work. The Keynes-Hicks risk premium of Section 4 depends on (1) aggregate hedging and (2) aggregate willingness to speculate. The "blurring effect" depends also on (3) the average speculative horizon (embodied in the $\bar{\bar{E}}$ at the end of Section 5). These factors are not directly observable, but may have observable proxies: (1) could be related to (forthcoming) changes to the current account, (2) could be related to perceived volatility of exchange rates, while (2) and (3) could be related to relative transactions volume at different maturities. The distinction between speculation and arbitrage emphasized throughout the analysis suggests that, if available, data on participation by different types of transactors (e.g., Commercial Banks can be regarded as primarily arbitrageurs, at least at more distant maturities) might also be useful.

To summarize, we have argued that the relationship between forward and subsequent spot FX rates is not as simple as it might seem. If our analysis inspires new empirical investigations, or even new, more explicit theoretical models of FX markets, it will have served its purpose.

Appendix:

I. The Accounting System

The *basic agents* in FX markets are denoted by the set $B = \{a,b,c,\ldots\}$. They may be thought of as the actual legal entities (individuals, firms, agencies, *etc*.) participating in the market.

I distinguish three roles -- Trader, Speculator, Arbitrageur, defined below -- and form three corresponding disjoint sets of *agent-roles* T, S, A. The agent-roles (generally referred to simply as *agents*) are related to the set B of basic agents by means of (1:1) inclusion maps $C_X\colon\ X \to B$, $X = T,S,A$. Thus if $a = C_T(k) = C_S(i) = C_A(j) \in B$ for some $k = k(a) \in T$, $i = i(a) \in S$ and $j = j(a) \in A$, then we say that the basic agent a (perhaps a multinational corporation) has been *decomposed* into his trader $(k(a))$, speculator $(i(a))$ and arbitrageur $(j(a))$ roles.

Time is discrete here; generally past dates are denoted by u, present by t and future by s, so $0 \leq u \leq t \leq s \leq \infty$.

Notation for contracts, *etc*.: Generally, *'s indicate *desired* values of variables, ^'s indicate some sort of adjustment, and undecorated variables indicate actual values. For each $i \in T \cup S \cup A$:

$G_i(t,s)$ = the quantity of foreign currency (£) for which agent i by means of a forward contract written on date t agrees to *accept* delivery at date s (*make* delivery if < 0).

$g_i(t,s) \equiv \sum_{u<t} G_i(t,s)$ = i's net £ position for date s as of date t.

$p(t,s)$ = the forward FX rate for £ in terms of \$ (home currency); *e.g.* on date s, i receives £$G_i(t,s)$ and pays \$$p(t,s)\, G_i(t,s)$.

$p(t) \equiv p(t,t)$ and $G_i(t) \equiv G_i(t,t)$ refer to *spot* contracts.

$r_{£}(t,s)$ = the per "date" (e.g. per diem, per month, or per annum) continuously compound interest rate as of date t Eurosterling deposits and loans maturing at date s.

$r_{\$}(t,s)$ = same for Eurodollars.

$r(t,s) = r_{\$}(t,s) - r_{£}(t,s)$ = the interest rate differential. Note: the (t,s) arguments will often be dropped when the context makes them clear.

$g_i^+(t) = \sum_{s>t} e^{-(s-t)r_{£}} g_i(t,s)$ = present £ value of i's net forward £ positions.

= i's "open position" as of date t.

$X_i(t,s)$ = the quantity of £ for which agent i on date t agrees to accept delivery at date s, other than by a forward contract (normally, a scheduled payment for goods or services).

$x_i(t,s) = \sum_{u<t} X_i(t,s)$ = net scheduled £ inflow for date s as of date t other than that due to FX contracts.

$z(p,e)$ = aggregate excess demand for spot £ by traders.

e = a shift parameter in z, thought of as a random supply or demand shock; more properly, a summary of real and nominal influences on spot excess demand not explicitly modelled.

$D_i^C(t,s)$ = New currency C (C = £ or \$) deposits for agent i at date t maturing at date s, measured in currency C value at maturity. If positive, D is often referred to as lending; if negative as borrowing. Note that the C-present value is $e^{-(s-t)r_C} D_i^C(t,s)$.

$d_i^C(t,s) = \sum_{u<t} D_i^C(u,s)$ = Net currency C deposits for agent i as of date t maturing at date s (currency C maturity value).

$V_i(t,s) = \sum_{u<t} (p(t,s)-p(u,s)) (G_i(u,s)+D_i^{£}(u,s))$ = current \$ market value of agent i's date s forward commitments, i.e. unrealized gains or losses on date s contracts accumulated as of date t.

$V_i^+(t) = \sum_{s \geq t} V_i(t,s)$ = current \$ market value of i's total £ position as of date t.

$m_i(t,s) = \sum_{u \leq t} (p(s)-p(u,s))G_i(u,s)+d_i^{\$}(t,s)+p(s)d_i^{£}(t,s)$ = the <u>\$ value of</u> agent i's scheduled <u>net cash flow</u> for date s as of date t;

$m_i(t) \equiv m_i(t,t)$ represents <u>realized</u> gains or losses.

The roles are defined by the following assumptions:

<u>DC</u>: $\sum_{i \in T \cup S \cup A} G_i(t,s) = 0$, all t,s -- all contracts have counterparties; all agents belong to $T \cup S \cup A$.

<u>DS</u>: $g_i(t) \equiv g_i(t,t) = 0$, all t and $i \in S$ -- speculators close out their positions; they don't actually accept delivery of £.

<u>DT</u>: $G_k(t,s) = 0$, all $k \in T$ and $s > t$ -- traders don't buy (or sell) £ forward.

<u>DD</u>: $D_i^C(t,s) = 0$, for $i \in T \cup S$; C = \$, £; t > 0; and $s \geq t$ -- we only count deposits by arbitrageurs.

<u>DA1</u>: $G_j(t,s) + D_j^{£}(t,s) = 0$, $\forall\, j \in A$, $t \geq 0$ and $s > t$ -- Arbitrageurs offset forward contracts by lending (or borrowing) £.

<u>DA2</u>: $p(t,s)G_j(t,s) + D_j^{\$}(t,s) \geq 0$, $\forall\, j \in A$, $t \geq 0$ and $s > t$ -- They will not schedule a negative \$ cash flow.

<u>DA3</u>: $G_j(t) = \sum_{s>t} e^{-(s-t)r_{£}} D_j^{£}(t,s)$, $\forall\, j \in A$, $t \geq 0$ -- They buy (or sell) spot the net quantity of £ necessary to fund their new £ loans (or borrowing).

These assumptions about arbitrageurs easily yield the fact that arbitrage is riskless in the following senses (we drop the index j for convenience):

CA1: $g(t) + d^{£}(t) = 0$ for all t; i.e., actual net £ cash flows are always 0. This follows by summing DA1 over $u \leq t$ for $g = t$.

CA2: $\sum_{u \leq t} p(u,t)G(u,t) + d^{\$}(t) \geq 0$ (and > 0 only if Interest Rate Parity ((A1) in text) fails for some $u < t$); i.e. actual net \$ cash flows are non-negative. This follows by summing DA2 over $u \leq t$ for $s = t$ (and noting that (A1) implies[1] equality in DA2).

CA3: $V^{+}(t) = 0$ for all t; i.e., the market value of arbitrageurs' £ positions is constant over time, hence there are no unrealized gains or losses. This follows by summing DA1 over $u \leq t$, multiplying by $p(t,s)$, and summing over $s \geq t$.

The assumptions DA1-3 also lead to (A1) and (A2) of the text. The argument in the text for A1 can be made completely formal using DA1-3. For (A2), note that

$$\sum_{s \geq t} e^{-(s-t)r_{£}} G(t,s) = G(t) + \sum_{s>t} e^{-(s-t)r_{£}} G(t,s)$$

$$\text{(by DA3 and DA1)} = \sum_{s>t} e^{-(s-t)r_{£}} D^{£}(t,s) - \sum_{s>t} e^{-(s-t)r_{£}} D^{£}(t,s) = 0 \quad .$$

Proposition 4: An arbitrary set of contracts $G_a(\cdot,\cdot)$ can be decomposed into trade, speculative and arbitrage contracts; i.e. there are contracts $G_{k(a)}$, $G_{i(a)}$ and $G_{j(a)}$ satisfying the appropriate assumptions such that $G_a(t,s) = G_{k(a)}(t,s) + G_{i(a)}(t,s) + G_{j(a)}(t,s)$ for all $0 \leq t \leq s \leq \infty$.

---

[1]Actually, this implication requires the "budget constraint" that the present \$ value of the sum of new sterling and dollar loans is zero.

<u>Proof</u>: The simplest way seems to be to define

$$G_{k(a)}(t,s) = \begin{cases} 0 \text{ if } s > t, \\ \sum_{s \geq t} G_a(t,s) \text{ if } s = t. \end{cases}$$

Then define $G_{i(a)}(t,s) = G_a(t,s) - G_{k(a)}(t,s)$ and $G_{j(a)}(t,s) = 0$, for all t,s. It is routine to verify that the required equations are satisfied.

Of course the decomposition is not unique; in fact, $G_{j(a)}$ satisfying DA1-3 can be chosen arbitrarily and $G_{i(a)}$ and $G_{k(a)}$ then evidently be chosen uniquely to satisfy the remaining equations. Thus, the decomposition could be extended to cover <u>deposits</u> as well as forward contracts, but we do not need this for present purposes.

II. Speculation

1. The <u>basic</u> <u>optimization</u> <u>problem</u> is: Speculator i seeks to

$$\max_{G_i(t,\cdot)} J = \sum_{s=t}^{\infty} E_{it} U(m_i(t,s))\, e^{-\rho_i(s-t)}$$, *i.e.* chose forward contracts $G_i(t_1\cdot)$

so as to maximize a discounted sum of current (conditional) expected utilities of profits; p_i is i's rate of time preference. To solve this problem, drop the i subscripts to reduce clutter and note that

$$\begin{aligned} m(t,s) &= \sum_{u\leq t} (p(s)-p(u,s))G(u,s) \\ &= \sum_{u\leq t} ([p(s)-p(t,s)] + [p(t,s)-p(u,s)])G(u,s) \\ &= \delta(t,s) \sum_{u\leq t} G(u,s) + \sum_{u<t} (p(t,s) - p(u,s))G(u,s) \\ &\equiv \delta(t,s)g(t,s) + V(t,s), \end{aligned}$$

so we have expressed the dollar profits m as depending on the prospective price discrepancy δ, the net position g and the market value of the inherited position V.

Now $U(m) = U(V) + (\delta g)U'(V) + (\delta g)^2U''(V)/2 + 0(\delta g)^3$, since U is smooth; we will henceforth assume $0(\delta g)^3$ is zero (*i.e.*, replace U by its 2^{nd} order Taylor approximation about m = V).

Then $E_tU(m) = U(V) + (E_t\delta)g\cdot U'(V) + \frac{1}{2}(E_t\delta^2)g^2U''(V)$.

The first-order conditions for maximizing J for given t are:

$$0 = \frac{\partial J}{\partial g(t,s)} = e^{-\rho(s-t)} \frac{\partial E_tU(m(t,s))}{\partial g(t,s)}, \text{ for all } s \geq t.$$

Dropping the (t,s) arguments, we then have

$$0 = \frac{\partial E_tU(m)}{\partial g} = E_t\delta\cdot U'(V) + g(E_t\delta^2)U''(V),$$

so $g^* = -\frac{U'(V)}{U''(V)} \frac{E_t\delta}{E_t\delta^2} = a_{ts}(E_t p(s)-p(t,s))$, where[1] $a_{ts} = \frac{R(V(t,s))}{Var_t p(s)+(E_t p(s)-p(t,s))^2}$

and $R = -\frac{U'}{U''}$, the reciprocal of the index of absolute risk aversion.

Since $E_t U(m)$ is concave in g, the first-order conditions are necessary and sufficient and the solution $g^*(t,\cdot)$ (resp. $G^*(t,\cdot)$) is unique.

2. Foreign Currency Based Speculation

Suppose our speculator uses £ as numeraire instead of \$. One can modify the variables in the basic optimization as follows. Set $p_£(t,s) = 1/p(t,s)$,

$$\text{so } \delta_£(t,s) = \frac{1}{p(s)} - \frac{1}{p(t,s)} = -\delta(t,s)/p(s)p(t,s).$$

Set $V_£(t,s) = V(t,s)/p(t,s)$, and

$$g_£(t,s) = -p(s)g(t,s), \text{ so}$$

$$M_£(t,s) = m(t,s)/p(t,s).$$

Also, $U_£(m_£(t,s)) = U(p(t,s)m_£(t,s)) = U(m(t,s))$,

$$\text{so } U_£'(m_£(t,s)) = p(t,s)U'(p(t,s)m_£(t,s)),$$

$$\text{and } U_£''(m_£(t,s)) = p(t,s)^2 U''(p(t,s)m_£(t,s)).$$

$$\text{Hence } R_£(V_£(t,s)) = -\frac{U_£'(V_£(t,s))}{U_£''(V_£(t,s))} = -\frac{p(t,s)U'(V(t,s))}{p(t,s)^2 U''(V(t,s))} = R(V(t,s))/p(t,s).$$

Using these redefinitions, the basic optimization problem yields

$$\begin{aligned} g_£^*(t,s) &= R_£(V_£(t,s))\ E_t\delta_£(t,s)/E_t\delta_£^2(t,s) \\ &= \frac{R(V(t,s))}{p(t,s)} \frac{E_t(-\delta(t,s)/p(s))/p(t,s)}{E_t(-\delta(t,s)/p(s))^2/(p(t,s))^2} \\ &= -R(V(t,s)) \frac{E_t(\delta(t,s)/p(s))}{E_t(\delta(t,s)/p(s))^2} \end{aligned}$$

$$= -p(t,s)R(V(t,s))\ \{\frac{E_t\delta(t,s)}{E_t\delta^2(t,s)} - \frac{2E_t\delta(t,s)}{p^2(t,s)} + \ldots\}$$

$$= -p(t,s)g^*(t,s) + 2R(V(t,s))E_t\delta(t,s)/p(t,s) + \ldots$$

The next-to-last equality is obtained from the expansion $\delta/p \equiv \delta/(p+\delta) = \delta/p - (\delta/p)^2 + (\delta/p)^3 - \ldots$, where $\delta \equiv \delta(t,s)$ and $p \equiv p(t,s)$.

Hence $g^*_£(t,s)$, the desired net \$ position for a £-based speculator, is essentially the same as the dollar value of desired net £ position of a \$-based speculator, $p(t,s)g^*(t,s)$, with the sign reversed (since "long" £ means "short" \$). There is a discrepancy, $2RE_t\delta/p + \ldots$, due to the usual Jensen's inequality problem, but since it is of order $E_t\delta/p$, it should ordinarily be quite small and bounded by the bid-ask spread on the forward rate p. Thus for our purposes the discrepancy is negligible and the speculator's choice of numeraire is immaterial.

3. Now suppose there is a scheduled £ payment $x(t,s)$. Set $\hat{V}(t,s) = V(t,s) + p(t,s)x(t,s)$, $\hat{g}(t,s) = g(t,s) + x(t,s)$ and $\hat{m}(t,s) = m(t,s) + p(s)x(t,s)$. Then one can readily verify that $\hat{m} = d\hat{g} + \hat{V}$, so the derivation in the basic optimization problem yields $\hat{g}^* = \hat{a}_{ts}(E_tp(s)-p(t,s))$, where $\hat{a}_{ts}$ is the same as a_{ts} except $\hat{V}$ replaces V. Hence one would generally expect $\hat{g}^*$ to be very close to g^*; the two would always coincide if U exhibited constant risk aversion. In any case, $\hat{g}(t,s) = \hat{g}(t-1,s) + X(t,s) + G(t,s)$, so

$$\hat{G}^*(t,s) = \hat{g}^*(t,s) - \hat{g}(t-1,s) - X(t,s) \simeq G^*(t,s) - X(t,s).$$

4. The specification for J in the basic optimization problem applies best to a situation in which speculators have other income which is constant

in each period, and in which they can't borrow or lend. If unlimited borrowing or lending at a market rate r were permitted, it would seem preferable to specify an objective function of the form

$$K = E_t U(\sum_{s \geq t} m(t,s)e^{-r(s-t)}),$$

i.e., speculators maximize expected utility of the *present value* of prospective cash flows, PV. The first-order conditions for K, however, are of the form

$$E_t(\delta(t,s) \cdot U'[PV]) = 0 \text{ for all } s \geq t.$$

My interpretation of these conditions is that the speculator should try to arrange his forward positions so that the vector of anticipated price discrepancies $\delta(t,\cdot)$ is orthogonal to (statistically independent of) his marginal utility of present value of cash flows, with anticipations and orthogonality taken with respect to his *joint* subjective probability distribution for $p(t,\cdot)$. In general, there will not be a unique, nontrivial solution to these first-order conditions.

Perhaps one escape route from some of the difficulties inherent in this last approach is to refine the objective function still further. Following Ben Eden, we might specify something like

$$L = E_t(\sum_{s \geq t} e^{-\rho(s-t)}U(C_s) + \lambda(\sum_{s \geq t} e^{-r(s-t)}(C_s - m(t,s)))$$

as the objective function; *i.e.*, speculators maximize utility of *consumption* C_s, subject to a life-long budget constraint. I do not yet know the implications of this approach.

A still more general approach would be to have the speculator act on his beliefs, not only about the future time path of the spot rate, but also on the time paths of forward rates. Such a speculator would solve the dynamic optimization problem of picking forward contracts to be entered into at the present time, and also planning forward contracts to be entered into in the future. For instance, if he believed that $p(t,s)$, although probably above the eventual realization of $p(s)$, is quite likely below some $p(\tau,s)$ for $t < \tau < s$, he might buy £ forward now ($G^*(t,s) > 0$) with the intention of reversing the sale later ($G^*(\tau,s) = -G^*(t,s)$).

Such refinements, although possibly useful in a fully articulated theory of speculation, are not really germane to the main purpose of this paper and therefore will not be pursued further.

III. Market Thinning

Suppose (A2a) applies, and R_i is bounded. Then the asymptotic behavior of a_{its} is dominated by the "forecast precision" $Var_{it}^{-1}p(s)$ as $s \to \infty$. One common model[1] of FX spot rate dynamics is lognormal diffusion; i.e. the log of the probability density of the future spot rate p(s), given the current spot rate p(t), is normal $N(\mu_i(t,s),\sigma_i^2(t,s))$, where $\sigma_i^2(t,s) \geq b_i(s-t)$ for some positive b_i. It can be shown[2] that lognormality implies that $Var_{it}p(s) = \exp(2\mu(t,s) + \sigma^2(t,s)) \{\exp(\sigma^2(t,s)) - 1\}$ so the precision $Var_{it}^{-1}p(s)$ falls to 0 at least as fast as $\exp(-2b_i(s-t))$ as $s - t \to \infty$. Hence (A2a), bounded risk acceptance and lognormality of beliefs imply exponentially decreasing individual willingness to speculate a_{its}, with increasing s.

If all a_{its} decrease exponentially as $s \to \infty$ then so will their sum $w_{ts} = \sum_{i\varepsilon a} a_{its}$. This implies that markets thin as increasingly distant maturities s are considered, as can be seen from the following argument.

The volume of new contracts satisfies $\frac{1}{2} \sum_{i\varepsilon SnA} |G_i(t,s)| \leq w_{ts} \max_{i\varepsilon S} \delta_{its}$, where $\delta_{its} = E_{it}p(s) - p(t,s)$. (The upper bound would be achieved in case all δ_i's coincided; then speculators would form one side of the market and arbitrageurs the other.) Assuming the δ_{its} are bounded (or at least do not grow more rapidly than $\exp(-2B(s-t))$, where $B = \min b_i$ -- seemingly a safe assumption) then indeed markets must eventually thin exponentially at longer maturities.

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[1]See for example Solnik (1977). An empirical examination of this model is contained in Friedman and Vandersteel (1980).

[2]See Clark (1973), for instance.

A similar condition is also necessary for the definition of W_t in Proposition 3 and other terms involving summations over s. Specifically, we need the condition that

$$(*) W_t = \sum_{s>t} e^{-(s-t)(r-r_{\pounds})} W_{ts} < \infty \quad .$$

It suffices that $r - r_{\pounds} = r_{\$}(t,s) - 2r_{\pounds}(t,s) < 2B$ for s sufficiently large -- again, a seemingly safe assumption.

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